Surface Family with a Common Natural Asymptotic or Geodesic Lift of a Spacelike Curve with Spacelike Binormal in Minkowski 3-Space

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Abstract

In the present paper, we find a surface family possessing the natural lift of a given spacelike curve with spacelike binormal as an asymptotic curve or geodesic in Minkowski 3-space. We express necessary and sufficient conditions for the given curve such that its natural lift is an asymptotic curve or geodesic on any member of the surface family. Finally, we illustrate the method with some examples.

Keywords: Asymptotic curve; Geodesic; Natural lift; Surface family.
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1. Introduction

The problem of finding surfaces with a given common curve as a special curve was firstly handled by Wang et.al. [1]. They constructed surfaces with a common geodesic. Li et.al. [2] obtained necessary and sufficient condition for a given curve to be a line of curvature on a surface pencil. Bayram et.al. [3] studied surface pencil with a common asymptotic curve. In 2014, Ergün et.al. [4] presented a constraint for surfaces with a common line of curvature in Minkowski 3-space. In [5-16] authors found necessary and sufficient conditions for a given curve to be a common special curve on a surface family in Euclidean and Minkowski spaces.

Inspired with the above studies, we find necessary and sufficient condition for a surface family possessing the natural lift of a given spacelike curve with spacelike binormal as a common asymptotic curve or geodesic. First, we start with fundamentals required for the paper. Minkowski 3-space $\mathbb{R}^3_1$ is the vector space $\mathbb{R}^3$ equipped with the Lorentzian inner product g given by

$$g(X,X) = -x_1^2 + x_2^2 + x_3^2$$

where $X = (x_1, x_2, x_3) \in \mathbb{R}^3$. A vector $X = (x_1, x_2, x_3) \in \mathbb{R}^3$ is said to be timelike if $g(X,X) < 0$, spacelike if $g(X,X) > 0$ or $X = 0$ and lightlike (or null) if $g(X,X) = 0$ and $x \neq 0$ [16]. Similarly, an arbitrary curve $\alpha = \alpha(s)$ in $\mathbb{R}^3_1$ can locally be timelike, spacelike or null (lightlike), if all of its velocity vectors $\alpha'(s)$ are respectively timelike, spacelike or null (lightlike), for every $s \in I \subset \mathbb{R}$. A lightlike vector X is said to be positive (resp. negative) if and only if $x_1 > 0$ (resp. $x_1 < 0$) and a timelike vector X is said to be positive (resp. negative) if and only if $x_1 > 0$ (resp. $x_1 < 0$). The norm of a vector X is defined by $\|X\|_{IL} = \sqrt{|g(X,X)|}$ [16].
The vectors $X = (x_1, x_2, x_3)$, $Y = (y_1, y_2, y_3) \in \mathbb{R}^3_1$ are orthogonal if and only if $g(X,X) = 0$ [17].
Now let X and Y be two vectors in $\mathbb{R}^3_1$, then the Lorentzian cross product is given by [18]

$$
\begin{aligned}
X \times Y &= \begin{vmatrix} \vec{e_1} & -\vec{e_2} & -\vec{e_3} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} \\
&= (x_2 y_3 - x_3 y_2, \ x_1 y_3 - x_3 y_1, \ x_2 y_1 - x_1 y_2).
\end{aligned}
$$

We denote by $\{T(s), N(s), B(s)\}$ the moving Frenet frame along the curve α. Then T, N and B are the tangent, the principal normal and the binormal vector of the curve α, respectively.

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Let α be a unit speed timelike curve with curvature κ and torsion τ. So, T is a timelike vector field, N and B are spacelike vector fields. For these vectors, we can write

$$T \times N = -B, \quad N \times B = T, \quad B \times T = -N,$$

where $\times$ is the Lorentzian cross product in $\mathbb{R}^3_1$ [19]. The binormal vector field $B(s)$ is the unique spacelike unit vector field perpendicular to the timelike plane $\{T(s), N(s)\}$ at every point $\alpha(s)$ of α, such that $\{T, N, B\}$ has the same orientation as $\mathbb{R}^3_1$. Then, Frenet formulas are given by [19]

$$T' = \kappa N, \ N' = \kappa T + \tau B, \ B' = -\tau N.$$

Let α be a unit speed spacelike curve with spacelike binormal. Now, T and B are spacelike vector fields and N is a timelike vector field. In this situation,

$$T \times N = -B, \quad N \times B = -T, \quad B \times T = N.$$

The binormal vector field $B(s)$ is the unique spacelike unit vector field perpendicular to the timelike plane $\{T(s), N(s)\}$ at every point $\alpha(s)$ of α, such that $\{T, N, B\}$ has the same orientation as $\mathbb{R}^3_1$. Then, Frenet formulas are given by [19]

