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On Minimal Generating Sets of Certain Subsemigroups of Isometries

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Abstract
Let DP_n and ODP_n be the semigroups of all isometries and of all order-preserving isometries on X_n, respectively. In this paper we investigate the structure of minimal generating sets of the subsemigroup $DP_{n,r} = \{\alpha \in DP_n : |\mathrm{im}\,(\alpha)| \leq r\}$ (similarly of the subsemigroup $ODP_{n,r} = \{\alpha \in ODP_n : |\mathrm{im}\,(\alpha)| \leq r\}$) for $2 \leq r \leq n-1$.

Keywords: *Isometry; order-preserving /order-reversing map; (minimal) generating set.*

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1. Introduction

Let I_n be the symmetric inverse semigroup on the finite chain $X_n = \{1, \ldots, n\}$, and let $\alpha \in I_n$. If $(\forall x \in \mathrm{dom}\,(\alpha))$ $x\alpha = x$ then α is called the *partial identity map* on $U = \mathrm{dom}\,(\alpha) \subseteq X_n$, denoted by id_U. If $(\forall x, y \in \mathrm{dom}\,(\alpha))$ $x \leq y \Rightarrow x\alpha \leq y\alpha$ ($x \leq y \Rightarrow x\alpha \geq y\alpha$) then α is called an *order-preserving map* (an *order-reversing map*), and if $(\forall x, y \in \mathrm{dom}\,(\alpha))$ $|x - y| = |x\alpha - y\alpha|$ then α is called an *isometry* (or *distance-preserving map*) on X_n, under its natural order. Then the subset of all isometries and the subset of all order-preserving isometries, denoted by DP_n and ODP_n respectively, that is,

$$\begin{aligned} DP_n &= \{\alpha \in I_n : (\forall x, y \in \mathrm{dom}\,(\alpha)) \quad |x - y| = |x\alpha - y\alpha|\,\} \text{ and} \\ ODP_n &= \{\alpha \in DP_n : (\forall x, y \in \mathrm{dom}\,(\alpha)) \quad x \leq y \Rightarrow x\alpha \leq y\alpha\,\}, \end{aligned}$$

are clearly subsemigroups of I_n and $ODP_n \subseteq DP_n \subseteq I_n$. Moreover, for $2 \leq r \leq n-1$, let

$$\begin{aligned} DP_{n,r} &= \{\alpha \in DP_n : |\mathrm{im}\,(\alpha)| \leq r\} \text{ and} \\ ODP_{n,r} &= \{\alpha \in ODP_n : |\mathrm{im}\,(\alpha)| \leq r\} \end{aligned}$$

which are clearly subsemigroups of DP_n and ODP_n, respectively.

Let S be any semigroup, and let W be any non-empty subset of S. Then the subsemigroup generated by W, that is, the smallest subsemigroup of S containing W, is denoted by $\langle W \rangle$. The *rank* of a finitely generated semigroup S, i.e., a semigroup generated by a finite subset, is defined by

$$\mathrm{rank}\,(S) = \min\{\,|W| : \langle W \rangle = S\}.$$

Moreover, the generating set of S with cardinality $\mathrm{rank}\,(S)$ is called a *minimal generating set* of S.

Al-Kharousi, Kehinde and Umar showed in [1, Theorems 3.1, 3.4 and 3.5] that

$$\begin{aligned} &\mathrm{rank}\,(ODP_{n,n-1}) = n, \qquad \mathrm{rank}\,(ODP_n) = n + 1, \\ &\mathrm{rank}\,(DP_{n,n-1}) = n, \text{ and } \mathrm{rank}\,(DP_n) = \lfloor \frac{n+3}{2} \rfloor. \end{aligned}$$

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Then, in [2], we introduced some properties of $DP_{n,r}$ and $ODP_{n,r}$, and also showed that

$$\operatorname{rank}(DP_{n,r}) = \binom{n}{r} \text{ and } \operatorname{rank}(ODP_{n,r}) = \binom{n}{r}.$$

However, there were no results about the structure of any minimal generating set of $DP_{n,r}$ ($ODP_{n,r}$) and no method for whether an arbitrary non-empty subset X of $DP_{n,r}$ ($ODP_{n,r}$) is a minimal generating set of $DP_{n,r}$ ($ODP_{n,r}$), or not for $2 \leq r \leq n-1$. Thereby, in this study we improve a useful method to respond this lack.

2. Preliminaries

In this section we remind some definitions and properties given also in [2], and without otherwise stated we take $2 \leq r \leq n-1$.

Let $\alpha \in DP_n$, and let $\operatorname{dom}(\alpha) = \{a_1 < \cdots < a_p\}$ with $2 \leq p \leq n$. Then the *gap* and the *reverse-gap* of α, denoted by $\mathrm{g}(\alpha)$ and $\mathrm{g}^R(\alpha)$, are defined by

$$\mathrm{g}(\alpha) = (d_1, \ldots, d_{p-1}) \text{ and } \mathrm{g}^R(\alpha) = (d_{p-1}, \ldots, d_1),$$

respectively, where $d_i = a_{(i+1)} - a_i$ for each $1 \leq i \leq p-1$. It is easy to see that $p-1 \leq \sum_{i=1}^{p-1} d_i \leq n-1$ for any gap $(d_1, \ldots, d_{p-1})$. Moreover, for any ordered $(p-1)$-tuple $(d_1, \ldots, d_{p-1})$, if

$$(d_1, \ldots, d_{p-1}) = (d_{p-1}, \ldots, d_1),$$

then $(d_1, \ldots, d_{p-1})$ is called *symmetric* (otherwise, *asymmetric*) ordered $(p-1)$-tuple.

