TURKISH JOURNAL OF SCIENCE
VOLUME 5, ISSUE 3, 170–185
ISSN: 2587–0971

http://dergipark.gov.tr/tjos

The alpha power Weibull Frechet distribution: properties and applications

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Abstract. Modeling everyday life processes play a great role in human existence. Thus, distribution theory has helped to understand how our everyday life processes are distributed. However, this depends on how researchers in distribution theory compound several distributions to derive a more flexible distribution. This study proposes the alpha power Weibull Frechet distribution for real-life datasets. However, some statistical structural properties of the model such as kurtosis, hazard rate and odd functions, cumulative, quantiles, reversed hazard, skewness, order statistics and survival function were derived. The parameters of the proposed model were obtained using the maximum likelihood method. The behavioural nature of the model was studied through simulation. Finally, a two real life data was used to investigate the performance of the proposed model. The results show that the new model performs better than some existing continuous models in statistical literature.

1. Introduction

Integral representations of solutions for differential equations and operators are used in many scientific fields [1, 2]. Several methods for generating family of univariate distributions were based on differential equation (Pearson 1895). Of most important, is the translation method proposed in [3] . This method is based on quantile function that was developed in [4]. Lifetime processes have received several attentions through modeling the way and manner in which they are distributed, thus developing a flexible distribution depending on how the researcher compounds one or more distribution(s) to form a better or a comparable distribution [5]. The Weibull distribution plays a very important role in modeling lifetime processes. The Weibull distribution was proposed by a famous statistician called Weibull [6]. This Weibull

distribution has a wide range of applications in modelling lifetime processes, failure time processes, survival time, mechanical and electrical systems and machine learning. More so, the Frechet distribution is used in modeling extreme value theory. Its applications ranging from horse racing accelerated life testing in earthquakes, floods, rainfall, queues in supermarkets, wind speed and sea waves. The Frechet distribution can also be used in modelling material properties in engineering materials.

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Received: 22 April 2020; Accepted: 17 October 2020; Published: 30 December 2020
Keywords. Alpha power, Compounding, Maximum likelihood, Statistical analysis, Weibull Frechet
2010 *Mathematics Subject Classification*. 60E05, 60E10, 60E15
Cited this article as: Eghwerido JT, Utoyo-Amrevugherea OT, Efe-Eyefia E. The alpha power Weibull Frechet distribution: properties and applications. Turkish Journal of Science. 2020, 5(3), 170-185.

Let S be a random variable, say $s > 0$. Then, the Frechet distribution is defined as

$$g(s, \alpha, \beta) = \beta\alpha^{\beta} s^{-beta-1} exp\Big[-(\frac{\alpha}{s})^{\beta}\Big] \quad \alpha, \beta > 0. \tag{1}$$

The corresponding cdf is expressed as

$$G(s, \alpha, \beta) = exp\Big[-(\frac{\alpha}{s})^{\beta}\Big] \quad \alpha, \beta > 0, \tag{2}$$

where α and β are the scale and shape parameters respectively.

More so, the Weibull pdf with the parameters $\alpha > 0$ and $\beta > 0$ is defined as

$$f(s, \lambda, \beta) = \lambda\gamma s^{\gamma-1} exp(-\gamma s^{\gamma-1}); \quad \lambda\ \gamma > 0. \tag{3}$$

The cdf that corresponds to the Weibull pdf is given as

$$F(s, \lambda, \beta) = 1 - exp(-\gamma s^{\gamma}); \quad \lambda\ \gamma > 0 \tag{4}$$

where λ and γ are the shaped and scale parameters respectively.

[7] Proposed the Weibull Frechet (WFr) distribution and obtained the its pdf as

$$f(s) = \psi b\beta\tau^{\beta} s^{-\beta-1} exp\Big[-b(\frac{\tau}{s})^{\beta}\Big]\Big\{1 - exp\Big[-(\frac{\tau}{s})^{\beta}\Big]\Big\}^{-b-1} exp\Big[-\psi\Big[exp\Big[(\frac{\tau}{s})^{\beta}\Big] - 1\Big]^{-b}\Big] \tag{5}$$

The corresponds cdf is expressed as

$$F(s) = 1 - exp\Big[-\psi\Big[exp\Big[(\frac{\tau}{s})^{\beta}\Big] - 1\Big]^{-b}\Big], \tag{6}$$

whereτ is the scale parameter,β, ψ and b are the shape parameters.

The alpha power transformation (AP) was proposed in [8]. The pdf of the alpha power transformed family of distribution is given as

$$f_{AP}(s) = \begin{cases} g(s)\frac{\log\alpha}{(\alpha-1)}\alpha^{G(s)}, & \text{if } \alpha \in (\Re^{+} - (1)) \\ \\ g(s), & \text{otherwise} \alpha = 1. \end{cases} \tag{7}$$

The corresponding cdf is defined as

$$F_{AP}(s) = \frac{\alpha^{G(s)} - 1}{\alpha - 1} \quad \alpha \in (\Re^{+} - (1)) \,. \tag{8}$$

Otherwise, $F(s)$, for $\alpha = 1$ where $g(s)$ is the baseline pdf and $G(s)$ is the baseline cdf.

Several research works have been done in literature researched. [9] Proposed the Weibull-G family of distribution. The alpha power inverted exponential distribution was proposed in [10]. Gompertz-G distribution was proposed in [11]. Gompertz alpha power inverted exponential distribution was proposed in [12]. The extended new generalized exponential distribution was proposed in [13]. The Weibull alpha power inverted exponential distribution was proposed in [14]. Alpha power Weibull distribution was proposed in [15].

However, many distributions have been proposed in literature to extend distributions that are significant to the progress of distribution frontiers and to make life more meaningful. Thus, this study set up a model called alpha power Weibull Frechet (APWF) distribution to push back the frontiers of knowledge in data science, data analysis and distribution theory.