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$$T' = \kappa N, \ N' = -\kappa T + \tau B, \ B' = \tau N.$$

Lemma 1.1. *Let X and Y be nonzero Lorentz orthogonal vectors in $\mathbb{R}^3_1$. If X is timelike, then Y is spacelike [17].*

Lemma 1.2. *Let X and Y be positive (negative) timelike vectors in $\mathbb{R}^3_1$. Then*

$$g(X,Y) \leq \|X\| \|Y\|$$

whit equality if and only if X and Y are linearly dependent [17].

Lemma 1.3. ***i)*** *Let X and Y be positive (negative) timelike vectors in $\mathbb{R}^3_1$. By Lemma 2, there is a unique nonnegative real number $\varphi(X,Y)$ such that*

$$g(X,Y) = \|X\| \|Y\| \cosh \varphi(X,Y).$$

The Lorentzian timelike angle between X and Y is defined to be $\varphi(X,Y)$ [17].
ii) *Let X and Y be spacelike vectors in $\mathbb{R}^3_1$ that span a spacelike vector subspace. Then we have*

$$|g(X,Y)| \leq \|X\| \|Y\|.$$

Hence, there is a unique real number $\varphi(X,Y)$ between 0 and π such that

$$g(X,Y) = \|X\| \|Y\| \cos \varphi(X,Y).$$

$\varphi(X,Y)$ is defined to be the Lorentzian spacelike angle between X and Y [17].
iii) *Let X and Y be spacelike vectors in $\mathbb{R}^3_1$ that span a timelike vector subspace. Then, we have*

$$g(X,Y) > \|X\| \|Y\|.$$

Hence, there is a unique positive real number $\varphi(X,Y)$ between 0 and π such that

$$|g(X,Y)| = \|X\| \|Y\| \cosh \varphi(X,Y).$$

$\varphi(X,Y)$ is defined to be the Lorentzian timelike angle between X and Y [17]. ***iv)*** *Let X be a spacelike vector and Y be a positive timelike vector in $\mathbb{R}^3_1$. Then there is a unique nonnegative real number $\varphi(X,Y)$ such that*

$$|g(X,Y)| = \|X\| \|Y\| \sinh \varphi(X,Y).$$

$\varphi(X,Y)$ is defined to be the Lorentzian timelike angle between X and Y [17].

Let P be a surface in $\mathbb{R}_1^3$ and let $\alpha : I \longrightarrow P$ be a parametrized curve. α is called an integral curve of X if

$$\frac{d}{ds}(\alpha(s)) = X(\alpha(s)), \ (\text{for all } t \in I),$$

where X is a smooth tangent vector field on P [16]. We have

$$TP = \bigcup_{p \in P} T_p P = \chi(P),$$

where $T_p P$ is the tangent space of P at p and $\chi(P)$ is the space of tangent vector fields on P.

For any parametrized curve $\alpha : I \longrightarrow P$, $\bar{\alpha} : I \longrightarrow TP$ is given by

$$\bar{\alpha}(s) = (\alpha(s), \alpha'(s)) = \alpha'(s)|_{\alpha(s)}$$

is called the natural lift of α on TP [20].
Let $\alpha(s)$, $L_1 \leq s \leq L_2$, be an arc length spacelike curve with spacelike binormal. Then, the natural lift $\bar{\alpha}$ of α is a timelike curve . We have following relations between the Frenet frame $\{T(s), N(s), B(s)\}$ of α and the Frenet frame $\{\bar{T}(s), \bar{N}(s), \bar{B}(s)\}$ of $\bar{\alpha}$. We have the following relationship for the Frenet frames of the curve α and its natural lift $\bar{\alpha}$

$$\begin{pmatrix} \bar{T}(s) \\ \bar{N}(s) \\ \bar{B}(s) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \cos\theta & 0 & \sin\theta \\ \sin\theta & 0 & -\cos\theta \end{pmatrix} \begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix}. \tag{1}$$