From [1, Lemma 1.2] we know that each element of DP_n is either order-preserving or order-reversing map. Let $\alpha \in DP_n$ such that $\operatorname{dom}(\alpha) = A = \{a_1 < \cdots < a_p\}$ and $\operatorname{im}(\alpha) = B = \{b_1 < \cdots < b_p\}$ for $2 \leq p \leq n$. If α is an order-preserving map, then $a_{i+1} - a_i = b_{i+1} - b_i$ for each $1 \leq i \leq p-1 \leq n-1$ and α has the following tabular form:

$$\alpha = \begin{pmatrix} a_1 & a_2 & \cdots & a_p \\ b_1 & b_2 & \cdots & b_p \end{pmatrix}, \text{ or shortly } \alpha = \begin{pmatrix} A \\ B \end{pmatrix}.$$

If $\alpha \in DP_n$ is an order-reversing map, then $a_{i+1} - a_i = b_{p-i+1} - b_{p-i}$ for each $1 \leq i \leq p-1 \leq n-1$ and α has the following tabular form:

$$\alpha = \begin{pmatrix} a_1 & a_2 & \cdots & a_p \\ b_p & b_{p-1} & \cdots & b_1 \end{pmatrix}, \text{ or shortly } \alpha = \begin{pmatrix} A \\ B^R \end{pmatrix}.$$

From the definitions of the Green's equivalences we clearly have

(i) $\alpha \mathcal{R} \beta \Leftrightarrow \operatorname{dom}(\alpha) = \operatorname{dom}(\beta)$,

(ii) $\alpha \mathcal{L} \beta \Leftrightarrow \operatorname{im}(\alpha) = \operatorname{im}(\beta)$ and

(iii) $\alpha \mathcal{H} \beta \Leftrightarrow \operatorname{dom}(\alpha) = \operatorname{dom}(\beta)$ and $\operatorname{im}(\alpha) = \operatorname{im}(\beta)$

for $\alpha, \beta \in DP_{n,r}$ or $\alpha, \beta \in ODP_{n,r}$, and we have

(iv) $\alpha \mathcal{D} \beta \Leftrightarrow \mathrm{g}(\alpha) = \mathrm{g}(\beta)$ or $\mathrm{g}(\alpha) = \mathrm{g}^R(\beta)$ for $\alpha, \beta \in DP_{n,r}$ and

(v) $\alpha \mathcal{D} \beta \Leftrightarrow \mathrm{g}(\alpha) = \mathrm{g}(\beta)$ for $\alpha, \beta \in ODP_{n,r}$.

(For the definitions of Green's equivalences and for the other terms in semigroup theory, which are not explained here, we refer to [3, 4]).

Let $K_p = \{\alpha \in DP_n : |\operatorname{im}(\alpha)| = p\}$ and let $L_p = \{\alpha \in ODP_n : |\operatorname{im}(\alpha)| = p\}$ for $0 \leq p \leq n$. Then K_p (L_p) is disjoint union of some $\mathcal{D}$-classes since $|\operatorname{im}(\alpha)| = |\operatorname{im}(\beta)|$ for $(\alpha, \beta) \in \mathcal{D}$, and there exist $\binom{n}{p}$ $\mathcal{R}$-classes and $\binom{n}{p}$ $\mathcal{L}$-classes in K_p (L_p). Moreover, $DP_{n,r}$ ($ODP_{n,r}$) is the disjoint union of $K_0, K_1, \ldots, K_r$ ($L_0, L_1, \ldots, L_r$).

For $2 \leq p \leq n$ let $(d_1, \ldots, d_{p-1})$ be a possible gap. Then, let $D_{(d_1,\ldots,d_{p-1})}$ denotes the $\mathcal{D}$-class which consists of the elements with gap or reverse-gap $(d_1, \ldots, d_{p-1})$ in K_p, and similarly denotes the $\mathcal{D}$-class which consists of the elements with gap $(d_1, \ldots, d_{p-1})$ in L_p. Notice that all the subsets of X_n with the gap $(d_1, \ldots, d_{p-1})$ are

$$A_k = \{k, k+d_1, k+d_1+d_2, \ldots, k+t\} \quad \text{for } 1 \leq k \leq n-t$$

and with the reverse-gap $(d_1, \ldots, d_{p-1})$ are

$$B_k \quad = \quad \{k, k + d_{p-1}, k + d_{p-1} + d_{p-2}, \ldots, k + t\} \quad \text{ for } 1 \leq k \leq n - t$$

where $t = \sum_{i=1}^{p-1} d_i$. If $(d_1, \ldots, d_{p-1})$ is symmetric then, since $A_k = B_k$ for each $1 \leq k \leq n-t$, the $\mathcal{D}$-class $D_{(d_1,\ldots,d_{p-1})}$ in K_p has the following egg box form:

