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A Fixed Point Theorem for Non-Self Mappings in Multiplicative Metric Spaces

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Abstract

In this paper, we define the concept of metrical convexity in the context of multiplicative metric spaces. Using this, we then prove a fixed point theorem for non-self mappings in multiplicative metric spaces. Furthermore, we give an illustrative case on the utilization of the hypothesis.

Keywords: *Fixed point, multiplicative metric space, multiplicative convex set, non-self mapping.*
2010 Mathematics Subject Classification: *47H10, 54H25.*

1. Introduction and Preliminaries

The concept of multiplicative calculus was initiated by Grossman and Katz [3]. Inspired by this new approach to calculus, Özavşar and Çevikel [5] came up with the idea of the multiplicative metric space. This space differs from the more familiar metric space in that the operation of addition is replaced by that of multiplication in the defining axioms of the space. The researchers Özavşar and Çevikel [5] also described various topological properties of this space. They proved that the Banach Contraction Principle is applicable for multiplicative metric spaces when the term "contraction" is suitably defined.
Researchers have developed fixed point theorems for self-mappings in multiplicative metric spaces. We have those by Abbas et. al [1] and Hxiaoju et. al [4] as examples. However, research on fixed point theorems for non-self mappings in multiplicative metric spaces is limited. In this study, we prove a fixed point theorem for non-self mappings in multiplicative metric spaces.
First, we introduce the preliminary concepts which are useful in this study. In this work, $\mathbb{R}_+$ and $\mathbb{N}$ represent the set of all positive real numbers and the set of natural numbers respectively. We also use the term MMS as an abbreviation of multiplicative metric space.
The following is the definition of a multiplicative metric space.

Definition 1.1. [1] Let X be a nonempty set. A function $d : X \times X \to \mathbb{R}_+$ is said to be a multiplicative metric on X if for any $x, y, z \in X$, the following conditions hold:

(m1) $d(x,y) \geq 1$ and $d(x,y) = 1$ if and only if $x = y$;
(m2) $d(x,y) = d(y,x)$;
(m3) $d(x,y) \leq d(x,z) \cdot d(z,y)$.

The pair (X,d) is called a multiplicative metric space.

Examples of multiplicative metric spaces are stated here.

Example 1.2. [5] Let $d^* : (\mathbb{R}_+)^n \times (\mathbb{R}_+)^n \to [1,+\infty)$ be defined as follows

$$d^*(x,y) = \left|\frac{x_1}{y_1}\right|^* \left|\frac{x_2}{y_2}\right|^* \cdots \left|\frac{x_n}{y_n}\right|^*,$$

where $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n) \in (\mathbb{R}_+)^n$ and $\left|\cdot\right|^* : \mathbb{R}_+ \to [1,+\infty)$ is defined as

$$|a|^* = \begin{cases} a, & \text{if } a \geq 1 \\ \frac{1}{a}, & \text{if } a < 1. \end{cases}$$

Then $\left((\mathbb{R}_+)^n, d^*\right)$ is a multiplicative metric space.

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The following example is modified from Özavşar and Çevikel [5].

Example 1.3. Let $a>1$ be a fixed number. Then $d_a:\mathbb{R}\times\mathbb{R}\to[1,+\infty]$ defined by $d_a(x,y)=a^{|x-y|}$ holds the multiplicative metric conditions.

Definition 1.4. [5] Let (X,d) be a multiplicative metric space, $x\in X$ and $\varepsilon>1$. Define the following set: $B_\varepsilon(x):=\{y\in X: d(x,y)<\varepsilon\}$, which is called the multiplicative open ball of radius ε with center x. Similarly, one can describe the multiplicative closed ball as follows: $\bar{B}_\varepsilon(x):=\{y\in X: d(x,y)\le\varepsilon\}$.