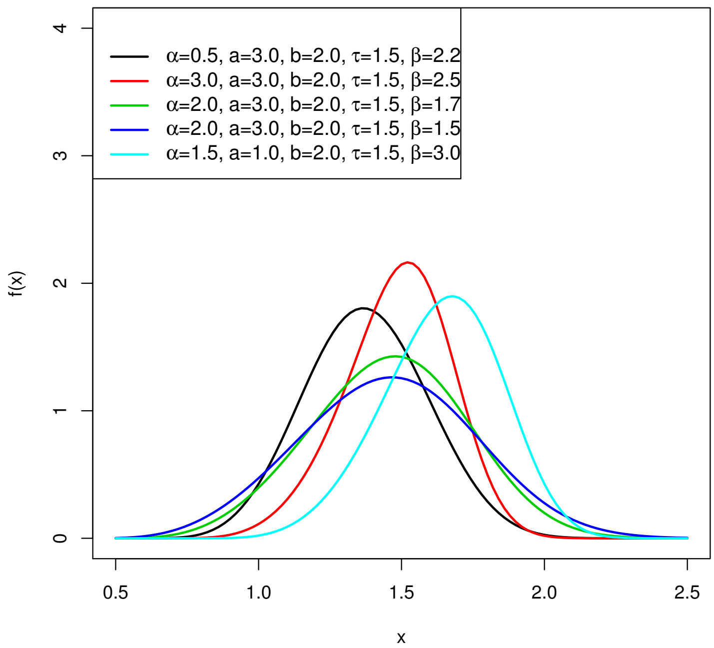

Figure 1: The APWF density for different parameter values cases

This study was motivated by studies and events obtained from some literature research in probability and distribution theories. However, the APWF model was proposed to push back the frontiers of knowledge in data science, data analysis and distribution theory by addition of a parameter to improve the existing models using the AP characterization.

The aim of this study was to introduce a class of Frechet distribution in distribution theory together with its mathematical properties. It worthy to note that this study was proposed to address APWF model, since, say , we obtained the usual WFr model.

2. The APWF Distribution

This section proposed a class of the Frechet family of distribution called APWF model. Let $s_1, s_2, s_3, \cdots s_n$ be a random sample of the APWF distribution. Then, the pdf of the APWF is given as

$$f_{APWF}(s) = \psi b\beta\tau^{\beta}s^{-\beta-1}exp\Big[-b(\frac{\tau}{s})^{\beta}\Big]\Big\{1-exp\Big[-(\frac{\tau}{s})^{\beta}\Big]\Big\}^{-b-1} exp\Big[-\psi\Big[exp\Big[(\frac{\tau}{s})^{\beta}\Big]-1\Big]^{-b}\Big]$$
$$\times\frac{\log\alpha}{(\alpha-1)}\alpha^{\Big[1-exp\Big[-\psi\Big[exp\Big[(\frac{\tau}{s})^{\beta}\Big]-1\Big]^{-b}\Big]\Big]}, \qquad \alpha\in(\Re^{+}-(1)). \tag{9}$$

Figure 1 shows the plot of the pdf for different parameter values cases. In Figure 1, the shape of the pdf could be increasing, decreasing, unimodal and symmetrical depending on the parameter values.

The cdf that corresponds to Equation (9) is given as

$$F_{APWF}(s) = \Big\{\alpha^{\Big[1-exp\Big[-\psi\Big[exp\Big[(\frac{\tau}{s})^{\beta}\Big]-1\Big]^{-b}\Big]\Big]}-1\Big\}(\alpha-1)^{-1}, \qquad \alpha\in(\Re^{+}-(1)). \tag{10}$$

3. Mathematical Mixture Representation

In this section, we expressed the APWF distribution in power series. First and foremost, we expressed the Weibull Frechet distribution before the proposed distribution is addressed. Thus, the Equation (5) can be defined as

$$f(s) = \psi b\beta\tau^{\beta}s^{-(\beta+1)}exp\Big[-b(\frac{\tau}{s})^{\beta}\Big]exp\Big[-\psi\Big[\frac{exp\Big[-(\frac{\tau}{s})^{\beta}\Big]}{1-exp\Big[-(\frac{\tau}{s})^{\beta}\Big]}\Big]^{b}\Big]\Big\{1-exp\Big[-(\frac{\tau}{s})^{\beta}\Big]\Big\}^{-(b+1)} \tag{11}$$

Let the middle quantity in Equation (11) be A. Then, expanding the exponential function in A, we expressed

$$A = \sum_{k=0}^{\infty}\frac{(-1)^{k}\alpha^{k}}{k!}\frac{exp\Big[-bk(\frac{\tau}{s})^{\beta}\Big]}{\Big[1-exp\Big[-b(\frac{\tau}{s})^{\beta}\Big]\Big]^{kb}}. \tag{12}$$

Inserting the Equation (12) into Equation (11), we have

$$f(s) = b\beta\tau^{\beta}s^{-(\beta+1)}\sum_{\xi=0}^{\infty}\frac{(-1)^{\xi}\alpha^{\xi+1}}{\xi!}exp\Big[-(\xi+1)b(\frac{\tau}{s})^{\beta}\Big]\Big[1-exp\Big[-(\frac{\tau}{s})^{\beta}\Big]\Big]^{-(\xi b+b+1)} \tag{13}$$

Further expansion of the last quantity in power series gives

$$f(s) = b\beta\tau^{\beta}s^{-(\beta+1)}\sum_{j=0}^{\infty}\sum_{\xi=0}^{\infty}\frac{(-1)^{\xi}\Psi^{\xi+1}\Big[(\xi+1)b+1\Big]^{j}}{j!\xi!}exp\Big[-[(\xi+1)b+j](\frac{\tau}{s})^{\beta}\Big], \tag{14}$$

where $\Psi^{j} = \frac{\Gamma(\Psi+j)}{\Gamma(\Psi)}$ is the rising factional for any real Ψ.

However, the Equation (14) can be expressed as

$$f(s) = \beta[(\xi+1)b+j]\tau^{\beta}\sum_{j=0}^{\infty}\sum_{\xi=0}^{\infty}\upsilon_{j,\xi}s^{-(\beta+1)}exp\Big[-[(\xi+1)b+j](\frac{\tau}{s})^{\beta}\Big], \tag{15}$$

where

$$\upsilon_{j,\xi} = \frac{(-1)^{\xi}\Psi^{\xi+1}\Big[(\xi+1)b+1\Big]^{j}}{j!\xi![(\xi+1)b+j]}. \tag{16}$$

Thus the Equation (11) reduces to

$$f(s) = \sum_{j=0}^{\infty}\sum_{\xi=0}^{\infty}\upsilon_{j,\xi}h_{(\xi+1)b+j}(s), \tag{17}$$

where is the scale parameter $\alpha[(\xi+1)b+j]^{\frac{1}{\beta}}$ of the Frechet distribution $h_{(\xi+1)b+j}(s)$ and shape parameter β .

