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# Monte Carlo and Quasi Monte Carlo Approach to Ulam's Method for Position Dependent Random Maps

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**Abstract**
We consider position random maps $T = \{\tau_1(x), \tau_2(x), \ldots, \tau_K(x); p_1(x), p_2(x), \ldots, p_K(x)\}$ on $I = [0,1]$, where $\tau_k, k = 1,2,\ldots,K$ is non-singular map on $[0,1]$ into $[0,1]$ and $\{p_1(x), p_2(x), \ldots, p_K(x)\}$ is a set of position dependent probabilities on $[0,1]$. We assume that the random map $T$ posses a density function $f^*$ of the unique absolutely continuous invariant measure (acim) $\mu^*$. In this paper, first, we present a general numerical algorithm for the approximation of the density function $f^*$. Moreover, we show that Ulam's method is a special case of the general method. Finally, we describe a Monte-Carlo and a Quasi Monte Carlo implementations of Ulam's method for the approximation of $f^*$. The main advantage of these methods is that we do not need to find the inverse images of subsets under the transformations of the random map $T$.

**Keywords:** Dynamical systems, Invariant measure, Invariant density, Monte Carlo approach, Position dependent random maps, Quasi Monte Carlo approach, Ulam's method.
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## 1. Introduction

A position dependent random map is a special type of random dynamical system involving a set of non-singular transformations on the state space and a set of position dependent probabilities on the state space. In each iteration of the process, one map from the set of maps with one position dependent probability from the set of probabilities [1] is selected and applied. There are applications of random maps in many areas of science and engineering [2]-[3]-[4]-[5]-[6]. In [2] the author applied the theory of random dynamical systems in the study of fractals. In [3], Boyarsky and Góra applied the theory of random dynamical systems in modelling interference effects in quantum mechanics. The authors in [4] applied random maps for computing metric entropy. Random maps have application in forecasting the financial markets [5] and in economics [6].

Invariant measures describe the statistical behaviour of trajectories of position dependent random maps [1]. In particular, invariant measures of random maps which are absolutely continuous with respect to Lebesgue measure are very useful for the study of chaotic nature of random dynamical systems [7]. The Frobenius-Perron operator [1, 8] of a random map is one of the important tools for the study of invariant measures. A Fixed point $f^*$ of the Frobenius-Perron operator of a position dependent random maps are the density function $f^*$ of invariant measures $\mu^*$ [1, 7]. It is difficult to solve the fixed point equation or the Frobenius–Perron equation [1] for a position dependent random map because it is a complicated functional equation except for some simple cases. Therefore, finite dimensional approximation of the Frobenius-Perron operator is necessary to approximate invariant measures for position dependent random maps. In [9], Lasota and Yorke proved the existence of absolutely continuous invariant measures (acims) for one dimensional deterministic dynamical systems. In his pioneering work[10], Ulam suggested finite dimensional approximation of the Frobenious Perron operator of dynamical systems for the approximation of invariant measures. It was T-Y Li who first proved in [11] the convergence of Ulam's approximation for piecewise expanding transformations $\tau$ on $[0,1]$. In [8], Pelikan proved a Lasota–Yorke type inequality random maps with i. i. d. probabilities using bounded variation techniques. Then, he used the Lasota-Yorke type inequality for proving the existence of absolutely continuous invariant measures for i. i. d. random maps. Góra and Boyarsky [1] proved the existence of absolutely continuous invariant measures (acim) for position dependent random maps. Moreover, they proved the convergence of Ulam's method for position dependent random maps.

Ulam's method is a simple, easy to implement and very useful method for approximating invariant measures for deterministic and random maps [1]-[14]. Note that each of the map $\tau_k, k=1,2,\dots,K$ of a position dependent random map $T=\{\tau_1,\tau_2,\cdots,\tau_K;p_1(x),p_2(x),\cdots,p_K(x)\}$ is a piecewise monotonic map on a finite partition $\mathscr{P}=\{I_1,I_2,\cdots,I_q\}$. The entries of an Ulam's matrix for a random map $T$ are related to inverse images of the transformations $\tau_k, k=1,2,\dots,K$. For non-linear $\tau_k, k=1,2,\dots,K$, it is difficult to find inverse images under $\tau_k, k=1,2,\dots,K$, and hence the computation of Ulam's matrix becomes challenging and complicated. In this paper, we describe a Monte Carlo method and a Quasi Monte Carlo approach to Ulam's method for approximating the entries of Ulam's matrix. The main advantage of Monte-Carlo method and Quasi Monte Carlo approach to Ulam's method is that we do not need to find the inverse images of subsets under the transformations of the random map $T$. Moreover, the evaluation of an entry of the Ulam's matrix is independent of their entries [12].

## 2. Invariant Measures for Position Dependent Random Maps and Ulam's Method

In this section, we review position dependent random maps, the Frobenius-Perron operator, density function of absolutely continuous invariant measures and Ulam's method. We closely follow [1, 13, 14].