Definition 1.5. [5] Let (X,d) be a multiplicative metric space, $\{x_n\}$ be a sequence in X and $x\in X$. If for every multiplicative open ball $B_\varepsilon(x)$ there exists a natural number N such that if $n\ge N\Rightarrow x_n\in B_\varepsilon(x)$, then the sequence $\{x_n\}$ is said to be multiplicative converging to x, denoted by $x_n\to_* x\,(n\to\infty)$.

Lemma 1.6. [5] *Let (X,d) be a multiplicative metric space, $\{x_n\}$ be a sequence in X and $x\in X$. Then $x_n\to_* x$ as $n\to\infty$ if and only if $d(x_n,x)\to_* 1$ as $n\to\infty$.*

Lemma 1.7. [5] *Let (X,d) be a multiplicative metric space and $\{x_n\}$ be a sequence in X. If the sequence $\{x_n\}$ is multiplicative convergent, then the multiplicative limit point is unique.*

Definition 1.8. [5] Let (X,d) be a multiplicative metric space and $\{x_n\}$ be a sequence in X. The sequence $\{x_n\}$ is called a multiplicative Cauchy sequence if for all $\varepsilon>1$, there exists $N\in\mathbb{N}$ such that $d(x_m,x_n)<\varepsilon$ for all $m,n\ge N$.

Lemma 1.9. [5] *Let (X,d) be a multiplicative metric space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a multiplicative Cauchy sequence if and only if $d(x_n,x_m)\to_* 1$ as $m,n\to\infty$.*

Definition 1.10. [5] Let (X,d) be a multiplicative metric space. A subset $S\subseteq X$ is called multiplicative closed in (X,d) if S contains all of its multiplicative limit points.

Theorem 1.11. [5] *Let (X,d) be a multiplicative metric space and $S\subseteq X$. Then the set S is multiplicative closed if and only if every multiplicative convergent sequence in S has a multiplicative limit point that belongs to S.*

Theorem 1.12. [5] *Let (X,d) be a multiplicative metric space and $S\subseteq X$. Then (S,d) is complete if and only if S is multiplicative closed.*

Using the common topological definition of the boundary of a set, we provide the following definition.

Definition 1.13. Let (X,d) be a multiplicative metric space and $S\subseteq X$. The boundary of S, denoted by ∂S, is the set of points $x\in X$ such that every open ball $B_\varepsilon(x)$ contains at least one point of S and at least one point not of S.

Inspired by the definition of a metrically convex metric space by Assad and Kirk [6], we define a multiplicative metrically convex MMS.

Definition 1.14. A complete MMS (X,d) is said to be multiplicative metrically convex if X has the property that for each $x,y\in X$ with $x\neq y$ there exists $z\in X, x\neq z\neq y$, such that $d(x,y)=d(x,z)\cdot d(z,y)$.

If (X,d) is a multiplicative metrically convex metric space, and $x,y\in X$, we term

$$\operatorname{seg}[x,y]:=\{z\in X: d(x,y)=d(x,z)\cdot d(z,y)\}. \tag{1.1}$$

We state an example of a multiplicative convex MMS.

Example 1.15. Let $d^*:\mathbb{R}_+\times\mathbb{R}_+\to[1,+\infty)$ be defined as

$$d^\star(x,y)=\left|\frac{x}{y}\right|^*,\ \text{where } |a|^*=\begin{cases} a, & \text{if } a\ge 1\\ \dfrac{1}{a}, & \text{if } a<1\end{cases}.$$

Then $(\mathbb{R}_+,d^*)$ is a multiplicative convex MMS. Without loss of generality, let $x,y\in\mathbb{R}_+$ be such that $x<y$. Then for all z such that $x<z<y$, we have $z\in seg[x,y]$.