D_s	L_1		L_{n-t}
R_1	$\begin{pmatrix} A_1 \\ A_1 \end{pmatrix}, \begin{pmatrix} A_1 \\ A_1^R \end{pmatrix}$	$\cdots$	$\begin{pmatrix} A_1 \\ A_{n-t} \end{pmatrix}, \begin{pmatrix} A_1 \\ A_{n-t}^R \end{pmatrix}$
		$\ddots$	
R_{n-t}	$\begin{pmatrix} A_{n-t} \\ A_1 \end{pmatrix}, \begin{pmatrix} A_{n-t} \\ A_1^R \end{pmatrix}$	$\cdots$	$\begin{pmatrix} A_{n-t} \\ A_{n-t} \end{pmatrix}, \begin{pmatrix} A_{n-t} \\ A_{n-t}^R \end{pmatrix}$

If $(d_1, \ldots, d_{p-1})$ is asymmetric, then the $\mathcal{D}$-class $D_{(d_1,\ldots,d_{p-1})}$ in K_p has the following egg box form:

D_{as}	L_1		L_{n-t}	L_{n-t+1}		$L_{2(n-t)}$
R_1	$\begin{pmatrix} A_1 \\ A_1 \end{pmatrix}$	$\cdots$	$\begin{pmatrix} A_1 \\ A_{n-t} \end{pmatrix}$	$\begin{pmatrix} A_1 \\ B_1^R \end{pmatrix}$	$\cdots$	$\begin{pmatrix} A_1 \\ B_{n-t}^R \end{pmatrix}$
		$\ddots$			$\ddots$	
R_{n-t}	$\begin{pmatrix} A_{n-t} \\ A_1 \end{pmatrix}$	$\cdots$	$\begin{pmatrix} A_{n-t} \\ A_{n-t} \end{pmatrix}$	$\begin{pmatrix} A_{n-t} \\ B_1^R \end{pmatrix}$	$\cdots$	$\begin{pmatrix} A_{n-t} \\ B_{n-t}^R \end{pmatrix}$
R_{n-t+1}	$\begin{pmatrix} B_1 \\ A_1^R \end{pmatrix}$	$\cdots$	$\begin{pmatrix} B_1 \\ A_{n-t}^R \end{pmatrix}$	$\begin{pmatrix} B_1 \\ B_1 \end{pmatrix}$	$\cdots$	$\begin{pmatrix} B_1 \\ B_{n-t} \end{pmatrix}$
		$\ddots$			$\ddots$	
$R_{2(n-t)}$	$\begin{pmatrix} B_{n-t} \\ A_1^R \end{pmatrix}$	$\cdots$	$\begin{pmatrix} B_{n-t} \\ A_{n-t}^R \end{pmatrix}$	$\begin{pmatrix} B_{n-t} \\ B_1 \end{pmatrix}$	$\cdots$	$\begin{pmatrix} B_{n-t} \\ B_{n-t} \end{pmatrix}$

Similarly, the $\mathcal{D}$-class $D_{(d_1,\ldots,d_{p-1})}$ in L_p has the following egg box form:

D_o	L_1		L_{n-t}
R_1	$\begin{pmatrix} A_1 \\ A_1 \end{pmatrix}$	$\cdots$	$\begin{pmatrix} A_1 \\ A_{n-t} \end{pmatrix}$
		$\ddots$	
R_{n-t}	$\begin{pmatrix} A_{n-t} \\ A_1 \end{pmatrix}$	$\cdots$	$\begin{pmatrix} A_{n-t} \\ A_{n-t} \end{pmatrix}$

Recall from [2], as a result of [2] Lemmas 1 and 2, that a non-empty subset W of K_r (L_r) is a generating set of $DP_{n,r}$ $(ODP_{n,r})$ if and only if $K_r \subseteq \langle W \rangle$ $(L_r \subseteq \langle W \rangle)$. Moreover, recall that

- **(I)** Let $\alpha_1, \ldots, \alpha_k$ be some elements of K_p (L_p) for $k \geq 2$ and $1 \leq p \leq n-1$. Then $\alpha_1 \cdots \alpha_k = \gamma$ is also an element of K_p (L_p) if and only if $\alpha_i \alpha_{i+1}$ is element of K_p (L_p), equivalently, $\operatorname{im}(\alpha_i) = \operatorname{dom}(\alpha_{i+1})$ for each $1 \leq i \leq k-1$.
- **(II)** Let D be a $\mathcal{D}$-class in $DP_{n,r}$ $(ODP_{n,r})$ for $2 \leq r \leq n-1$, and let $\alpha_1, \ldots, \alpha_k \in D$ for $k \geq 2$. Then $\alpha_1 \cdots \alpha_k \in D$ if and only if $\alpha_i \alpha_{i+1} \in D$, equivalently, $\operatorname{im}(\alpha_i) = \operatorname{dom}(\alpha_{i+1})$ for each $1 \leq i \leq k-1$.
- **(III)** For $2 \leq r \leq n-1$, a non-empty subset W of K_r (L_r) is a generating set of $DP_{n,r}$ $(ODP_{n,r})$ if and only if $D \subseteq \langle W \cap D \rangle$ for each $\mathcal{D}$-class D in K_r (L_r).

