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Subclasses of analytic functions associated with Pascal distribution series

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Abstract

In the present paper we determine necessary and sufficient conditions for the Pascal distribution series to be in the subclasses $\mathcal{S}(k,\lambda)$ and $\mathcal{C}(k,\lambda)$ of analytic functions. Further, we consider an integral operator related to Pascal distribution series. Some interesting special cases of our main results are also considered.

Keywords: Analytic functions ; Hadamard product; Pascal distribution series.
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1. Introduction and definitions

Let $\mathcal{A}$ denote the class of the normalized functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Further, let $\mathcal{T}$ be a subclass of $\mathcal{A}$ consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| \, z^n. \tag{2}$$

A function f of the form (2) is in $\mathcal{S}(k,\lambda)$ if it satisfies the condition

$$\left| \frac{\frac{zf'(z)}{(1-\lambda)f(z)+\lambda zf'(z)} - 1}{\frac{zf'(z)}{(1-\lambda)f(z)+\lambda zf'(z)} + 1} \right| < k, \quad (0 < k \le 1,\ 0 \le \lambda < 1, z \in \mathbb{U})$$

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and $f \in \mathcal{C}(k,\lambda)$ if and only if $zf' \in \mathcal{S}(k,\lambda)$.The class $\mathcal{S}(k,\lambda)$ was studied by Frasin et al. [7].

We note that $\mathcal{S}(k,0) = \mathcal{S}(k)$ and $\mathcal{C}(k,0) = \mathcal{C}(k)$, where the classes $\mathcal{S}(k)$ and $\mathcal{C}(k)$ were introduced and studied by Padmanabhan [17] (see also, [11], [16]) .

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{R}^{\tau}(A,B)$,$\tau \in \mathbb{C}\backslash\{0\}$, $-1 \le B < A \le 1$, if it satisfies the inequality

$$\left|\frac{f'(z)-1}{(A-B)\tau - B[f'(z)-1]}\right| < 1, \quad z \in \mathbb{U}.$$

This class was introduced by Dixit and Pal [3].

A variable X is said to be Pascal distribution if it takes the values $0, 1, 2, 3, \ldots$ with probabilities $(1-q)^m$, $\frac{qm(1-q)^m}{1!}$, $\frac{q^2m(m+1)(1-q)^m}{2!}$, $\frac{q^3m(m+1)(m+2)(1-q)^m}{3!}$, $\ldots$, respectively, where q and m are called the parameters, and thus

$$P(X=r) = \binom{r+m-1}{m-1} q^r(1-q)^m, \quad (m \ge 1, 0 \le q \le 1, r = 0, 1, 2, 3, \ldots.).$$

Very recently, El-Deeb et al.[5] (see also, [14, 1]) introduced a power series whose coefficients are probabilities of Pascal distribution, that is

$$\Psi_q^m(z) := z + \sum_{n=2}^{\infty} \binom{n+m-2}{m-1} q^{n-1}(1-q)^m z^n, \; z \in \mathbb{U},$$

where $m \ge 1$, $0 \le q \le 1$, and we note that, by ratio test the radius of convergence of above series is infinity. We also define the series

$$\Phi_q^m(z) := 2z - \Psi_q^m(z) = z - \sum_{n=2}^{\infty} \binom{n+m-2}{m-1} q^{n-1}(1-q)^m z^n, \; z \in \mathbb{U}. \tag{3}$$

Let consider the linear operator $\mathcal{I}_q^m : \mathcal{A} \to \mathcal{A}$ defined by the convolution or Hadamard product

$$\mathcal{I}_q^m f(z) := \Psi_q^m(z) * f(z) = z + \sum_{n=2}^{\infty} \binom{n+m-2}{m-1} q^{n-1}(1-q)^m a_n z^n, \; z \in \mathbb{U},$$

where $m \ge 1$ and $0 \le q \le 1$.

Motivated by several earlier results on connections between various subclasses of analytic and univalent functions, by using hypergeometric functions (see for example, [2, 7, 10, 19, 20]) and by using various distributions such as Yule-Simon distribution, Logarithmic distribution, Poisson distribution, Binomial distribution, Beta-Binomial distribution, Zeta distribution, Geometric distribution and Bernoulli distribution (see for example, [4, 6, 8, 9, 12, 13, 18, 15]), in this paper, we determine the necessary and sufficient conditions for $\Phi_q^m(z)$ to be in our classes $\mathcal{S}(k,\lambda)$ and $\mathcal{C}(k,\lambda)$ and connections of these subclasses with $\mathcal{R}^{\tau}(A,B)$. Finally, we give conditions for the integral operator $\mathcal{G}_q^m(m,z) = \int_0^z \frac{\Phi_q^m(t)}{t}dt$ belonging to the above classes.

2. Preliminary lemmas

To establish our main results, we need the following Lemmas.