Lemma 1.16. *Let C be a multiplicative closed subset of the complete and multiplicative metrically convex MMS (X,d). If $x\in C$ and $y\notin C$, then there exists a point $z\in\partial C$ (the boundary of C) such that $z\in\operatorname{seg}[x,y]$.*

Proof. From Definition 1.14, for $x,y\in X$ and $z\in seg[x,y]$, we have

$$\begin{aligned} &d(x,z)\cdot d(z,y)=d(x,y)\\ &\quad\Rightarrow d(x,z)\le d(x,y),\ \text{because } d(y,z)\ge 1\\ &\quad\Rightarrow d(x,z)=d(x,y)^t \text{ for some } t\in(0,1).\end{aligned}$$

We can write z as a function of t as

$$z(t)=\{z\in\operatorname{seg}[x,y]: d(x,z)=d(x,y)^t\}.$$

When $t\to 0$, we have

$$d(x,z)\to 1\Rightarrow z\to x,\ \text{by (m1) of Definition 1.1}.$$

When $t\to 1$, we have

$$d(x,z)\to d(x,y)\Rightarrow z\to y.$$

The power function is a continuous function. Thus z traces a continuous curve from x to y. If $x\in C$ and $y\in X\backslash C$, the continuous curve traced by z will intersect the boundary of C on at least one point. This proves the lemma. □

We introduce the following lemma.

Lemma 1.17. *Let (X,d) be a multiplicative metrically convex MMS and let $x,y\in X$. If $z\in \mathrm{seg}[x,y]$ then*

(i) $d(x,z)\le d(x,y)$ *and*
(ii) $d(y,z)\le d(x,y)$.

Proof. From Definition 1.14, for $x,y\in X$ and $z\in seg[x,y]$, we have

$$d(x,z)\cdot d(z,y)=d(x,y)$$
$$\Rightarrow d(x,z)=\frac{d(x,y)}{d(z,y)}\le d(x,y) \text{ because } d(z,y)\ge 1.$$

Similarly,

$$d(y,z)=\frac{d(x,y)}{d(x,z)}\le d(x,y) \text{ because } d(x,z)\ge 1.$$

□

In this study, the following lemma from Rugumisa and Kumar [8] is used.

Lemma 1.18. [8] *Consider a sequence $\{w_n\}_{n\in\mathbb{N}}\in\mathbb{R}_+$ such that, for all $n\ge 2$, we have*

$$w_n\le k\max\{w_{n-2},w_{n-1}\},k\in(0,1), \tag{1.2}$$

then

$$w_n\le k^{n/2}k^{-1/2}\max\{w_0,w_1\}. \tag{1.3}$$

Recently Khan and Imdad [7] proved the following fixed point theorem for self mappings in MMS.

Theorem 1.19. [7] *Let A,B,I and J be self-mappings of a multiplicative metric space (X,d) satisfying $A(X)\subset J(X)$, $B(X)\subset I(X)$ and*

$$d(Ax,By)\le\left[\max\{d(Ix,Jy),d(Ix,Ax),d(By,Jy),d(Ax,Jy),d(Ix,By)\}\right]^{\lambda},\lambda\in(0,\frac{1}{2})\text{ for all } x,y\in X.$$

If one of $A(X),B(X),I(X),J(X)$ is a complete subspace of X, then the following conclusions hold
(i) (A,I) has coincidence point,
(ii) (B,J) has coincidence point.
Further, if the pairs (A,I) and (B,J) are coincidently commuting, then A,B,I and J have a unique common fixed point.

Theorem 1.19 is proved for *self* mappings. In this study, we modify Theorem 1.19 so that it applies to *non-self* mappings.

2. Main Result.

We prove the following theorem.

Theorem 2.1. *Let (X,d) be a multiplicative metric space which is complete and multiplicative metrically convex. Let the mapping $T:C\to X$, where C is a multiplicative closed subset of X with a non-empty boundary ∂C, obey the following conditions:*

(i) $d(Tx,Ty)\le[\max\{d(x,y),d(x,Tx),d(y,Ty),d(x,Ty),(d(y,Tx)\}]^{\lambda}$, *where* $\lambda\in\left(0,\frac{1}{3}\right)$,
(ii) $\partial C\subset TC$,
(iii) $x\in\partial C$ *implies* $Tx\in C$, *and*
(iv) TC *is multiplicative closed.*

Then T has a unique fixed point.