As a final of this section we give some definitions about digraphs. Let $\Pi = (V(\Pi), \overrightarrow{E}(\Pi))$ be a digraph where $V(\Pi)$ is the set of vertices and $\overrightarrow{E}(\Pi) \subseteq V(\Pi) \times V(\Pi)$ is the list of directed edges. For any $u_1, \ldots, u_k \in V(\Pi)$ $(k \geq 2)$ (they have not to be distinct) if $(u_1, u_2), (u_2, u_3), \ldots, (u_{k-1}, u_k) \in \overrightarrow{E}(\Pi)$, then $u_1 \to u_2 \to \cdots \to u_k$ is called *a walk from u_1 to u_k*. In particular, for distinct vertices $u_1, \ldots, u_k \in V(\Pi)$ where $k \geq 1$, the closed walk $u_1 \to \cdots \to u_k \to u_1$ is called a *cycle*, and the cycle consists of a unique vertex is called *loop*. Also, for any vertices $u, v \in V(\Pi)$ if $u = v$ or there exists a walk from u to v we say u is *connected* to v, and respectively, the vertex $u = v$ or the walk $u \to \cdots \to v$ is also called *connection* from u to v. Let W_D be a non-empty subset of any $\mathcal{D}$-class D in K_r (L_r). Then we define the digraph Γ_{W_D} as follows:

- the vertex set of Γ_{W_D}, denoted by $V = V(\Gamma_{W_D})$, is W_D; and
- the directed edge set of Γ_{W_D}, denoted by $\overrightarrow{E} = \overrightarrow{E}(\Gamma_{W_D})$, is

$$\overrightarrow{E} = \{(\alpha, \beta) \in V \times V : \alpha\beta \in D\}.$$

(For unexplained terms about digraphs, see [5].)

Theorem 2.1. *[[2] Theorem 3] Let D be a $\mathcal{D}$-class in K_p for $2 \le p \le n-1$, and let $\emptyset \neq W_D \subseteq D$. Then $D \subseteq \langle W_D \rangle$ if and only if*

(i) *for each order-preserving map $\gamma \in D \setminus W_D$ there exist $\alpha, \beta \in W_D$ such that $\mathrm{dom}\,(\alpha) = \mathrm{dom}\,(\gamma)$ and $\mathrm{im}\,(\beta) = \mathrm{im}\,(\gamma)$, and at least one walk ρ, from α to β in Γ_{W_D} such that the number of order-reversing maps in ρ is even, and*

(ii) *for each order-reversing map $\gamma' \in D \setminus W_D$ there exist $\alpha', \beta' \in W_D$ such that $\mathrm{dom}\,(\alpha') = \mathrm{dom}\,(\gamma')$ and $\mathrm{im}\,(\beta') = \mathrm{im}\,(\gamma')$, and at least one walk ρ', from α' to β' in Γ_{W_D} such that the number of order-reversing maps in ρ' is odd.* □

Theorem 2.2. *[[2] Theorem 4] For $2 \le r \le n-1$ $\mathrm{rank}\,(DP_{n,r}) = \binom{n}{r}$.* □

Theorem 2.3. *[[2] Theorem 5] Let D be a $\mathcal{D}$-class in L_p for $2 \le p \le n-1$, and let $\emptyset \neq W_D \subseteq D$. Then $D \subseteq \langle W_D \rangle$ if and only if, for each $\gamma \in D \setminus W_D$ there exist $\alpha, \beta \in W_D$ such that $\mathrm{dom}\,(\alpha) = \mathrm{dom}\,(\gamma)$ and $\mathrm{im}\,(\beta) = \mathrm{im}\,(\gamma)$, and there exists at least one walk from α to β in Γ_{W_D}.* □

Theorem 2.4. *[[2] Theorem 6] For $2 \le r \le n-1$ $\mathrm{rank}\,(ODP_{n,r}) = \binom{n}{r}$.* □

3. Minimal generating sets of $DP_{n,r}$

Lemma 3.1. *Let D be a $\mathcal{D}$-class in K_p, for $1 \le p \le n-1$, and let $\emptyset \neq W_D \subseteq D$. For any possible subset A of X_n let R_A and L_A be the $\mathcal{R}$-class and $\mathcal{L}$-class, which contain id_A, in D, respectively. Moreover, let $H_A = R_A \cap L_A$.*

(i) *If $R_A \cap W_D \subseteq H_A$, then $R_A \cap \langle W_D \rangle \subseteq H_A$.*

(ii) *If $L_A \cap W_D \subseteq H_A$, then $L_A \cap \langle W_D \rangle \subseteq H_A$.*

Proof. Let $D = D_{(d_1,\ldots,d_{p-1})}$ be a $\mathcal{D}$-class in K_p, and then notice that

$$H_A = \begin{cases} \{\begin{pmatrix} A \\ A \end{pmatrix}, \begin{pmatrix} A \\ A^R \end{pmatrix}\} & \text{if the gap } (d_1,\ldots,d_{p-1}) \text{ is symmetric,} \\ \{\begin{pmatrix} A \\ A \end{pmatrix}\} & \text{if the gap } (d_1,\ldots,d_{p-1}) \text{ is asymmetric.} \end{cases}$$