Lemma 2.1. *[7]A function f of the form (2) is in $\mathcal{S}(k,\lambda)$ if and only if it satisfies*

$$\sum_{n=2}^{\infty} [n((1-\lambda)+k(1+\lambda)) - (1-\lambda)(1-k)]\,|a_n| \le 2k \tag{4}$$

where $0 < k \leq 1$ *and* $0 \leq \lambda < 1$. *The result is sharp.*

Lemma 2.2. *[7]A function* f *of the form (2) is in* $\mathcal{C}(k,\lambda)$ *if and only if it satisfies*

$$\sum_{n=2}^{\infty} n[n((1-\lambda)+k(1+\lambda))-(1-\lambda)(1-k)]\,|a_n| \leq 2k \tag{5}$$

where $0 < k \leq 1$ *and* $0 \leq \lambda < 1$. *The result is sharp.*

Lemma 2.3. *[3] If* $f \in \mathcal{R}^{\tau}(A,B)$ *is of the form (1) , then*

$$|a_n| \leq (A-B)\frac{|\tau|}{n}, \qquad n \in \mathbb{N}-\{1\}.$$

The result is sharp for the function

$$f(z) = \int_0^z (1+(A-B)\frac{\tau t^{n-1}}{1+Bt^{n-1}})dt, \qquad (z \in \mathbb{U}; n \in \mathbb{N}-\{1\}).$$

3. Necessary and sufficient conditions

For convenience throughout in the sequel, we use the following identities that hold at least for $m \geq 2$ and $0 \leq q < 1$:

$$\sum_{n=0}^{\infty}\binom{n+m-1}{m-1}q^n = \frac{1}{(1-q)^m}, \quad \sum_{n=0}^{\infty}\binom{n+m-2}{m-2}q^n = \frac{1}{(1-q)^{m-1}},$$
$$\sum_{n=0}^{\infty}\binom{n+m}{m}q^n = \frac{1}{(1-q)^{m+1}}, \quad \sum_{n=0}^{\infty}\binom{n+m+1}{m+1}q^n = \frac{1}{(1-q)^{m+2}}.$$

By simple calculations we derive the following relations:

$$\sum_{n=2}^{\infty}\binom{n+m-2}{m-1}q^{n-1} = \sum_{n=0}^{\infty}\binom{n+m-1}{m-1}q^n - 1 = \frac{1}{(1-q)^m} - 1,$$
$$\sum_{n=2}^{\infty}(n-1)\binom{n+m-2}{m-1}q^{n-1} = qm\sum_{n=0}^{\infty}\binom{n+m}{m}q^n = \frac{qm}{(1-q)^{m+1}},$$

and

$$\begin{aligned}\sum_{n=3}^{\infty}(n-1)(n-2)\binom{n+m-2}{m-1}q^{n-1} &= q^2m(m+1)\sum_{n=0}^{\infty}\binom{n+m+1}{m+1}q^n \\ &= \frac{q^2m(m+1)}{(1-q)^{m+2}}.\end{aligned}$$

Unless otherwise mentioned, we shall assume in this paper that $0 < k \leq 1$, $0 \leq \lambda < 1$, while $m \geq 1$ and $0 \leq q < 1$.

Firstly, we obtain the necessary and sufficient conditions for Φ_q^m to be in the class $\mathcal{S}(k,\lambda)$.

Theorem 3.1. *We have* $\Phi_q^m \in \mathcal{S}(k,\lambda)$, *if and only if*

$$((1-\lambda)+k(1+\lambda))\frac{q\ m}{(1-q)^{m+1}} \leq 2k. \tag{6}$$

Proof. Since

$$\Phi_q^m(z) = z - \sum_{n=2}^{\infty} \binom{n+m-2}{m-1} q^{n-1}(1-q)^m z^n \tag{7}$$

in view of Lemma 2.1, it suffices to show that

$$\sum_{n=2}^{\infty}[n((1-\lambda)+k(1+\lambda))-(1-\lambda)(1-k)]\binom{n+m-2}{m-1}q^{n-1}(1-q)^m \leq 2k. \tag{8}$$

Writing

$$n = (n-1)+1$$

in (8)) we have

$$\begin{aligned}
&\sum_{n=2}^{\infty}[n((1-\lambda)+k(1+\lambda))-(1-\lambda)(1-k)]\binom{n+m-2}{m-1}q^{n-1}(1-q)^m \\
&\sum_{n=2}^{\infty}[(n-1)((1-\lambda)+k(1+\lambda))+2k]\binom{n+m-2}{m-1}q^{n-1}(1-q)^m \\
= \;& [(1-\lambda)+k(1+\lambda)]\sum_{n=2}^{\infty}(n-1)\binom{n+m-2}{m-1}q^{n-1}(1-q)^m \\
&+2k\sum_{n=2}^{\infty}\binom{n+m-2}{m-1}q^{n-1}(1-q)^m \\
= \;& ((1-\lambda)+k(1+\lambda))\frac{q\ m}{1-q} + 2k\left(1-(1-q)^m\right).
\end{aligned}$$