Proof. We construct a sequence $\{x_n\}_{n\in\mathbb{N}}$ in the following way. We commence at arbitrary point $x_0\in\partial C$. By (ii), we find $x_1=Tx_0\in C$. We then find Tx_1. We proceed inductively as follows:
If $Tx_n\in C$, then $x_{n+1}=Tx_n$. If however $Tx_n\notin C$ then we choose $x_{n+1}\in\partial C$ such that $x_{n+1}\in\mathrm{seg}[x_n,Tx_n]$. We partition the sequence $\{x_n\}_{n\in\mathbb{N}}$ into two sets P and Q where $P=\{x_n : x_n=Tx_{n-1}, i\ge 1\}$ and $Q=\{x_n : x_n\ne Tx_{n-1}\}$. From the construction of sequence, we note that when $x_n\in Q$, then $x_n\in\mathrm{seg}[x_{n-1},Tx_{n-1}]$ and $x_n\in\partial C$. We consider the following cases.
Case 1. Consider $(x_n,x_{n+1})\in P\times P$, $n\ge 1$. This implies $x_n=Tx_{n-1}$ and $x_{n+1}=Tx_n$. From (i) in the assumption, we have

$$\begin{aligned}
d(x_n,x_{n+1}) &= d(Tx_{n-1},Tx_n)\\
&\le[\max\{d(x_{n-1},x_n),d(x_{n-1},Tx_{n-1}),d(x_n,Tx_n),d(x_{n-1},Tx_n),d(x_n,Tx_{n-1})\}]^{\lambda}\\
&=[\max\{d(x_{n-1},x_n),d(x_{n-1},x_n),d(x_n,x_{n+1}),d(x_{n-1},x_{n+1}),d(x_n,x_n)\}]^{\lambda}\\
&=[\max\{d(x_{n-1},x_n),d(x_n,x_{n+1}),d(x_{n-1},x_{n+1})\}]^{\lambda},\text{using (m1) of Definition 1.1},\\
&\le[\max\{d(x_{n-1},x_n),d(x_{n-1},x_n)\cdot d(x_n,x_{n+1})\}]^{\lambda},\text{ by (m3)},\\
&=[d(x_{n-1},x_n)\cdot d(x_n,x_{n+1})]^{\lambda},\text{ by (m1)}.
\end{aligned}$$

The expression above implies

$$d(x_n,x_{n+1}) \le d(x_{n-1},x_n)^{\frac{\lambda}{1-\lambda}} = d(x_{n-1},x_n)^k, \text{ where } k = \frac{\lambda}{1-\lambda}. \tag{2.1}$$

Case 2. Consider $(x_n,x_{n+1}) \in P\times Q$, $n\ge 1$. From the construction of sequence, this means $x_n = Tx_{n-1}$ and $x_{n+1} \in \mathrm{seg}[Tx_{n-1},Tx_n]$. From (i), we have

$$\begin{aligned}
d(x_n,Tx_n) &= d(Tx_{n-1},Tx_n)\\
&\le [\max\{d(x_{n-1},x_n),d(x_{n-1},Tx_{n-1}),d(x_n,Tx_n),d(x_{n-1},Tx_n),d(x_n,Tx_{n-1})\}]^{\lambda}\\
&= [\max\{d(x_{n-1},x_n),d(x_{n-1},x_n),d(x_n,Tx_n),d(x_{n-1},Tx_n),d(x_n,x_n)\}]^{\lambda}\\
&= [\max\{d(x_{n-1},x_n),d(x_{n-1},Tx_n)\}]^{\lambda}\\
&\le [\max\{d(x_{n-1},x_n),d(x_{n-1},x_n)\cdot d(x_n,Tx_n)\}]^{\lambda}, \text{ by (m3)}\\
&= [d(x_{n-1},x_n)\cdot d(x_n,Tx_n)]^{\lambda}, \text{ by (m1)}.
\end{aligned}$$

The above expression implies $d(x_n,Tx_n) \le [d(x_{n-1},x_n)]^{\frac{\lambda}{1-\lambda}}$. Because $x_{n+1}\in \mathrm{seg}[x_n,Tx_n)]$, by Lemma 1.17, we have

$$d(x_n,x_{n+1}) \le d(x_n,Tx_n) \le [d(x_{n-1},x_n)]^k, \text{ where } k = \frac{\lambda}{1-\lambda}. \tag{2.2}$$

Case 3. Consider $(x_n,x_{n+1}) \in Q\times P$. In this scenario, we have $x_{n+1} = Tx_n$. We also have $x_n \in \mathrm{seg}[x_{n-1},Tx_{n-1}]$.