(i) If $R_A \cap W_D = \emptyset$ then $R_A \cap \langle W_D \rangle = \emptyset$ since $\mathrm{dom}\,(\beta) \neq A$ for each $\beta \in \langle W_D \rangle$. Now let $\emptyset \neq R_A \cap W_D \subseteq H_A$, and let $\beta \in R_A \cap \langle W_D \rangle$. Then there exist $\beta_1, \ldots, \beta_k \in W_D$ such that $\beta = \beta_1 \cdots \beta_k$ ($k \in \mathbb{Z}^+$). It follows from (II) that $\mathrm{im}\,(\beta_i) = \mathrm{dom}\,(\beta_{i+1})$ for each $1 \le i \le k-1$, and so $\mathrm{dom}\,(\beta) = \mathrm{dom}\,(\beta_1)$. Thus $\beta_1 \in R_A$, and so $\beta_1 \in R_A \cap W_D$. Then, from the assumption, we have $\beta_1 \in H_A$. Similarly, since $\mathrm{dom}\,(\beta_{i+1}) = \mathrm{im}\,(\beta_i) = A$ for each $1 \le i \le k-1$, it follows that $\beta_1, \ldots, \beta_k \in H_A$, and so $\beta \in H_A$, as required.

(ii) It can be proved similarly. □

Lemma 3.2. *Let $D = D_{(d_1,\ldots,d_{p-1})}$ be a $\mathcal{D}$-class in K_p for $2 \le p \le n-1$, such that $(d_1, \ldots, d_{p-1})$ is asymmetric, and let $\emptyset \neq W_D \subseteq D$. If W_D contains at least one order-reversing map, and if Γ_{W_D} is a cycle then the number of order-reversing maps in W_D is a positive even number.*

Proof. Notice that D has the form as D_{as} given above. Then, with the same notations, the set of all order-reversing maps with gap $(d_1, \ldots, d_{p-1})$, and the set of all order-reversing maps with reverse-gap $(d_1, \ldots, d_{p-1})$ are

$$\begin{aligned} U &= \{\begin{pmatrix} A_1 \\ B_1^R \end{pmatrix}, \ldots, \begin{pmatrix} A_1 \\ B_{n-t}^R \end{pmatrix}, \ldots, \begin{pmatrix} A_{n-t} \\ B_1^R \end{pmatrix}, \ldots, \begin{pmatrix} A_{n-t} \\ B_{n-t}^R \end{pmatrix}\} \text{ and} \\ V &= \{\begin{pmatrix} B_1 \\ A_1^R \end{pmatrix}, \ldots, \begin{pmatrix} B_1 \\ A_{n-t}^R \end{pmatrix}, \ldots, \begin{pmatrix} B_{n-t} \\ A_1^R \end{pmatrix}, \ldots, \begin{pmatrix} B_{n-t} \\ A_{n-t}^R \end{pmatrix}\} \end{aligned}$$

where $t = \sum_{i=1}^{p-1} d_i$, respectively.

First of all, let $\mu_1 \to \cdots \to \mu_l$ be any walk in Γ_{W_D}, for any $l \geq 3$, such that $\mu_1,\ \mu_l \in U$ and $\mu_2, \ldots, \mu_{l-1} \notin U$. Then it is clear that, since $\operatorname{im}(\mu_i) = \operatorname{dom}(\mu_{i+1})$ for each $1 \leq i \leq l-1$, there exists a unique $2 \leq j \leq l-1$ such that $\mu_j \in V$. That is, there exists a unique order-reversing map with reverse-gap $(d_1, \ldots, d_{p-1})$ between two order-reversing maps with gap $(d_1, \ldots, d_{p-1})$ in Γ_{W_D}. Similarly, there exists a unique order-reversing map with gap $(d_1, \ldots, d_{p-1})$ between two order-reversing maps with reverse-gap $(d_1, \ldots, d_{p-1})$ in Γ_{W_D}.

Now, without loss of generality, suppose that $W_D = \{\lambda_1, \ldots, \lambda_s\}$ for any $s \geq 2$. If $s = 2$, since W_D contains at least one order-reversing map and Γ_{W_D} is a cycle, then it is clear that, without loss of generality, λ_1 has a form $\begin{pmatrix} A \\ B^R \end{pmatrix}$ and λ_2 has a form $\begin{pmatrix} B \\ A^R \end{pmatrix}$, as required, where $A \in \{A_1, \ldots, A_{n-t}\}$ and $B \in \{B_1, \ldots, B_{n-t}\}$. Now let $s \geq 3$, and suppose that there exist only $k \geq 1$ order-reversing maps with gap $(d_1, \ldots, d_{p-1})$ in W_D, say $\lambda_{i_1}, \ldots, \lambda_{i_k}$. Then, since Γ_{W_D} is a cycle, without loss of generality Γ_{W_D} has the form

$$\lambda_{i_1} \to \cdots \to \lambda_{i_2} \to \cdots \to \lambda_{i_k} \to \cdots \to \lambda_{i_1}.$$

Since there exists a unique order-reversing map with reverse-gap $(d_1, \ldots, d_{p-1})$ between two order-reversing maps with gap $(d_1, \ldots, d_{p-1})$ in Γ_{W_D}, also there exist only k order-reversing maps with reverse-gap $(d_1, \ldots, d_{p-1})$ in W_D, and so the number of order-reversing maps in W_D is $2k$, as required. ☐