Integrating Equation (17), the cdf of can be expressed as

$$F(s) = \sum_{j=0}^{\infty}\sum_{\xi=0}^{\infty}\upsilon_{j,\xi}H_{(\xi+1)b+j}(s), \tag{18}$$

where

$$h(s) = \psi b\beta\tau^{\beta}s^{-(\beta+1)}exp\Big[-b(\frac{\tau}{s})^{\beta}\Big]\Big\{1 - exp\Big[-(\frac{\tau}{s})^{\beta}\Big]\Big\}^{-(b+1)} \tag{19}$$

and

$$H(s) = \Psi\Big\{exp\Big[(\frac{\tau}{s})^{\beta}\Big] - 1\Big\}^{-b}. \tag{20}$$

Also, $\alpha^{G(s)}$ can be written as

$$\alpha^{G(s)} = \sum_{i=0}^{\infty}\frac{(\log\alpha)^{i}G(s)^{i}}{i}, \tag{21}$$

where $G(s)$ is the baseline pdf. Hence, $F(s)^i$ in Equation (18) can be expressed as

$$F(s)^{i} = \sum_{j=0}^{\infty}\sum_{\xi=0}^{\infty}\upsilon^{i}_{j,\xi}H^{i}_{(\xi+1)b+j}(s). \tag{22}$$

Hence, Equation (21) becomes

$$\alpha^{G(s)} = \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\sum_{\xi=0}^{\infty}\frac{(\log\alpha)^{i}}{i}\upsilon^{i}_{j,\xi}H^{i}_{(\xi+1)b+j}(s). \tag{23}$$

However, the pdf of the APWF distribution is given in mixture representation as

$$f_{APWF}(s) = \frac{\log\alpha}{(\alpha-1)}g(s)\alpha^{G(s)} = \frac{1}{\alpha-1}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{\xi=0}^{\infty}\frac{(\log\alpha)^{i+1}}{i}\upsilon^{i+1}_{j,\xi}h_{(\xi+1)b+j}H^{i}_{(\xi+1)b+j}(s) \tag{24}$$

The corresponding cdf is defined as

$$F_{APWF}(s) = \frac{1}{\alpha-1}\Big(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{\xi=0}^{\infty}\frac{(\log\alpha)^{i+1}}{i}\upsilon^{i}_{j,\xi}H^{i}_{(\xi+1)b+j}(s) - 1\Big) \tag{25}$$

where $H_{(\xi+1)b+j}(s)$ is the Frechet cdf with scale parameter $\alpha[(\xi+1)b+j]^{\frac{1}{\beta}}$ and shape parameter β .

4. Mathematical Properties

This section investigates the properties of the APWF density. The structural properties of the APWF density was computed efficiently by using programming software like R, Maple, Matlab and Mathematical.

4.1. The Quantile and Random Number Generation of the APWF Distribution

Let S be a random variable such that $S \sim APWF(\psi, b, \beta, \tau, \alpha)$. Then, the quantile function of the variable S for $\mu \in (0.1)$ is given as

$$s_{\mu} = \tau\Big[\log\Big[\Big[-\psi^{-1}\log\Big[1-\Big(\log\alpha\Big)^{-1}\log\Big[\mu\Big(\alpha-1\Big)+1\Big]\Big]\Big]^{\frac{1}{b}}+1\Big]^{\frac{1}{\beta}}\Big]. \tag{26}$$

By setting $\mu = 0.5$ in Equation (26), we obtain the median of the random variable S is obtained as

$$s_{0.5} = \tau\Big[\log\Big[\Big[-\psi^{-1}\log\Big[1-\Big(\log\alpha\Big)^{-1}\log\Big[0.5\Big(\alpha-1\Big)+1\Big]\Big]\Big]^{\frac{1}{b}}+1\Big]^{\frac{1}{\beta}}\Big]. \tag{27}$$

However, the 25th and 75th percentile for the random variable of the APWF distribution are obtained as

$$s_{0.25} = \tau\Big[\log\Big[\Big[-\psi^{-1}\log\Big[1-\Big(\log\alpha\Big)^{-1}\log\Big[0.25\Big(\alpha-1\Big)+1\Big]\Big]\Big]^{\frac{1}{b}}+1\Big]^{\frac{1}{\beta}}\Big], \tag{28}$$

$$s_{0.75} = \tau\Big[\log\Big[\Big[-\psi^{-1}\log\Big[1-\Big(\log\alpha\Big)^{-1}\log\Big[0.75\Big(\alpha-1\Big)+1\Big]\Big]\Big]^{\frac{1}{b}}+1\Big]^{\frac{1}{\beta}}\Big]. \tag{29}$$

Simulating the APWF random variable deviate from a uniform variates on the interval $(0,1)$. The Bowley's formula for finding the coefficient of skewness is given as

$$S_k(s) = \frac{x_{0.75}+x_{0.25}-2x_{0.5}}{x_{0.75}-x_{0.25}}. \tag{30}$$

The corresponding Moor's formula for coefficient of Kurtosis is given as

$$K_k(s) = \frac{x_{0.875}-x_{0.625}+x_{0.125}-x_{0.375}}{x_{0.75}-x_{0.25}}. \tag{31}$$

4.2. *Survival and Reliability Function*

The reliability function of the APWF random variable X is given as

$$R_{APWF}(s) = \frac{1}{(\alpha-1)}\Big(\alpha-\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\sum_{\xi=0}^{\infty}\Big(\frac{\log\alpha}{i!}v_{j,\xi}H_{(\xi+1)b+j}(s)\Big)^{i}\Big). \tag{32}$$

4.3. *Hazard Rate Function of the APWF Distribution*

The failure rate function of the APWF random variable is given as

$$h_{APWF}(s) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{\xi=0}^{\infty}\Bigg(\frac{(\log\alpha)^{i+1}v_{j,\xi}^{i+1}h_{(\xi+1)b+j}(s)H^{i}_{(\xi+1)b+j}(s)\Big)}{i!\Big(\alpha-\Big(\frac{\log\alpha}{i!}v_{j,\xi}H_{(\xi+1)b+j}(s)\Big)^{i}\Big)}\Bigg) \tag{33}$$

Figure 2 shows the plot for the hazard rate function of the APWF distribution.

4.4. *APWF Cumulative Hazard Function*

The Cumulative hazard function of the APWF distribution is given as

$$H_{APWF}(s) = \log(\alpha-1)-\log\Big[\alpha-\sum_{i,j,\xi=0}^{\infty}\Big(\log\alpha v_{j,\xi}+H_{(\xi+1)b+j}(s)\Big)^{i}\Big] \tag{34}$$

4.5. *APWF Reversed Hazard Function*

The Reversed Hazard Function of the APWF distribution is the ratio of the pdf of the APWF distribution to the cdf of the APWF distribution. Thus,

$$r_{APWF}(s) = \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\sum_{\xi=0}^{\infty}\Big[\frac{[\log(\alpha)]^{i+1}}{i!}v_{j,\xi}^{i+1}h_{(\xi+1)b+1}(s)H^{i}_{(\xi+1)b+j}(s)\Big]\Big[\frac{[\log(\alpha)]^{i}}{i!}v_{j,\xi}^{i}H^{i}_{(\xi+1)b+j}(s)-1\Big]^{-1}. \tag{35}$$

4.6. *APWF Odds Function*

The Odds function of the APWF distribution is given as

$$O_{APWF}(s) = F_{APWF}(s)R_{APWF}(s)^{-1}, \tag{36}$$

where $R_{APWF}(s)$ is the APWF reliability function.