2. Surface family with a common natural asymptotic or geodesic lift of a spacelike curve with spacelike binormal in Minkowski 3-space

Suppose we are given a 3-dimensional spacelike curve with spacelike binormal $\alpha(s)$, $L_1 \leq s \leq L_2$, in which s is the arc length and $\|\alpha''(s)\| \neq 0$, $L_1 \leq s \leq L_2$. Let $\bar{\alpha}(s)$, $L_1 \leq s \leq L_2$, be the natural lift of the given curve $\alpha(s)$. Now, $\bar{\alpha}$ is a timelike curve.
Surface family that interpolates $\bar{\alpha}(s)$ as a common curve is given in the parametric form as

$$P(s,t) = \bar{\alpha}(s) + u(s,t)\bar{T}(s) + v(s,t)\bar{N}(s) + w(s,t)\bar{B}(s), \tag{2.1}$$

$L_1 \leq s \leq L_2$, $T_1 \leq t \leq T_2$ where $u(s,t)$, $v(s,t)$ and $w(s,t)$ are C^1 functions, called *marching-scale functions,* and $\{\bar{T}(s), \bar{N}(s), \bar{B}(s)\}$ is the Frenet frame of the curve $\bar{\alpha}$.

Remark 2.1. *Observe that choosing different marching-scale functions yields different surfaces possessing $\bar{\alpha}(s)$ as a common curve.*

Our goal is to find the necessary and sufficient conditions for which the curve $\bar{\alpha}(s)$ is isoparametric and asymptotic curve (or geodesic) on the surface $P(s,t)$. Firstly, as $\bar{\alpha}(s)$ is an isoparametric curve on the surface $P(s,t)$, there exists a parameter $t_0 \in [T_1, T_2]$ such that

$$u(s,t_0) = v(s,t_0) = w(s,t_0) \equiv 0, \ L_1 \leq s \leq L_2, \ T_1 \leq t_0 \leq T_2. \tag{2.2}$$

The normal vector of $P(s,t)$ can be written as

$$n(s,t) = \frac{\partial P(s,t)}{\partial s} \times \frac{\partial P(s,t)}{\partial t}.$$

Along the curve $\bar{\alpha}$, one can obtain the normal vector $n(s,t_0)$ using Eqns. $(1, 2.1, 2.2)$ as

$$n(s,t_0) = \kappa \left[\frac{\partial w}{\partial t}(s,t_0)\bar{N}(s) - \frac{\partial v}{\partial t}(s,t_0)\bar{B}(s) \right], \tag{2.3}$$

where κ is the curvature of the curve α.

Theorem 2.2. *Let $\alpha(s)$, $L_1 \leq s \leq L_2$, be a unit speed spacelike curve with spacelike binormal with nonvanishing curvature and $\bar{\alpha}(s)$, $L_1 \leq s \leq L_2$, be its natural lift. $\bar{\alpha}$ is an asymptotic curve on the surface (2.1) if and only if*

$$\begin{cases} u(s,t_0) = v(s,t_0) = w(s,t_0) = \frac{\partial w}{\partial t}(s,t_0) \equiv 0, \\ \qquad \frac{\partial v}{\partial t}(s,t_0) \neq 0, \end{cases} \tag{2.4}$$

where $L_1 \leq s \leq L_2$, $T_1 \leq t$, $t_0 \leq T_2$ (t_0 fixed).

Corollary 2.3. *Let $\alpha(s)$, $L_1 \leq s \leq L_2$, be a unit speed spacelike curve with spacelike binormal with nonvanishing curvature and $\bar{\alpha}(s)$, $L_1 \leq s \leq L_2$, be its natural lift. If*

$$\begin{cases} u(s,t) = v(s,t) = t - t_0, \ w(s,t) \equiv 0 \\ \qquad or \\ u(s,t) = w(s,t) \equiv 0, \ v(s,t) = t - t_0, \end{cases} \tag{2.5}$$

where $L_1 \leq s \leq L_2$, $T_1 \leq t$, $t_0 \leq T_2$ (t_0 fixed) then (2.1) is a ruled surface possessing $\bar{\alpha}$ as an asymptotic curve.