Theorem 3.1. *For $2 \leq r \leq n-1$, let W be a non-empty subset of K_r with cardinality $\binom{n}{r}$. Then W is a minimal generating set of $DP_{n,r}$ if and only if the following conditions are satisfied for each $\mathcal{D}$-class $D = D_{(d_1,\ldots,d_{r-1})}$ in K_r.*

(i) *$|R \cap W| = |L \cap W| = 1$ for each $\mathcal{R}$-class R and $\mathcal{L}$-class L in D.*

(ii)
- *If $(d_1, \ldots, d_{r-1})$ is symmetric, then the digraph $\Gamma_{W\cap D}$ is a cycle with $n-t$ vertices and the number of order-reversing maps in $W \cap D$ is an odd number.*
- *If $(d_1, \ldots, d_{r-1})$ is asymmetric, then the digraph $\Gamma_{W\cap D}$ is a cycle with $2(n-t)$ vertices and the number of order-reversing maps in $W \cap D$ is a positive even number*

where $t = \sum_{i=1}^{r-1} d_i$.

Proof. (⇒) Suppose that $\emptyset \neq W \subseteq K_r$ is a minimal generating set of $DP_{n,r}$ with cardinality $\binom{n}{r}$. Then, from (III), $D \subseteq \langle W \cap D \rangle$ for each $\mathcal{D}$-class D in K_r. Now let $D = D_{(d_1,\ldots,d_{r-1})}$ be any $\mathcal{D}$-class in K_r and let $t = \sum_{i=1}^{r-1} d_i$.

(i) The claim is provided from (III), Theorems 2.1 and 2.2.

(ii) **Case 1.** Suppose that $(d_1, \ldots, d_{r-1})$ is symmetric. Then D has the form as D_s and it is clear that $|W \cap D| = n - t \geq 1$ since the condition (i) is satisfied. If $|W \cap D| = 1$ then we have

	L_1
$D:\ R_1$	$\begin{pmatrix} A \\ A \end{pmatrix}, \begin{pmatrix} A \\ A^R \end{pmatrix}$

where A is the unique subset of X_n with symmetric gap $(d_1, \ldots, d_{r-1})$. It is clear that $D \subseteq \langle W \cap D \rangle$ if and only if $W \cap D = \{\alpha = \begin{pmatrix} A \\ A^R \end{pmatrix}\}$, and so $\Gamma_{W\cap D}$ is a cycle with a unique vertex α, which is an order-reversing map, as required.

If $|W \cap D| \geq 2$ then, from the first condition and Lemma 3.1, there is no element in $W \cap D$ which has a form $\begin{pmatrix} A \\ A \end{pmatrix}$ or $\begin{pmatrix} A \\ A^R \end{pmatrix}$ for any possible non-empty subset A of X_n. Hence there is no loop in $\Gamma_{W\cap D}$. Now let α and β be two distinct elements of $W \cap D$. Then consider any (order-preserving or order-reversing) map $\gamma \in D$ such that $\operatorname{dom}(\gamma) = \operatorname{dom}(\alpha)$ and $\operatorname{im}(\gamma) = \operatorname{im}(\beta)$. Notice that α and β are not in the same $\mathcal{R}$-class and not in the same $\mathcal{L}$-class in D, from the first condition, and so $\alpha \neq \gamma$, $\beta \neq \gamma$, and moreover $\gamma \notin W \cap D$. Since W is a generating set of $DP_{n,r}$, from (III), there exist $\lambda_1, \ldots, \lambda_k \in W \cap D$ such that $\lambda_1 \cdots \lambda_k = \gamma$ for $k \geq 2$. Then, from (II), we have $\operatorname{dom}(\lambda_1) = \operatorname{dom}(\gamma) = \operatorname{dom}(\alpha)$ and $\operatorname{im}(\lambda_k) = \operatorname{im}(\gamma) = \operatorname{im}(\beta)$, and so $\lambda_1 \mathcal{R} \alpha$ and $\lambda_k \mathcal{L} \beta$. From the first condition $\lambda_1 = \alpha$ and $\lambda_k = \beta$, and so there exists a walk from α to β in the digraph $\Gamma_{W\cap D}$. Moreover, for any $\alpha \in W \cap D$, there exists a unique $\lambda \in (W \cap D)\backslash\{\alpha\}$ such that $\operatorname{im}(\alpha) = \operatorname{dom}(\lambda)$ and a unique $\mu \in (W \cap D)\backslash\{\alpha\}$ such that $\operatorname{dom}(\alpha) = \operatorname{im}(\mu)$ from the first condition. That is, there exists a unique edge from α and a unique edge to α in $\Gamma_{W\cap D}$. Therefore, $\Gamma_{W\cap D}$ is a cycle with $n - t$ vertices.