But this last expression is bounded above by $2k$ if and only if (6) holds. □

Theorem 3.2. *We have $\Phi_q^m \in \mathcal{C}(k,\lambda)$ if and only if*

$$\begin{aligned}
&[(1-\lambda)+k(1+\lambda)]\frac{q^2\ m(m+1)}{(1-q)^{m+2}} \\
&+[(1-\lambda)(2+k)+3k(1+\lambda)]\frac{q\ m}{(1-q)^{m+1}} \leq 2k.
\end{aligned} \tag{9}$$

Proof. In view of Lemma 2.2, we must show that

$$\sum_{n=2}^{\infty} n[n((1-\lambda)+k(1+\lambda))-(1-\lambda)(1-k)]\binom{n+m-2}{m-1}q^{n-1}(1-q)^m \leq 2k. \tag{10}$$

Writing

$$n^2 = (n-1)(n-2)+3(n-1)+1 \qquad \text{and} \quad n = (n-1)+1,$$

we get

$$
\begin{aligned}
&\sum_{n=2}^{\infty} n[n((1-\lambda)+k(1+\lambda))-(1-\lambda)(1-k)]\binom{n+m-2}{m-1}q^{n-1}(1-q)^m \\
= \; & [(1-\lambda)+k(1+\lambda)]\sum_{n=3}^{\infty}(n-1)(n-2)\binom{n+m-2}{m-1}q^{n-1}(1-q)^m \\
& +[(1-\lambda)(2+k)+3k(1+\lambda)]\sum_{n=2}^{\infty}(n-1)\binom{n+m-2}{m-1}q^{n-1}(1-q)^m \\
& +2k\sum_{n=2}^{\infty}\binom{n+m-2}{m-1}q^{n-1}(1-q)^m \\
= \; & [(1-\lambda)+k(1+\lambda)]\frac{q^2\, m(m+1)}{(1-q)^2} \\
& +[(1-\lambda)(2+k)+3k(1+\lambda)]\frac{q\, m}{(1-q)}+2k\left(1-(1-q)^m\right).
\end{aligned}
$$

Therefore, we see that the last expression is bounded above by $2k$ if (9) is satisfied. □

4. Inclusion Properties

Making use of Lemma 2.3, we will study the action of the Pascal distribution series on the classes $\mathcal{S}(k,\lambda)$ and $\mathcal{C}(k,\lambda)$.

Theorem 4.1. *Let $m>1$. If $f\in\mathcal{R}^{\tau}(A,B)$, then $\mathcal{I}_q^m\in\mathcal{S}(k,\lambda)$ if*

$$
\begin{aligned}
& (A-B)|\tau|\Big[[(1-\lambda)+k(1+\lambda)]\left(1-(1-q)^m\right) \\
& -\frac{(1-\lambda)(1-k)}{q(m-1)}\left[(1-q)-(1-q)^m-q(m-1)(1-q)^m\right]\Big] \\
\leq \; & 2k. \qquad (11)
\end{aligned}
$$

Proof. In view of Lemma 2.1, it suffices to show that

$$
\sum_{n=2}^{\infty}[n((1-\lambda)+k(1+\lambda))-(1-\lambda)(1-k)]\binom{n+m-2}{m-1}q^{n-1}(1-q)^m\,|a_n|\leq 2k.
$$

Since $f\in\mathcal{R}^{\tau}(A,B)$, then by Lemma 2.3, we have

$$
|a_n|\leq\frac{(A-B)\,|\tau|}{n}. \qquad (12)
$$

Thus, we have

$$
\begin{aligned}
&\sum_{n=2}^{\infty}[n((1-\lambda)+k(1+\lambda))-(1-\lambda)(1-k)]\binom{n+m-2}{m-1}q^{n-1}(1-q)^m\,|a_n| \\
\leq \; & (A-B)\,|\tau|\left[\sum_{n=2}^{\infty}[(1-\lambda)+k(1+\lambda)]\binom{n+m-2}{m-1}q^{n-1}(1-q)^m\right. \\
& \left.-\sum_{n=2}^{\infty}\frac{1}{n}[(1-\lambda)(1-k)]\binom{n+m-2}{m-1}q^{n-1}(1-q)^m\right] \\
= \; & (A-B)|\tau|\Big[[(1-\lambda)+k(1+\lambda)]\left(1-(1-q)^m\right) \\
& -\frac{(1-\lambda)(1-k)}{q(m-1)}\left[(1-q)-(1-q)^m-q(m-1)(1-q)^m\right]\Big]
\end{aligned}
$$