Proof. By taking marching scale functions as $u(s,t) = v(s,t) = t - t_0$, $w(s,t) \equiv 0$ or $u(s,t) = w(s,t) \equiv 0$, $v(s,t) = t - t_0$, the surface (2.1) takes the form

$$\begin{cases} P(s,t) = \bar{\alpha}(s) + (t - t_0)[\bar{T}(s) + \bar{N}(s)] \\ \qquad\qquad or \\ \quad P(s,t) = \bar{\alpha}(s) + (t - t_0)\bar{N}(s), \end{cases}$$

which is a ruled surface satisfying Eqn. (2.4). □

Theorem 2.4. *Let $\alpha(s)$, $L_1 \leq s \leq L_2$, be a unit speed spacelike curve with spacelike binormal with nonvanishing curvature and $\bar{\alpha}(s)$, $L_1 \leq s \leq L_2$, be its natural lift. $\bar{\alpha}$ is a geodesic on the surface (2.1) if and only if*

$$\begin{cases} u(s,t_0) = v(s,t_0) = w(s,t_0) = \frac{\partial v}{\partial t}(s,t_0) \equiv 0, \\ \qquad\qquad \frac{\partial w}{\partial t}(s,t_0) \neq 0, \end{cases} \tag{2.6}$$

where $L_1 \leq s \leq L_2$, $T_1 \leq t$, $t_0 \leq T_2$ (t_0 fixed).

Corollary 2.5. *Let $\alpha(s)$, $L_1 \leq s \leq L_2$, be a unit speed spacelike curve with spacelike binormal with nonvanishing curvature and $\bar{\alpha}(s)$, $L_1 \leq s \leq L_2$, be its natural lift. If*

$$\begin{cases} u(s,t) = w(s,t) = t - t_0, \ v(s,t) \equiv 0 \\ \qquad\qquad or \\ u(s,t) = v(s,t) \equiv 0, \ w(s,t) = t - t_0, \end{cases} \tag{2.7}$$

where $L_1 \leq s \leq L_2$, $T_1 \leq t$, $t_0 \leq T_2$ (t_0 fixed) then (2.1) is a ruled surface possessing $\bar{\alpha}$ as a geodesic.

Proof. By taking marching scale functions as $u(s,t) = w(s,t) = t - t_0$, $v(s,t) \equiv 0$ or $u(s,t) = v(s,t) \equiv 0$, $w(s,t) = t - t_0$, the surface (2.1) takes the form

$$\begin{cases} P(s,t) = \bar{\alpha}(s) + (t - t_0)[\bar{T}(s) + \bar{B}(s)] \\ \qquad\qquad or \\ \quad P(s,t) = \bar{\alpha}(s) + (t - t_0)\bar{B}(s), \end{cases}$$

which is a ruled surface satisfying Eqn. (2.6). □

3. Examples

3.1. Example 1

Let $\alpha(s) = (\cosh s, 0, \sinh s)$ be a spacelike curve with spacelike binormal. It is easy to show that

$$\begin{aligned} T(s) &= (\sinh s, 0, \cosh s), \\ N(s) &= (\cosh s, 0, \sinh s), \\ B(s) &= (0, 1, 0). \end{aligned}$$

The natural lift $\bar{\alpha}(s) = (\sinh s, 0, \cosh s)$ of the curve α is a timelike curve and its Frenet vectors

$$\begin{aligned} \bar{T}(s) &= (\cosh s, 0, \sinh s), \\ \bar{N}(s) &= (\sinh s, 0, \cosh s), \\ \bar{B}(s) &= (0, -1, 0). \end{aligned}$$

Choosing marching scale functions as $u(s,t) = t$, $v(s,t) = \sinh t$, $w(s,t) = t^2$ Eqn. (2.4) is satisfied and we obtain the surface

$$P_1(s,t) = \left(t \cosh s + (1 + \sinh t) \sinh s, -t^2, t \sinh s + (1 + \sinh t) \cosh s \right),$$

$-1 \leq s \leq 1$, $-1 \leq t \leq 0$, possessing $\bar{\alpha}$ as a common natural asymptotic lift (Fig. 1).
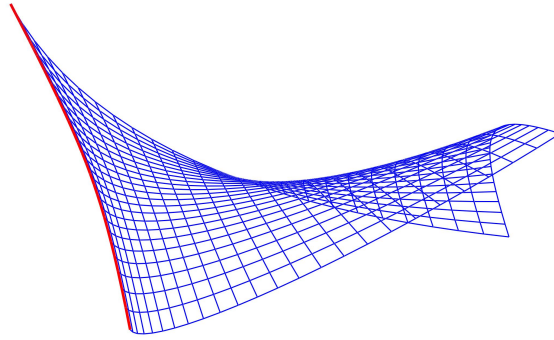
Fig. 1. $P_1(s,t)$ as a member of the surface family with a common natural asymptotic lift $\bar{\alpha}$.