Now let $W \cap D = \{\mu_1, \ldots, \mu_{n-t}\}$ and without loss of generality suppose that the cycle $\Gamma_{W\cap D}$ is $\mu_1 \to \cdots \to \mu_{n-t} \to \mu_1$. Since any product of some order-preserving maps is also an order-preserving map, it is clear that $W \cap D$ must contain at least one order-reversing map. Now consider the map

$$\delta = \begin{cases} \begin{pmatrix} A \\ B^R \end{pmatrix} & \text{if } \mu_1 = \begin{pmatrix} A \\ B \end{pmatrix}, \\ \begin{pmatrix} A \\ B \end{pmatrix} & \text{if } \mu_1 = \begin{pmatrix} A \\ B^R \end{pmatrix} \end{cases}$$

for two possible different subsets A and B with symmetric gap $(d_1, \ldots, d_{r-1})$. It is easy to see from (II) that, to generate the map δ we have to use the walk $\mu_1 \to \cdots \to \mu_{n-t} \to \mu_1$ in $\Gamma_{W\cap D}$, and δ can be written only as the product $(\mu_1 \cdots \mu_{n-t})^k \mu_1$ for some $k \geq 1$. If the number of order-reversing maps in $W \cap D$ is even, then $\mu_1 \cdots \mu_{n-t}$ is the partial identity map with domain set $\mathrm{dom}\,(\mu_1)$, and so $(\mu_1 \cdots \mu_{n-t})^k \mu_1 = \mu_1$ for each $k \geq 1$. Thus we have $\delta \notin \langle W \cap D \rangle$, which is a contradiction, and so the number of order-reversing maps in $W \cap D$ is odd.

Case 2. Suppose that $(d_1, \ldots, d_{r-1})$ is asymmetric. Then D has the form as D_{as} and it is clear that $|W \cap D| = 2(n-t) \geq 2$ since the condition (i) is satisfied. Similarly we can show that $\Gamma_{W\cap D}$ is a cycle with $2(n-t)$ vertices and $W \cap D$ must contain at least one order-reversing map. Then, from Lemma 3.2, the result is clear.

($\Leftarrow$) Suppose that the conditions are satisfied. Now let $D = D_{(d_1,\ldots,d_{r-1})}$ be any $\mathcal{D}$-class in K_r and let $\gamma \in D$. Then, from the first condition, there exist a unique $\alpha \in W \cap D$ and a unique $\beta \in W \cap D$ such that $\mathrm{dom}\,(\gamma) = \mathrm{dom}\,(\alpha)$ and $\mathrm{im}\,(\gamma) = \mathrm{im}\,(\beta)$.

Case 1. Suppose that $(d_1, \ldots, d_{r-1})$ is symmetric and recall that $|W \cap D| = n - t \geq 1$. From the second condition $\Gamma_{W\cap D}$ is a cycle with $n-t$ vertices and the number of order-reversing maps in $W \cap D$ is odd. Now suppose that $|W \cap D| = 1$. Then we similarly have

$$D:\ R_1 \ \overset{L_1}{\boxed{\begin{pmatrix} A \\ A \end{pmatrix}, \begin{pmatrix} A \\ A^R \end{pmatrix}}}$$

and $W \cap D = \{\alpha = \beta = \begin{pmatrix} A \\ A^R \end{pmatrix}\}$. Notice that $\gamma = \alpha$ or $\gamma = \alpha^2$, and so $D \subseteq \langle W \cap D \rangle$, as required.

Next suppose that $|W \cap D| = n - t \geq 2$. If $\gamma \in W \cap D$ then $\gamma = \alpha = \beta$, as required. If $\gamma \notin W \cap D$ and $\alpha = \beta$, then $\mathrm{dom}\,(\gamma) = \mathrm{dom}\,(\alpha)$, $\mathrm{im}\,(\gamma) = \mathrm{im}\,(\alpha)$ and $\gamma \neq \alpha$, that is $H \setminus \{\alpha\} = \{\gamma\}$ where H is the $\mathcal{H}$-class contains α. Then, without loss of generality, suppose that $W \cap D = \{\alpha, \lambda_1, \ldots, \lambda_{n-t-1}\}$ and that $\Gamma_{W\cap D}$ has a form

$$\alpha \to \lambda_1 \to \cdots \to \lambda_{n-t-1} \to \alpha.$$

It is clear that $\alpha\lambda_1 \cdots \lambda_{n-t-1}$ is an order-reversing map, and so

$$\gamma = \alpha\lambda_1 \cdots \lambda_{n-t-1}\alpha \in \langle W \cap D \rangle.$$

Finally, if $\gamma \notin W \cap D$ and $\alpha \neq \beta$, then, without loss of generality, suppose that $W \cap D = \{\alpha, \lambda_1, \ldots, \lambda_k, \beta, \mu_1, \ldots, \mu_l\}$ for $k, l \geq 0$ (notice that $k + l + 2 = n - t$), and that $\Gamma_{W\cap D}$ has a form

$$\alpha \to \lambda_1 \to \cdots \to \lambda_k \to \beta \to \mu_1 \to \cdots \to \mu_l \to \alpha.$$