Remark 1. *We claim that $x_n \in Q$ implies $x_{n-1}\in P$. We prove this claim by contradiction. Suppose $x_{n-1}\in Q$. This implies that $x_{n-1}\in \partial C$. From (iii) in the assumption, this means $x_n = Tx_{n-1}\in C$. This implies $x_n \in P$ which is a contradiction. Hence $x_n\in Q$ implies $x_{n-1}\in P$.*
As $x_{n-1}\in P$, we have $x_{n-1} = Tx_{n-2}$.
Applying (m3) of Definition 1.1, we have

$$d(x_n,x_{n+1}) = d(x_n,Tx_n) \le d(x_n,Tx_{n-1})\cdot d(Tx_{n-1},Tx_n). \tag{2.3}$$

Subcase 3.1. Suppose $d(x_n,Tx_{n-1}) \le d(Tx_{n-1},Tx_n)$. Then (2.3) leads to

$$d(x_n,x_{n+1}) \le d(Tx_{n-1},Tx_n)\cdot d(Tx_{n-1},Tx_n) = [d(Tx_{n-1},Tx_n)]^2. \tag{2.4}$$

Let us consider $d(Tx_{n-1},Tx_n)$. Using an argument similar to that in Case 2, we get

$$d(Tx_{n-1},Tx_n) \le [d(x_{n-1},x_n)]^k, \text{ where } k = \frac{\lambda}{1-\lambda}. \tag{2.5}$$

Applying (2.5) to (2.4), we get

$$d(x_n,x_{n+1}) \le [d(x_{n-1},x_n)]^{2k}. \tag{2.6}$$

Subcase 3.2. Suppose $d(x_n,Tx_{n-1}) > d(Tx_{n-1},Tx_n)$. Then (2.3) leads to

$$d(x_n,x_{n+1}) \le d(x_n,Tx_{n-1})\cdot d(x_n,Tx_{n-1}) = [d(x_n,Tx_{n-1})]^2. \tag{2.7}$$

By Lemma 1.17, we have that $d(x_n,Tx_{n-1}) \le d(x_{n-1},Tx_{n-1})$. Also, by Remark 1, we have $x_{n-1} = Tx_{n-2}$. Hence (2.7) becomes

$$d(x_n,x_{n+1}) \le [d(Tx_{n-2},Tx_{n-1})]^2. \tag{2.8}$$

Using (i) in the assumption, we get

$$\begin{aligned}
d(Tx_{n-2},Tx_{n-1}) &\le [\max\{d(x_{n-2},x_{n-1}),d(x_{n-2},Tx_{n-2}),d(x_{n-1},Tx_{n-1}),d(x_{n-2},Tx_{n-1}),d(x_{n-1},Tx_{n-2})\}]^{\lambda}\\
&= [\max\{d(x_{n-2},Tx_{n-2}),d(x_{n-2},Tx_{n-2}),d(Tx_{n-2},Tx_{n-1}),d(x_{n-2},Tx_{n-1}),d(Tx_{n-2},Tx_{n-2})\}]^{\lambda}\\
&= [\max\{d(x_{n-2},Tx_{n-2}),d(x_{n-2},Tx_{n-1})\}]^{\lambda}.
\end{aligned} \tag{2.9}$$