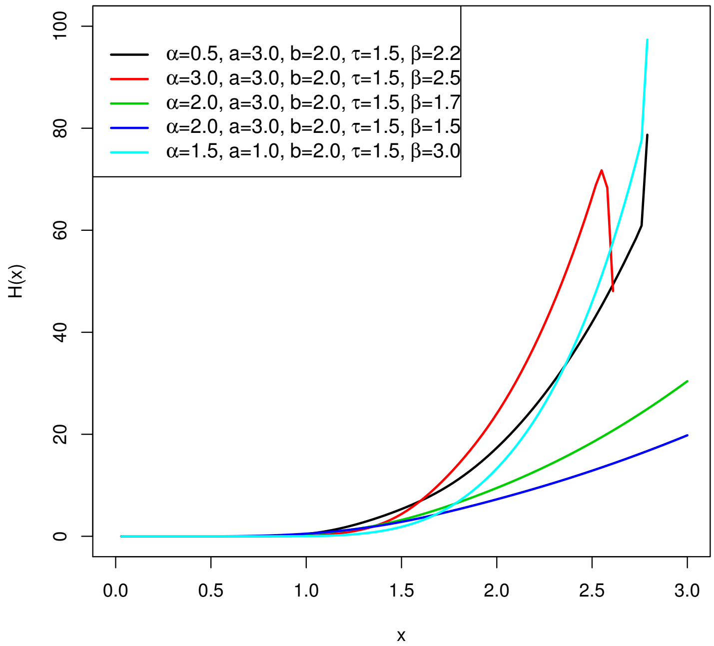

Figure 2: The hazard rate function of the APWF distribution for different parameter values

4.7. The APWF Order Statistics

Let $s_1, s_2, s_3, \cdots, s_n$ be a APWF random variable from a finite population which has the value $f(s)$ at s, then the pdf of the p^{th} order statistics is given as

$$\begin{aligned} g_p(s) =& \frac{n!}{(p-1)!(n-p)!}\left[\frac{1}{(\alpha-1)}\right]^n \left[\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\sum_{\xi=0}^{\infty}\frac{[\log(\alpha)]^i}{i!}v_{j,\xi}^i H_{(\xi+1)b+j}^i(s) - 1\right]^{p-1} \\ &\times \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\sum_{\xi=0}^{\infty}\frac{[\log(\alpha)]^{i+1}}{i!}v_{j,\xi}^{i+1} h_{(\xi+1)b+j(s)} H_{(\xi+1)b+j}^i(s) \\ &\left(\alpha - \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\sum_{\xi=0}^{\infty}\left(\frac{\log \alpha v_{j,\xi} H_{(\xi+1)b+j}(s)}{i!}\right)^i\right)^{n-p}. \end{aligned} \tag{37}$$

The following is observed for $p = 1$, we obtained the minimum order statistics distribution as

$$\begin{aligned} g_1(s) =& \frac{n!}{(n-p)!}\left[\frac{1}{(\alpha-1)}\right]^n \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\sum_{\xi=0}^{\infty}\frac{[\log(\alpha)]^{i+1}}{i!}v_{j,\xi}^{i+1} h_{(\xi+1)b+j(s)} H_{(\xi+1)b+j}^i(s) \\ &\left(\alpha - \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\sum_{\xi=0}^{\infty}\left(\frac{\log \alpha v_{j,\xi} H_{(\xi+1)b+j}(s)}{i!}\right)^i\right)^{n-1}. \end{aligned} \tag{38}$$

$p = n$ we obtained the maximum order statistics distribution as

$$g_n(s) = \frac{n!}{(n-1)!}\Big[\frac{1}{(\alpha-1)}\Big]^n \Big[\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\sum_{\xi=0}^{\infty} \frac{[\log(\alpha)]^i}{i!} \upsilon^i_{j,\xi} H^i_{(\xi+1)b+j}(s) - 1\Big]^{n-1}$$
$$\times \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\sum_{\xi=0}^{\infty} \frac{[\log(\alpha)]^{i+1}}{i!} \upsilon^{i+1}_{j,\xi} h_{(\xi+1)b+j(s)} H^i_{(\xi+1)b+j}(s). \quad (39)$$

When n is odd. $n = 2m + 1$,and setting $p = m + 1$, then the distribution of median is given as

$$g_p(s) = \frac{(2m+)!}{m!m!}\frac{1}{(\alpha-1)^{2m+1}}\Big[\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\sum_{\xi=0}^{\infty} \frac{[\log(\alpha)]^i}{i!} \upsilon^i_{j,\xi} H^i_{(\xi+1)b+j}(s) - 1\Big]^{m}$$
$$\times \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\sum_{\xi=0}^{\infty} \frac{[\log(\alpha)]^{i+1}}{i!} \upsilon^{i+1}_{j,\xi} h_{(\xi+1)b+j(s)} H^i_{(\xi+1)b+j}(s) \quad (40)$$
$$\Big(\alpha - \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\sum_{\xi=0}^{\infty}\Big(\frac{\log \alpha \upsilon_{j,\xi} H_{(\xi+1)b+j}(s)}{i!}\Big)^i\Big)^m.$$

when n is even, $n = m2m$ and $p = m + 1$

$$g_{m+1}(s) = \frac{2m!}{m!m!}\Big[\frac{1}{(\alpha-1)}\Big]^{2m}\Big[\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\sum_{\xi=0}^{\infty} \frac{[\log(\alpha)]^i}{i!} \upsilon^i_{j,\xi} H^i_{(\xi+1)b+j}(s) - 1\Big]^{m}$$
$$\times \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\sum_{\xi=0}^{\infty} \frac{[\log(\alpha)]^{i+1}}{i!} \upsilon^{i+1}_{j,\xi} h_{(\xi+1)b+j(s)} H^i_{(\xi+1)b+j}(s) \quad (41)$$
$$\Big(\alpha - \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\sum_{\xi=0}^{\infty}\Big(\frac{\log \alpha \upsilon_{j,\xi} H_{(\xi+1)b+j}(s)}{i!}\Big)^i\Big)^{m-1}.$$