If the number of order-reversing maps in $\{\alpha, \lambda_1, \ldots, \lambda_k, \beta\}$ is even, then

$$\gamma = \begin{cases} \alpha\lambda_1 \cdots \lambda_k\beta & \text{if } \alpha \text{ is an order-preserving map,} \\ \alpha\lambda_1 \cdots \lambda_k\beta\mu_1 \cdots \mu_l\alpha\lambda_1 \cdots \lambda_k\beta & \text{if } \alpha \text{ is an order-reversing map,} \end{cases}$$

and so $\gamma \in \langle W \cap D \rangle$. If the number of order-reversing maps in $\{\alpha, \lambda_1, \ldots, \lambda_k, \beta\}$ is odd, then

$$\gamma = \begin{cases} \alpha\lambda_1 \cdots \lambda_k\beta\mu_1 \cdots \mu_l\alpha\lambda_1 \cdots \lambda_k\beta & \text{if } \alpha \text{ is an order-preserving map,} \\ \alpha\lambda_1 \cdots \lambda_k\beta & \text{if } \alpha \text{ is an order-reversing map,} \end{cases}$$

and so $\gamma \in \langle W \cap D \rangle$. Thus $D \subseteq \langle W \cap D \rangle$, as required.

Case 2. Suppose that $(d_1, \ldots, d_{r-1})$ is asymmetric, and recall that $|W \cap D| = 2(n-t) \geq 2$. If $\gamma \in W \cap D$ then $\gamma = \alpha = \beta$, as required. If $\gamma \notin W \cap D$ then, since each $\mathcal{H}$-class in D consist of a unique element, we have $\alpha \neq \beta$, otherwise $\gamma = \alpha \in W \cap D$ which is a contradiction. Since $\Gamma_{W\cap D}$ is a cycle, from the second condition, there exists a unique shortest walk ρ in $\Gamma_{W\cap D}$ from α to β. Then it is easy to see that $\gamma = \xi \in \langle W \cap D\rangle$ where ξ is the consecutive product of all elements of ρ. Thus $D \subseteq \langle W \cap D\rangle$, as required.

Notice that $|W| = \binom{n}{r}$ from the first condition. Therefore, it follows from (III) and Theorem 2.2 that W is a minimal generating set of $DP_{n,r}$. □

Corollary 3.1. *Let W is a minimal generating set of $DP_{n,r}$ for $2 \le r \le n-1$, and let $D = D_{(d_1,\ldots,d_{r-1})}$ be a $\mathcal{D}$-class in K_r.*

(i) *If $(d_1, \ldots, d_{r-1})$ is symmetric and $|W \cap D| \ge 2$, then $W \cap D$ does not contain any partial map which has a form $\begin{pmatrix} A \\ A \end{pmatrix}$ or $\begin{pmatrix} A \\ A^R \end{pmatrix}$ for any possible subset A of X_n.*

(ii) *If $(d_1, \ldots, d_{r-1})$ is symmetric and $|W \cap D| = 1$, or if $(d_1, \ldots, d_{r-1})$ is asymmetric, then $W \cap D$ does not contain any partial identity map.* □

4. Minimal generating sets of $ODP_{n,r}$

Lemma 4.1. *Let D be a $\mathcal{D}$-class in L_p for $1 \le p \le n-1$, and let $\emptyset \ne W_D \subseteq D$. For any possible subset A of X_n let R_A and L_A be the $\mathcal{R}$-class and $\mathcal{L}$-class, which contain id_A, in D, respectively. Moreover, let $H_A = R_A \cap L_A$, that is $H_A = \{\mathrm{id}_A\}$.*

(i) *If $R_A \cap W_D \subseteq H_A$, then $R_A \cap \langle W_D\rangle \subseteq H_A$.*

(ii) *If $L_A \cap W_D \subseteq H_A$, then $L_A \cap \langle W_D\rangle \subseteq H_A$.*

Proof. The proof is similar to the proof of Lemma 3.1. □

Theorem 4.1. *For $2 \le r \le n-1$, let W be a non-empty subset of L_r with cardinality $\binom{n}{r}$. Then W is a minimal generating set of $ODP_{n,r}$ if and only if*

(i) *$|R \cap W| = |L \cap W| = 1$ for each $\mathcal{R}$-class R and $\mathcal{L}$-class L in L_r, and*

(ii) *for each $\mathcal{D}$-class D in L_r, the digraph $\Gamma_{W\cap D}$ is a cycle.*

Proof. The proof is similar to the proof of Theorem 3.1, by using the fact that, $|H| = 1$ for each $\mathcal{H}$-class H in $ODP_{n,r}$. □

Corollary 4.1. *For $2 \le r \le n-1$, any minimal generating set of $ODP_{n,r}$ does not contain any partial identity map except partial identities of singleton $\mathcal{D}$-classes in L_r.* □

References

[1] Al-Kharousi, F., Kehinde, R., Umar, A.: *On the semigroup of partial isometries of a finite chain*. Comm. Algebra. 44, 639–647 (2016).

[2] Bugay, L., Yağcı M., Ayık, H.: *The ranks of certain semigroups of partial isometries*. Semigroup Forum. 97, 214–222 (2018).

[3] Ganyushkin, O., Mazorchuk, V.: Classical finite transformation semigroups. Springer-Verlag. London (2009).

[4] Howie, J. M.: Fundamentals of semigroup theory. Oxford University Press. New York (1995).

[5] Wilson, R.J., Watkins, J.J.: Graphs, An Introductory Approach, A First Course in Discrete Mathematics. Jon Wiley & Sons Inc. Toronto (1990).

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