We apply (m3) of Definition 1.1 to (2.9), and get

$$\begin{aligned}
d(Tx_{n-2},Tx_{n-1}) &\le [\max\{d(x_{n-2},Tx_{n-2}),d(x_{n-2},Tx_{n-2})\cdot d(Tx_{n-2},Tx_{n-1})\}]^{\lambda}\\
&= [d(x_{n-2},Tx_{n-2})\cdot d(Tx_{n-2},Tx_{n-1})]^{\lambda}, \text{ by (m1)}\\
&\le [d(x_{n-2},Tx_{n-2})]^k, \text{ where } k = \frac{\lambda}{1-\lambda}\\
&\Rightarrow d(Tx_{n-2},Tx_{n-1}) \le [d(x_{n-2},x_{n-1})]^k.
\end{aligned} \tag{2.10}$$

Applying (2.10) to (2.8), we get

$$d(x_n,x_{n+1}) \le [d(x_{n-2},x_{n-1})]^{2k}. \tag{2.11}$$

From (2.6) and (2.11), we conclude that, for $(x_n,x_{n+1})\in P\times Q$, we have

$$d(x_n,x_{n+1}) \le [\max\{d(x_{n-2},x_{n-1}),d(x_{n-1},x_n)\}]^{2k}, \text{ where } k = \frac{\lambda}{1-\lambda}. \tag{2.12}$$

As is evident from Remark 1, the case where $(x_n, x_{n+1}) \in Q \times Q$ is not possible. Hence, for all possible cases, the equations (2.1), (2.2) and (2.12) imply

$$d(x_n, x_{n+1}) \leq [\max\{d(x_{n-2}, x_{n-1}), d(x_{n-1}, x_n)\}]^{2k}, \ n \geq 2. \tag{2.13}$$

Because the logarithm function is an increasing function, (2.13) leads to

$$\log\left(d(x_n, x_{n+1})\right) \leq 2k[\max\{\log\left(d(x_{n-2}, x_{n-1})\right), \log\left(d(x_{n-1}, x_n)\right)\}]. \tag{2.14}$$

From the assumption, we have $\lambda \in (0, 1/3)$. Because $k = \dfrac{\lambda}{1-\lambda}$, this means $2k \in (0,1)$. We apply Lemma 1.18 with $w_n = \log\left(d(x_n, x_{n+1})\right)$ and get

$$\log\left(d(x_n, x_{n+1})\right) \leq (2k)^{n/2}\delta, \ \text{where } \delta = (2k)^{-1/2}\max\{\log\left(d(x_0, x_1)\right), \log\left(d(x_1, x_2)\right)\}. \tag{2.15}$$

The exponential function is an increasing function. When we use the exponential function on both sides of equation (2.15), we get

$$d(x_n, x_{n+1}) \leq \exp\left((2k)^{n/2}\delta\right). \tag{2.16}$$

Let $m, n \in \mathbb{N}$ with $m < n$. Using (m3) of Definition 1.1 inductively, we get

$$\begin{aligned}
d(x_m, x_n) &\leq \prod_{i=m}^{n-1} d(x_i, x_{i+1}) \\
&\leq \prod_{i=m}^{+\infty} d(x_i, x_{i+1}) \\
&\leq \prod_{i=m}^{+\infty} \exp\left((2k)^{i/2}\delta\right), \ \text{by (2.16)} \\
&= \exp\left(\delta \sum_{i=m}^{+\infty} (2k)^{i/2}\right) \\
&= \exp\left(\delta (2k)^{m/2} \frac{1}{1-(2k)^{1/2}}\right), \ \text{sum of G.P.}
\end{aligned}$$

We take the limits $m, n \to +\infty$. In doing so, we use the continuity of the exponential function. We also recall that $2k \in (0,1)$. We get

$$\lim_{m,n\to+\infty} d(x_m, x_n) = 1.$$

By Lemma 1.9, this shows that the sequence $\{x_n\} \in C$ is a multiplicative Cauchy sequence. By Theorem 1.12, because C is a multiplicative closed subset of X, it is also complete. This means there is $z \in C$ such that $x_n \to_* z$ as $n \to +\infty$.
Consider the subsequence $\{x_{n_k}\}$ of $\{x_n\}$ for which $x_{n_k} \in P$ for all $k \in \mathbb{N}$. For $n_k \geq 1$, we have

$$\begin{aligned}
&Tx_{n_k-1} = x_{n_k} \\
\Rightarrow &\lim_{k\to+\infty} Tx_{n_k-1} = \lim_{k\to+\infty} x_{n_k} = z.
\end{aligned} \tag{2.17}$$