Choosing marching scale functions as $u(s,t) = t$, $v(s,t) = t^2$, $w(s,t) = \sinh t$ Eqn. (2.6) is satisfied and we obtain the surface

$$P_2(s,t) = \left(t\cosh s + \left(1+t^2\right)\sinh s, -\sinh t, t\sinh s + \left(1+t^2\right)\cosh s \right),$$

$-1 \le s \le 1$, $-1 \le t \le 0$, possessing $\bar{\alpha}$ as a common natural geodesic lift (Fig. 2).
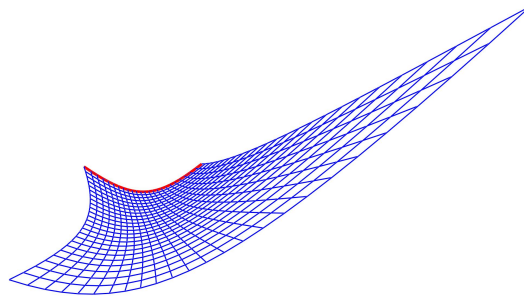

Fig. 2. $P_2(s,t)$ as a member of the surface family with a common natural geodesic lift $\bar{\alpha}$.

3.2. Example 2

Given the arclength spacelike curve with spacelike binormal $\alpha(s) = \left(\frac{1}{2}\cosh\sqrt{2}s, \frac{\sqrt{2}}{2}s, \frac{1}{2}\sinh\sqrt{2}s \right)$ its Frenet apparatus are

$$
\begin{aligned}
T(s) &= \left(\frac{\sqrt{2}}{2}\sinh\sqrt{2}s, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\cosh\sqrt{2}s \right), \\
N(s) &= \left(\cosh\sqrt{2}s, 0, \sinh\sqrt{2}s \right), \\
B(s) &= \left(-\frac{\sqrt{2}}{2}\sinh\sqrt{2}s, \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\cosh\sqrt{2}s \right).
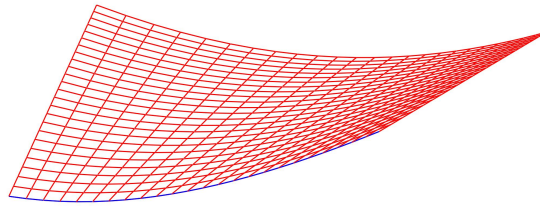\end{aligned}
$$

The natural lift of the curve α is $\bar{\alpha}(s) = \left(\frac{\sqrt{2}}{2}\sinh\sqrt{2}s, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\cosh\sqrt{2}s \right)$ and its Frenet vectors

$$
\begin{aligned}
\bar{T}(s) &= \left(\cosh\sqrt{2}s, 0, \sinh\sqrt{2}s \right), \\
\bar{N}(s) &= \left(\sinh\sqrt{2}s, 0, \cosh\sqrt{2}s \right), \\
\bar{B}(s) &= (0,-1,0).
\end{aligned}
$$

If we let marching scale functions as $u(s,t) = w(s,t) \equiv 0$, $v(s,t) = t$, we get the ruled surface

$$P_3(s,t) = \left(\left(\frac{\sqrt{2}}{2}+t \right)\sinh\sqrt{2}s, \frac{\sqrt{2}}{2}, \left(\frac{\sqrt{2}}{2}+t \right)\cosh\sqrt{2}s \right),$$

$0 \le s \le 1$, $-1 \le t \le 1$, satisfying Eqn. (2.5) and passing through $\bar{\alpha}$ as a common natural asymptotic lift (Fig. 3).
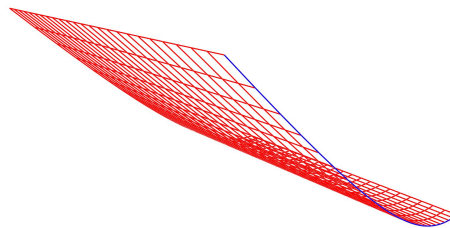
Fig. 3. Ruled surface $P_3(s,t)$ as a member of the surface family with a common natural asymptotic lift $\bar{\alpha}$.