4.8. Probability Weighted Moments (PWM)

The PWM is a function can be used to obtain the parameter and quantiles function of a particular distribution that may not be obtained in a closed form. The $(\mu, \upsilon)^{th}$ of PWM of random variable S is defined as

$$\rho(\mu,\upsilon) = \int_0^{\infty} s^{\mu} f(s) F^{\upsilon}(s) ds = \sum_{i,m=0}^{\infty}\sum_{j,\xi=0}^{\infty} \Gamma\Big(1 - \frac{\mu}{\beta}\Big) t_{i,j,\xi,m} \tau^{\mu}\Big[(\xi+1)b + j\Big]^{\frac{\mu}{\beta}} \frac{(\log\alpha)^{i+1}}{(\alpha-1)i!},$$

where

$$t_{i,j,\xi,m} = \Big[(\xi+1)b + j + 1\Big]^j \frac{(-1)^{\xi+m+1} b\psi^{\xi+1}(j+1)^{\xi}}{j!\xi!((\xi+1)b+j)}\Big[(\xi+1)b+1\Big]^{\frac{\mu}{\beta-1}}\binom{\upsilon}{i}\binom{i}{m}.$$

4.9. Parameter Estimation of the APWF Distribution

The parameter of the APWF distribution are obtained by maximum likelihood (MLE) method as follows: Let $s_1, s_2, s_3, \cdots, s_n$ be a APWF random sample from an infinite population with a pdf $f(s)$ at the point s with distribution of the vector APWF of parameter $\theta(\psi, b, \beta, \tau, \alpha)^T$, then the likelihood function is given as

$$\prod_{i=1}^{n} f(s, \psi, b, \beta, \tau, \alpha) = \psi^n b^n \beta^n \tau^{n\beta} (\log\alpha)^n \frac{1}{(\alpha-1)^n}$$
$$\times \prod_{i=1}^{n} s_i^{-(\beta+1)} exp\Big[\sum_{i=1}^{n}\Big[-b(\frac{\tau}{s_i})^{\beta}\Big]\Big] \prod_{i=1}^{n}\Big\{1 - exp\Big[-(\frac{\tau}{s_i})^{\beta}\Big]\Big\}^{-b-1} \tag{42}$$
$$\times exp\Big[\sum_{i=1}^{n} -\psi\Big\{exp\Big[\Big(\frac{\tau}{s_i}\Big)^{\beta}\Big] - 1\Big\}^{-b}\Big]\alpha^{\sum_{i=1}^{n}\Big[1 - exp\Big[-\psi\Big\{exp\Big[(\frac{\tau}{s_i})^{\beta}\Big]-1\Big\}^{b}\Big]\Big]}$$

Let ℓ denotes the log-likelihood function, then

$$\ell = n\log\psi + n\log b + n\log\beta + n\beta\log\tau - n\log(\alpha-1) + n\log(\log\alpha) - (\beta+1)\sum_{i=1}^{n}\log s_i$$
$$\sum_{i=1}^{n}\Big[-b\Big(\frac{\tau}{s_i}\Big)^{\beta}\Big] + (1-b)\sum_{i=1}^{n}\log\Big[1 - exp\Big[-\Big(\frac{\tau}{s_i}\Big)^{b}\Big]\Big] - \sum_{i=1}^{n}\psi\Big[exp\Big[\Big(\frac{\tau}{s_i}\Big)^{\beta}\Big] - 1\Big]^{-b} \tag{43}$$
$$+ \sum_{i=1}^{n}\Big[1 - exp\Big[-\psi\Big\{exp\Big[(\frac{\tau}{s_i})^{\beta}\Big] - 1\Big\}^{b}\Big]\Big]\log\alpha$$

However, taking the partial derivation of the Equation (43) with respect to the parameter ψ, b, β, τ and α and equation to zero, we have

$$\frac{\partial\ell}{\partial\psi} = \frac{n}{\psi} - \sum_{i=1}^{n}\Big[exp\Big[\Big(\frac{\tau}{s_i}\Big)^{\beta}\Big] - 1\Big]^{-b} = 0, \tag{44}$$

$$\frac{\partial\ell}{\partial b} = \frac{n}{b} - \sum_{i=0}^{n}\Big(\frac{\tau}{s_i}\Big)^{\beta} - \sum_{i=0}^{n}\log\Big[1 - exp\Big[-\Big(\frac{\tau}{s_i}\Big)^{\beta}\Big]\Big] + \sum_{i=1}^{n}\psi\Big[exp\Big[\Big(\frac{\tau}{s_i}\Big)^{\beta}\Big] - 1\Big]^{-b}\log\Big[\sum_{i=1}^{n}\psi\Big[exp\Big[\Big(\frac{\tau}{s_i}\Big)^{\beta}\Big] - 1\Big]\Big], \tag{45}$$

$$\frac{\partial\ell}{\partial\beta} = \frac{n}{\beta} + n\log\tau - \sum_{i=1}^{n}\log s_i + \sum_{i=1}^{n}\Big[-b\Big(\frac{\tau}{s_i}\Big)^{\beta}\Big]\log\Big[-b\Big(\frac{\tau}{s_i}\Big)^{\beta}\Big] + (1-b)\sum_{i=1}^{n}\frac{S'_{i\beta}}{S_i} - \sum_{i=0}^{n}p'_{i\beta} + \sum_{i=1}^{n}z'_{i\beta}, \tag{46}$$

$$\frac{\partial\ell}{\partial\tau} = \frac{n\beta}{\tau} - \sum_{i=1}^{n}\Big(\frac{b}{s_i}\Big)^{\beta}\tau^{\beta-1} + (1-b)\sum_{i=1}^{n}\frac{S'_{i\tau}}{S_i} - \sum_{i=0}^{n}p'_{i\tau} + \sum_{i=1}^{n}z'_{i\tau}, \tag{47}$$

$$\frac{\partial\ell}{\partial\alpha} = -\frac{n}{\alpha-1} + \psi'_{\alpha} + \alpha^{-1}\sum_{i=1}^{n}\frac{z_i}{log\alpha} = 0, \tag{48}$$

where

$$\psi = n\log(\log\alpha),$$

$$S_i = 1 - exp\Big[-\Big(\frac{\tau}{s_i}\Big)^{\beta}\Big],$$

$$p_i = \psi\Big[exp\Big[\Big(\frac{\tau}{s_i}\Big)^{\beta}\Big] - 1\Big]^{\beta},$$

$$z_i = \Big[1 - exp\Big[-\psi\Big\{exp\Big[(\frac{\tau}{s_i})^{\beta}\Big] - 1\Big\}^{b}\Big]\Big]\log\alpha.$$

5. Simulation Study and Real Life Applications

A simulation was carried out to test the flexibility and efficiency of the APWF distribution. Table 1 shows the simulation for different values of parameters for the APWF distribution. The simulation is performed as follows:

- Data are generated using
- $x_{\mu} = \tau\left[\log\left[\left[-\psi^{-1}\log\left[1-\left(\log\alpha\right)^{-1}\log\left[\mu\left(\alpha-1\right)+1\right]\right]\right]^{\frac{1}{b}}+1\right]^{\frac{1}{\beta}}\right] \qquad 0 < u < 1$
- The values of the parameters are set as $\alpha = 0.5, \tau = 2.0, \psi = 1.5, b = 0.5$, and $\beta = 3.0$.
- The APWF random sample sizes were taken as $n = 50, 100, 150$, and 350.
- Each APWF random sample is replicated 5000 times.