Because $z \in C$, there is $u \in TC$ such that $u = Tz$. Using (i) in the assumption, we have

$$\begin{aligned}
d(u, Tx_{n_k-1}) &= d(Tz, Tx_{n_k-1}) \\
&\leq [\max\{d(z, x_{n_k-1}), d(z, Tz), d(x_{n_k-1}, Tx_{n_k-1}), d(z, Tx_{n_k-1}), d(x_{n_k-1}, Tz)\}]^{\lambda} \\
&= [\max\{d(z, x_{n_k-1}), d(z, u), d(x_{n_k-1}, Tx_{n_k-1}), d(z, Tx_{n_k-1}), d(x_{n_k-1}, u)\}]^{\lambda}.
\end{aligned}$$

Taking limits $k \to +\infty$, we get

$$\begin{aligned}
d(u,z) &\leq [\max d(z,z), d(z,u), d(z,z), d(z,z), d(z,u)]^{\lambda} \\
&= [d(z,u)]^{\lambda} \\
&= [d(u,z)]^{\lambda} \\
\Rightarrow d(u,z) &= 1, \ \text{because } \lambda \in (0, 1/3) \\
&\Rightarrow u = z, \ \text{by (m1) of Definition 1.1.}
\end{aligned}$$

Thus $z = Tz$, making z a fixed point of mapping T.
We show that z is unique. Suppose z' is also a fixed point of T. From the assumption, we have

$$\begin{aligned}
d(z,z') &= d(Tz, Tz') \\
&\leq [\max\{d(z,z'), d(z,Tz), d(z',Tz'), d(z,Tz'), d(z',Tz)\}]^{\lambda} \\
&= [\max\{d(z,z'), d(z,z), d(z',z'), d(z,z'), d(z',z)\}]^{\lambda} \\
&= [d(z,z')]^{\lambda} \\
\Rightarrow d(z,z') &= 1, \ \text{because } \lambda \in (0, 1/3) \\
&\Rightarrow z = z', \ \text{by (m1) of Definition 1.1.}
\end{aligned}$$

Hence the fixed point z is unique. □

We now show an example on the use of this theorem.

Example 2.2. Consider the multiplicative space metric space $(\mathbb{R})+, d)$, with $d = d_a$ defined as in Example 1.3 with $a = 2$. The space (X,d) is complete and multiplicative metrically convex.

Let $C = [0,6] \cup [10,40]$. Define the mapping $T : C \to \mathbb{R}_+$ as $Tx = \frac{1}{4}x$. From the given information, $TC = [0,1.5] \cup [2.5,10]$ is closed. For $x \in \partial C = \{0,6,10,40\}$, we have $Tx = \{0,1/5,2.5,10\} \subset C$. We note that for some values $c \in C$, (such as $c = 32$), we have $Tc \notin C$, making T a non-self mapping. Without loss of generality let $x, y \in C,\ x \geq y$. Thus

$$\begin{aligned}
d(Tx,Ty) &= d\big(\frac{1}{4}x, \frac{1}{4}y\big)\\
&= 2^{\frac{1}{4}|x-y|}\\
&\leq 2^{0.3|x-y|}\\
&= \big[2^{|x-y|}\big]^{0.3}\\
&= [d(x,y)]^{0.3}\\
&\leq \big[\max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}\big]^{\lambda},
\end{aligned}$$

where $\lambda = 0.3 \in \left(0, \frac{1}{3}\right)$.

The conditions for the assumption have been met, and $z = 0$ is the unique fixed point of T because $T(0) = 0$.

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