But this last expression is bounded by $2k$, if (11) holds, which completes the proof of Theorem 4.1. □

Applying Lemma 2.2 and using the same technique as in the proof of Theorem 4.1 we have the following result:

Theorem 4.2. *Let $m \geq 1$. If $f \in \mathcal{R}^{\tau}(A,B)$, then $\mathcal{I}_q^m \in \mathcal{C}(k,\lambda)$ if*

$$(A-B)|\tau| \left[((1-\lambda)+k(1+\lambda))\frac{q\ m}{1-q} + 2k\left(1-(1-q)^m\right) \right] \leq 2k. \tag{13}$$

5. An integral operator

Theorem 5.1. *If $m \geq 1$, then the integral operator*

$$\mathcal{G}_q^m(m,z) = \int_0^z \frac{\Phi_q^m(t)}{t} dt \tag{14}$$

is in $\mathcal{C}(k,\lambda)$ if and only if inequality (6) *is satisfied.*

Proof. According to ((14)) it follows that

$$\mathcal{G}_q^m(m,z) = z - \sum_{n=2}^{\infty} \binom{n+m-2}{m-1} q^{n-1}(1-q)^m \frac{z^n}{n}$$

then by Lemma 2.2, we need only to show that

$$\sum_{n=2}^{\infty} n[n((1-\lambda)+k(1+\lambda)) - (1-\lambda)(1-k)] \times \frac{1}{n}\binom{n+m-2}{m-1} q^{n-1}(1-q)^m \leq 2k,$$

or, equivalently

$$\sum_{n=2}^{\infty} [n((1-\lambda)+k(1+\lambda)) - (1-\lambda)(1-k)] \binom{n+m-2}{m-1} q^{n-1}(1-q)^m \leq 2k. \tag{15}$$

The remaining part of the proof of Theorem 5.1 is similar to that of Theorem 3.1, and so we omit the details. □

Theorem 5.2. *If $m > 1$, then the integral operator $\mathcal{G}_q^m(m,z)$given by ((14)) is in $\mathcal{S}(k,\lambda)$ if and only if*

$$\begin{aligned} & [(1-\lambda)+k(1+\lambda)]\left(1-(1-q)^m\right) \\ & -\frac{(1-\lambda)(1-k)}{q(m-1)}\left[(1-q)-(1-q)^m - q(m-1)(1-q)^m\right] \\ \leq\ & 2k. \end{aligned}$$

The proof of Theorem 5.2 is lines similar to the proof of Theorem 5.1, so we omitted the proof of this theorem.

6. Corollaries and consequences

By specializing the parameter $\lambda = 0$ in the above theorems we obtain the following corollaries.

Corollary 6.1. *We have $\Phi_q^m \in \mathcal{S}(k)$, if and only if*

$$\frac{q\ m(1+k)}{(1-q)^{m+1}} \leq 2k. \tag{16}$$

Corollary 6.2. *We have* $\Phi_q^m \in \mathcal{C}(k)$ *if and only if*

$$\frac{q^2\ m(m+1)(1+k)}{(1-q)^{m+2}} + \frac{2q\ m(1+2k)}{(1-q)^{m+1}} \leq 2k. \tag{17}$$

Corollary 6.3. *Let* $m > 1$. *If* $f \in \mathcal{R}^\tau(A,B)$, *then* $\mathcal{I}_q^m \in \mathcal{S}(k)$ *if*

$$\begin{aligned} & (A-B)|\tau| \left[[(1+k)\,(1-(1-q)^m) \right. \\ & \left. -\frac{(1-k)}{q(m-1)}\,[(1-q)-(1-q)^m - q(m-1)(1-q)^m] \right] \\ \leq \;& 2k. \end{aligned} \tag{18}$$

Corollary 6.4. *Let* $m \geq 1$. *If* $f \in \mathcal{R}^\tau(A,B)$, *then* $\mathcal{I}_q^m f \in \mathcal{C}(k)$ *if*

$$(A-B)|\tau| \left[(1+k)\frac{q\ m}{1-q} + 2k\,(1-(1-q)^m) \right] \leq 2k. \tag{19}$$

Corollary 6.5. *If* $m \geq 1$, *then the integral operator* $\mathcal{G}_q^m(m,z)$ *given by* (14) *is in* $\mathcal{C}(k)$ *if and only if inequality* (16) *is satisfied.*

If $m > 1$, then the integral operator $\mathcal{G}_q^m(m,z)$ given by (14) is in $\mathcal{S}(k)$ if and only if

$$\begin{aligned} & (1+k)\,(1-(1-q)^m) \\ & -\frac{(1-k)}{q(m-1)}\,[(1-q)-(1-q)^m - q(m-1)(1-q)^m] \\ \leq \;& 2k. \end{aligned}$$

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