In this simulation study, we investigated the mean estimates (MEs), variance, biases and means squared errors (RMSEs) of the maximum likelihood estimate (MLEs).

The bias is calculated by for $(S = \alpha, \tau, \psi, b, \beta)$

$$\hat{B}ias = \frac{1}{5000}\sum_{i=1}^{5000}\left(\hat{S}_i - S\right).$$

Also, the MSE is obtained as

$$\hat{M}SE = \frac{1}{5000}\sum_{i=1}^{5000}\left(\hat{S}_i - S\right)^2.$$

Table 1 shows the simulation results for the Mean, Biases, Variances and MSE of the MLEs of APWF model for some fixed parameter values. The results of the APWF Monte Carlo study in Table 1 shows the MSEs and the biases decrease as the sample size increases and approach zero that corresponds to the first-order asymptotic theory. The mean estimates of the parameters approach the true parameter values as the sample size increases. The variance decreases in all the cases as the sample size increases.

5.1. Real life applications

The performance of the APWF model was examined with other competing distributions using the gas fiber and carbon data real-life datasets. We considered the Akaike Information Criteria (AIC), Consistent Akaike Information Criteria (CAIC), Bayesian Information Criteria (BIC), Hannan-Quinn Information Criteria (HQIC), The Anderson Darling (A) statistic, Cramer-von Mises statistic (W), Kolmogorov Smirnov (KS) statistic, Log-likelihood and the P value to compare the fits of the APWF model to other competing models such as the Gompertz Weibull (GOW), Weibull Frechet (WFr), Kumaraswamy Lomax (KL), Gompertz (GL), Beta Lomax (BL), and the Alpha Power Inverted Exponential (APIE) distributions.

5.1.1. First set of data is glass fiber data

Datasets were collected for 1.5 cm strengths of glass fibres data at the UK National Physical Laboratory and was used to test the performance of the APWF distribution as used in [16- 20] .

Table 2 is the measure of comparison for the various distributions under consideration with APIE as alpha power inverted exponential.

Table 1: Simulation results: mean estimates (AE), biases, Variance and mean squared errors (MSE) of $\hat{\alpha}, \hat{\psi}, \hat{b}, \hat{\tau}$ and $\hat{\beta}$

Sample size	Parameter	AE	Bias	Variance	MSE
	$\alpha = 0.5$	0.3788	-0.1212	0.0484	0.0631
	$\tau = 2.0$	1.8534	-0.1466	0.3809	0.4024
50	b= 0.5	0.5646	-2.4354	0.2211	6.1521
	$\psi = 1.5$	1.2534	-0.2466	0.2564	0.3172
	$\beta = 3.0$	1.6367	1.1367	0.3772	1.6692
	$\alpha = 0.5$	0.3866	-0.1134	0.0408	0.0537
	$\tau = 2.0$	1.9041	-0.0959	0.2558	0.2650
100	b= 0.5	0.4993	-2.5007	0.1585	6.4120
	$\psi = 1.5$	1.2571	-0.2429	0.1206	0.1795
	$\beta = 3.0$	1.5858	1.0858	0.1951	1.3741
	$\alpha = 0.5$	0.4062	-0.0938	0.0433	0.0521
	$\tau = 2.0$	1.9177	-0.0823	0.1878	0.1945
150	b= 0.5	0.5215	-2.4785	0.1457	6.2888
	$\psi = 1.5$	1.2847	-0.2153	0.0692	0.1155
	$\beta = 3.0$	1.5570	1.0570	0.1239	1.2412
	$\alpha = 0.5$	0.4575	-0.0425	0.0439	0.0457
	$\tau = 2.0$	1.9665	-0.0335	0.0858	0.0869
350	b= 0.5	0.5285	-2.4715	0.0992	6.2074
	$\psi = 1.5$	1.3219	-0.1781	0.0255	0.0572
	$\beta = 3.0$	1.4698	0.9698	0.0325	0.9731
	$\alpha = 0.5$	0.4841	-0.0159	0.0393	0.0396
	$\tau = 2.0$	1.9681	-0.0319	0.0671	0.0681
500	b= 0.5	0.5089	-2.4911	0.0993	6.3051
	$\psi = 1.5$	1.3464	-0.1536	0.0149	0.0385
	$\beta = 3.0$	1.4609	0.9609	0.0238	0.9472

Table 2: The performace rating of the APWF distribution with glass fibres dataset

Distribution	Parameter MLEs	AIC	CAIC	BIC	HQIC	W	A
APWF	$\hat{\psi} = 11.049$						
	$\hat{b} = 0.1156$						
	$\hat{\beta} = 0.3353$	37.3734	38.4260	48.0891	41.5880	0.1808	0.9911
	$\hat{\tau} = 10.098$						
	$\hat{\alpha} = 0.3012$						
Gompertz Weibull	$\hat{\alpha} = 0.2245$						
	$\hat{\beta} = 0.0092$						
		38.3769	39.0666	46.9495	41.7486	0.2330	1.2832
	$\hat{\psi} = 0.7973$						
	$\hat{b} = 5.6176$						
Gompertz Lomax	$\hat{\alpha} = 0.0046$						
	$\hat{\beta} = 8.1791$						
		39.0055	37.6951	45.5780	40.3771	0.1685	0.9462
	$\hat{a} = 0.5070$						
	$\hat{b} = 1.5158$						
Weibull Frechet	$\hat{\alpha} = 3.61218$						