If we let marching scale functions as $u(s,t) = v(s,t) \equiv 0$, $w(s,t) = t$, we get the ruled surface

$$P_4(s,t) = \left(\frac{\sqrt{2}}{2} \sinh \sqrt{2}s, \frac{\sqrt{2}}{2} - t, \frac{\sqrt{2}}{2} \cosh \sqrt{2}s \right),$$

$0 \le s \le 1$, $-1 \le t \le 1$, satisfying Eqn. (2.7) and passing through $\bar{\alpha}$ as a common natural geodesic lift (Fig. 4).

Fig. 4. Ruled surface $P_4(s,t)$ as a member of the surface family with a common natural geodesic lift $\bar{\alpha}$.

Acknowledgement

References

[1] G. J. Wang, K. Tang and C. L. Tai, Parametric representantion of a surface pencil with a common spatial geodesic, Comput. Aided Des., 36, (2004), 447–459.
[2] C. Y. Li, R. H. Wang and C. G. Zhu, Parametric representation of a surface pencil with a common line of curvature, Comput. Aided Des., 43 (2011), 1110–1117.
[3] E. Bayram and E. Kasap, Hypersurface family with a common isoasymptotic curve, Geometry, No. 2014 (2014).
[4] E. Ergün, E. Bayram and E. Kasap, Surface pencil with a common line of curvature in Minkowski 3-space, Acta Mathematica Sinica-English Series, 30, 12 (2014), 2103-2118.
[5] Ergün, E., Bayram, E., and Kasap, E. Surface family with a common natural line of curvature lift. Journal of Science and Arts, no. 4(31) (2015).
[6] E. Ergün, E. Bayram and E. Kasap, Surface family with a common natural asymptotic lift, Journal of Science and Arts, no. 2(31) (2015), 117-124.
[7] E. Bayram and M. Bilici, Surface family with a common involute asymptotic curve, International Geometric Methods in Modern Physics, 13 (2016), 1650062.
[8] G. Şaffak Atalay and E. Kasap, Surfaces with common null asymptotic, Applied Mathematics and Computation, no. 260 (2015), 135-139.
[9] E. Kasap andF. T. Akyıldız, Surfaces with common geodesic in Minkowski 3-space, Appl. Math. Comput., 177 (2008), 260–270.
[10] G. Şaffak and E. Kasap, Family of surface with a common null geodesic, Int. J. Phys. Sci., Vol 4 8 (2009), 428–433.
[11] E. Bayram, E. Ergün and E. Kasap, Surface family with a common natural asymptotic lift, Journal of Science and Arts, No. 2(31) (2015), 117-124.
[12] E. Ergün and E. Bayram, Surface family with a common natural geodesic lift, Int. J. Math. Combin., Vol 1 (2016), 34-41.
[13] S. Kahyaoğlu and E. Kasap, Timelike minimal surface family prescribed by a spacelike curve in Minkowski 3-space, International Electronic Journal of Pure and Applied Mathematics, 10, 1 (2016).
[14] S. Kahyaoğlu and E. Kasap, Spacelike maximal surface family prescribed by a spacelike curve in 3-dimensional Minkowski space, International Journal of Contemporary Mathematical Sciences, 11, 3 (2016).
[15] S. Kahyaoğlu S and E. Kasap, An approach for minimal surface family passing a curve, International Journal of Contemporary Mathematical Sciences, 10, 5 (2015).
[16] O'Neill, B., *Semi Riemannian geometry with applications to relativity*, Academic Press, Inc. New York, 1983.
[17] Ratcliffe, J. G., *Foundations of Hyperbolic Manifolds*, Springer-Verlag, New York, 1994.

[18] M. Önder and H. H. Uğurlu, Frenet frames and invariants of timelike ruled surfaces, Ain Shams Engineering Journal, 4 (2013), 502-513.
[19] Walrave J. 1995, Curves and surfaces in Minkowski space, PhD. Thesis, K. U. Leuven Faculteit Der Wetenschappen.
[20] E. Ergün and M. Çalışkan, On geodesic sprays In Minkowski 3-space, Int. J. Contem. Math. Sci., 6, 39 (2011), 1929-1933.