Table 2 – *Continued from previous page*

Distribution	Parameter MLEs	AIC	CAIC	BIC	HQIC	W	A
	$\hat{m}$ = 25.1859						
		39.0276	39.7812	47.3686	42.1676	0.2472	1.3566
	$\hat{\beta}$ = 0.1623						
	$\hat{a}$ = 0.2131						
Kumaraswamy Lomax	$\hat{\alpha}$ = 9.8352						
	$\hat{\beta}$ = 45.3107						
		44.2055	44.8951	52.7779	47.5771	1.6446	1.9915
	$\hat{a}$ = 15.1182						
	$\hat{b}$ = 0.0483						
Beta Lomax	$\hat{\alpha}$ = 18.1737						
	$\hat{\beta}$ = 26.7645						
		56.8068	57.4964	65.3793	60.1784	2.5426	3.1986
	$\hat{a}$ = 10.8769						
	$\hat{b}$ = 0.0329						
APIE	$\hat{\alpha}$ = 53.5634						
	$\hat{\lambda}$ = 0.3509	196.3253	196.5253	200.611	198.0111	0.7775	4.2384

Table 3: Test statistic for the APWF distribution with glass fibres dataset

Distribution	KS	p-Value	Log-likelihood
APWF	0.1236	0.2910	13.6867
Gompertz Weibull	0.1521	0.1087	15.1887
Gompertz Lomax	0.1542	0.0998	14.5027
Weibull Frechet	0.1552	0.0960	14.8177
Kumaraswamy Lomax	0.1854	0.0263	18.1027
Beta Lomax	0.2182	0.0049	24.4034
Alpha power inverted exponential	0.4646	3.0e-12	96.1627

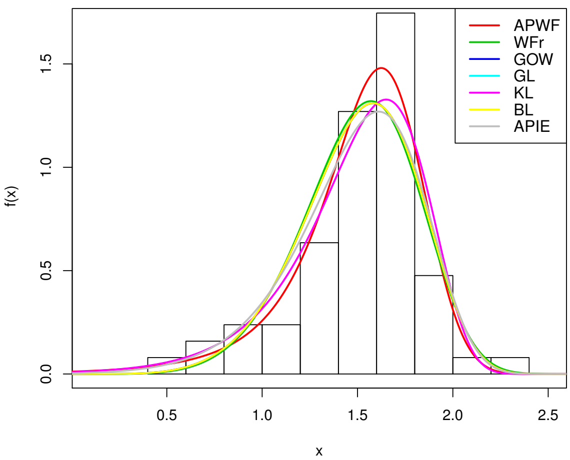

Figure 3: A plot of APWF distributions with the empirical histogram of the glass fibres data

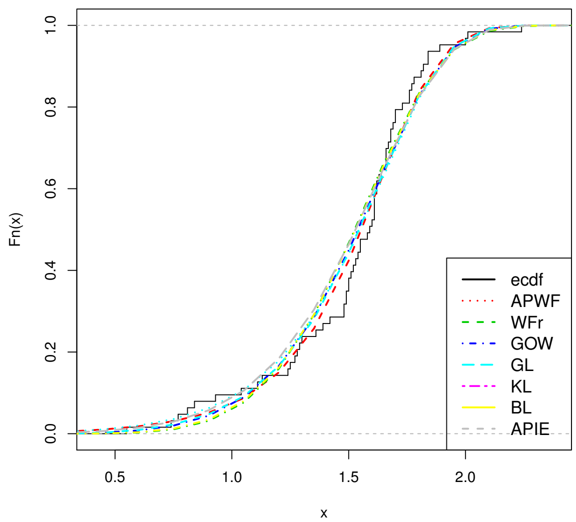

Figure 4: The fitted cdf of the APWF model for the glass data set

5.1.2. Second set of data carbon data

Our second set of data is from [21]. It consists of 100 observations taken on breaking stress of carbon fibers (in Gba). Table 4 and Table 5 are the goodness-of-fit and the performance rating of the APWF distribution using several test statistics for the carbon fibers dataset.

Table 4: Test statistic for the APWF distribution with glass fibres dataset

Distribution	KS	p-Value	Log-likelihood
APWF	0.06082131	0.8687617	141.3111
Gompertz Weibull	0.0632502	0.8185524	141.2822
Gompertz Lomax	0.06365319	0.8125448	142.4323
Weibull Frechet	0.06251348	0.8293575	141.3857
Kumaraswamy Lomax	0.07543761	0.6198049	141.484
Beta Lomax	0.17654926	0.00459718	156.7625
Alpha power inverted exponential	0.3503104	4.384659e-11	209.1656

Table 5: The performace rating of the APWF distribution with glass fibres dataset

Distribution	Parameter MLEs	AIC	CAIC	BIC	HQIC	W	A
APWF	$\hat{\psi} = 0.4603$ $\hat{b} = 2.7010$ $\hat{\beta} = 0.6398$ $\hat{\tau} = 0.9554$ $\hat{\alpha} = 6.1598$	282.3754	283.0137	295.4013	287.6472	0.0609	0.3719
Gompertz Weibull	$\hat{\alpha} = 2.2594$ $\hat{\beta} = -0.2017$ $\hat{\psi} = 0.2650$ $\hat{b} = 2.9808$	290.6544	290.9854	300.985	294.7818	0.0648	0.3834
Gompertz Lomax	$\hat{\alpha} = 0.0091$ $\hat{\beta} = 5.0656$ $\hat{a} = 1.9848$ $\hat{b} = 0.6471$	292.8646	293.2857	303.2853	297.0821	0.0611	0.4763
Weibull Frechet	$\hat{\alpha} = 0.6942$ $\hat{m} = 3.5178$ $\hat{\beta} = 0.6178$ $\hat{a} = 0.0947$	294.6000	295.0000	305.0000	298.8000	0.06892	0.4169
Kumaraswamy Lomax	$\hat{\alpha} = 3.7970$ $\hat{\beta} = 24.367$ $\hat{a} = 0.0334$ $\hat{b} = 6.0885$	295.9681	291.3891	301.3888	295.1855	0.0842	0.4532
Beta Lomax	$\hat{\alpha} = 18.1737$ $\hat{\beta} = 26.7645$ $\hat{a} = 10.8769$ $\hat{b} = 0.0329$	315.0974	317.4653	320.1753	317.4653	1.0896	2.0088
APIE	$\hat{\alpha} = 11.0025$ $\hat{\lambda} = 0.8694$	422.3312	422.455	427.5416	424.44	0.3726	2.0427

6. Discussion

The performance of a model is determined by the value that corresponds to the highest Log-likelihood or the lowest Akaike Information Criteria (AIC) value is considered as the best model. In the two real life cases considered, the APWF distribution has the lowest AIC value with 37.37339 in glass fibres data and 282.3754 in carbon data respectively. Also, the APWF has the value of log-likelihood as 13.68669 and 136.1877 for glass fibres and carbon data respectively. Hence, it competes favourably with other existing model for the data used.

7. Conclusion

The concept of the APWF distribution has been defined, introduced and studied. The mathematical expression for the pdf and cdf were examined. The statistical properties which include the order statistics

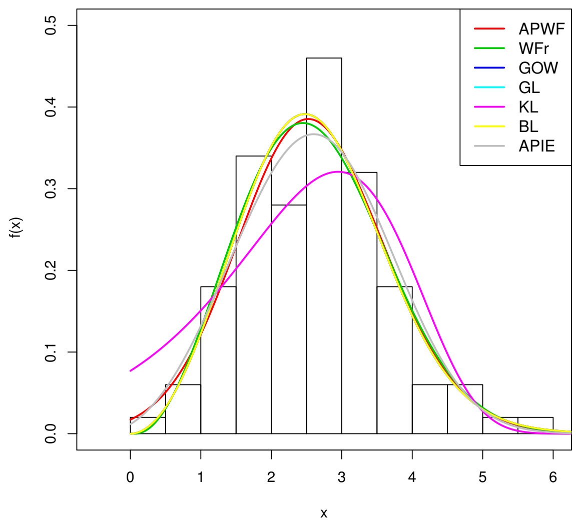

Figure 5: A plot of APWF distributions with the empirical histogram for the carbon data

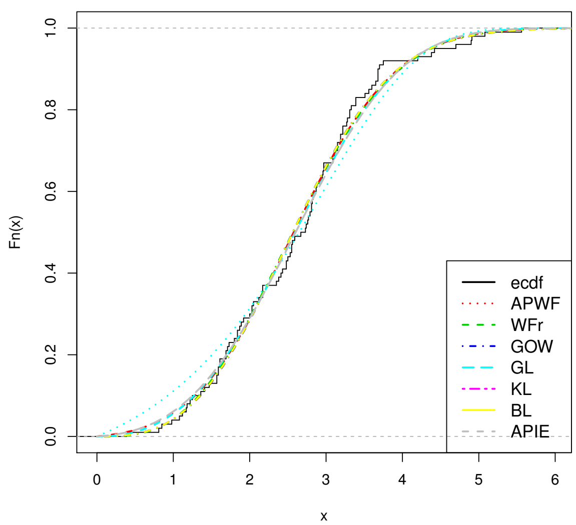

Figure 6: The fitted cdf of the APWF model for the carbon data set

distribution, cumulative hazard function, quantile, reversed hazard function, median, hazard rate function and odds function have been derived. The shape of the distribution could be inverted bathtub or decreasing. An application of the APWF model on a two real life data shows that the APWF distribution competes favourably with the Gompertz Weibull and Exponential, and better than the Kumaraswamy Lomax distribution, Beta Lomax distribution and some other families of distributions.

References

[1] Ergun A. Integral Representation for Solution of Discontinuous Diffusion Operator with Jump Conditions, Cumhuriyet Science Journal. 39 (1), 2018, 842–863.

[2] Ergun A, Amirov R Kh. Dırect and Inverse problems for diffusion operatör with discontinuity points, TWMS J. App. Eng. Math. 2019:9 (1) : 9–21.

[3] Johnson NL. Systems of frequency curves generated by methods of translation. Biometrika. 36, 1949, 49–176.

[4] Tukey JW. The practical relationship between the common transformations of percentages of counts and amounts. Technical report 36. Princeton University, Princeton N J, Statistical Technique Research Group.

[5] Lee C, Famoye F, Alzaatreh AY. Methods for generating families of univariate continuous distributions in the recent decades. Wiley Interdisciplinary Reviews: Computational Statistics 2013; 5 (3), 219 - 238.

[6] Weibull W. A statistical distribution functions of wide applicability. Journal of Applied Mechanics, Transactions. 18, 1951, 293–297.

[7] Afify AZ, Yousof HM, Cordeiro GM, Ortega EMM, Nofal ZM. The Weibull Frechect distribution and its applications. Journal of Applied Statistics 2016. Doi.org/10.1080/02664763.2016.1142945

[8] Mahdavi A, Kundu D. A new method for generating distributions with an application to exponential distribution. Communications in Statistics - Theory and Methods.46 (13), 2017, 6543 - 6557.

[9] Bourguignon MB, Silva R. Cordeiro, GM. The Weibull-G family of probability distributions. Journal of Data Science 2014; 12:53–68.

[10] Unal C, Cakmakyapan S, Ozel G. Alpha Power Inverted Exponential Distribution: Properties and Application, Gazi University Journal of Science. 31(3), 2018, 954–965.

[11] Alizadeh M, Cordeiro GM, Pinho LGB, Ghosh I. The Gompertz-G family of distributions, Journal of Statistical Theory and Practice. 11(1), 2017, 179–207.

[12] Eghwerido JT, Zelibe SC, Efe-Eyefia E. Gompertz alpha power inverted exponential: properties and applications 2020a; Article in the press.

[13] Eghwerido JT, Zelibe SC, Ekuma-Okereke E, Efe-Eyefia E. On the extended new generalized exponential distribution: properties and applications. FUPRE Journal of Scientific and Industrial Research. 3(1), 2019, 112–122.

[14] Efe-Eyefia E, Eghwerido JT, Zelibe SC. Theoritical Analysis of the Weibull alpha power inverted exponential Distribution: properties and applications. Gazi University Journal of Science, 2020; forthcoming.

[15] Nassar M, Alzaatreh A, Mead A, Abo-Kasem O. Alpha power Weibull distribution: Properties and applications. Communications in Statistics-Theory and Methods. 46, 2017, 10236–10252.

[16] Smith RL, and Naylor J. C. A comparison of maximum likelihood and Bayesian estimates for the three-parameter Weibull distribution. Applied Statistics. 36, 1987, 258–369.

[17] Haq MA, Butt NS, Usman RM, Fattah AA. Transmuted power function distribution, Gazi University Journal of Science. 29(1), 2016, 177–185.

[18] Merovci F, Khaleel M, AIbrahim NA, Shitan M. The beta type-X distribution: properties with application. Springer-Plus. 5, 2016, 697.

[19] Rastogi MK, Oguntunde PE. Classical and bayes estimation of reliability characteristics of the Kumaraswamy-inverse exponential distribution. International Journal of System Assurance Engineering and Management 2017.

[20] Oguntunde PE, Khaleel MA, Ahmed MT, Adejumo AO, Odetunmibi OA. A new generalization of the Lomax distribution with increasing, decreasing, and constant failure rate. Hindawi Modelling and Simulation in Engineering 2017, 1-7.

[21] Nichols MD, Padgett WJ. A bootstrap control chart for Weibull percentiles. Quality and Reliability Engineering International. 22, 2006, 141–151.