### 2.1 Position dependent random maps and their invariant measures

Let $(I=[0,1],\mathscr{B},\lambda)$ be a measure space and $\tau_k:[0,1]\to[0,1], k=1,2,\cdots,K$, be piecewise one-to-one and differentiable, non-singular maps on a common partition $\mathscr{I}=\{I_1,I_2,\cdots,I_q\}$ of $[0,1]$. We denote $V(.)$ for the standard one dimensional variation of a function, and $BV([0,1])$ for the space of functions of bounded variation on $I$ equipped with the norm $\|\,.\,\|_{BV}=V(.)+\|\,.\,\|_1$, where $\|\,.\,\|_1$ denotes the $L^1$ norm of a function. A position dependent random map $T$ on $I$ with position dependent probabilities is defined as

$$T=\{\tau_1,\tau_2,\cdots,\tau_K;p_1(x),p_2(x),\cdots,p_K(x)\}$$

where $\{p_1(x),p_2(x),\cdots,p_K(x)\}$ is a set of position dependent probabilities on $I$. For any $x\in I, T(x)=\tau_k(x)$ with probability $p_k(x)$ and, for any non-negative integer $N, T^N(x)=\tau_{k_N}\circ\tau_{k_{N-1}}\circ\cdots\circ\tau_{k_1}(x)$ with probability $p_{k_N}(\tau_{k_{N-1}}\circ\cdots\circ\tau_{k_1}(x))p_{k_{N-1}}(\tau_{k_{N-2}}\circ\cdots\circ\tau_{k_1}(x))\dots p_{k_1}(x)$. It is shown in [1] that a measure $\mu$ is invariant under the

$$\mu(A)=\sum_{k=1}^{K}\int_{\tau_k^{-1}(A)}p_k(x)d\mu(x) \tag{2.1}$$

for any $A\in\mathscr{B}$.

The Frobenius–Perron operator of the position dependent random map $T$ is given by [1]:

$$(P_T f)(x)=\sum_{k=1}^{K}\left(P_{\tau_k}(p_k f)\right)(x) \tag{2.2}$$

where $P_{\tau_k}$ in (2.2) is the Frobenius-Perron operator of $\tau_k$ [14] defined by

$$P_{\tau_k}f(x)=\sum_{z\in\{\tau_k^{-1}(x)\}}\frac{f(z)}{|\tau_k'(z)|} \tag{2.3}$$

where, for any $x$, the set $\{\tau_k^{-1}(x)\}$ consists of at most $q$ points. The Frobenius-Perron operator $P_T$ has the following properties

(i) $P_T : L^1([0,1]) \to L^1([0,1])$ is a linear operator;
(ii) $P_T$ is non-negative, i.e., $f \in L^1([0,1])$ and $f \geq 0 => P_T f \geq 0$;
(iii) $P_T$ is a contractive, i.e., $\| P_T f \|_1 \leq \| f \|_1$, for any $f \in L^1([0,1])$;
(iv) $P_T$ satisfies the composition property, i.e., if $T$ and $R$ are two position dependent random maps on $[0,1]$, then $P_{T\circ R} = P_T \circ P_R$. In particular, for any $n \geq 1, P_T^n = P_{T^n}$;
(v) $P_T f = f$ if and only if $\mu = f \cdot \lambda$ is T-invariant.

The following Lemmas (Lemma 2.1 and Lemma 2.2) are key Lemmas for proving the existence of invariant measures for position dependent random maps. These Lemmas are proved by Bahsoun and Góra in[13].

**Lemma 2.1.** *[13] Consider the position dependent random maps $T = \{\tau_1, \tau_2, \ldots, \tau_K; p_1(x), p_2(x), ..., p_K(x)\}$, where $\tau_k : [0,1] \to [0,1], k = 1,2,...,K$ are piecewise one-to-one and differentiable, nonsingular maps on a common partition $\mathscr{J} = \{J_1, J_2, ......, J_q\}$ of $[0,1]$. Let $g_k(x) = \frac{p_k(x)}{|\tau_k'(x)|}, k = 1,2,...,K$. Assume that the random map $T$ satisfies the following conditions: (i) $\sum_{k=1}^{K} g_k(x) < \alpha < 1, x \in [0,1]$; (ii) $g_k \in BV([0,1]), k = 1,2,...,K$. Then, for any $f \in BV([0,1])$, $P_T$ satisfies the following Lasota-Yorke type inequality:*

$$V_{[0,1]} P_T f \leq A V_{[0,1]} f + B \| f \|_1 \tag{2.4}$$

*where* $A = 3\alpha + \max_{1\leq i\leq q} \sum_{k=1}^{K} V_{J_i} g_k$ *and* $B = 2\beta\alpha + \beta \max_{1\leq i\leq q} \sum_{k=1}^{K} V_{J_i} g_k$ *with* $\beta = \max_{1\leq i\leq q} \frac{1}{\lambda(J_i)}$.

*Proof.* See [13] □

Note that for $x \in [0,1]$ and for any $N \geq 1$ we have, $T^N(x) = \tau_{k_N} \circ \tau_{k_{N-1}} \circ \cdots \circ \tau_{k_1}(x)$ with probability

$$p_{k_N}(\tau_{k_{N-1}} \circ \cdots \circ \tau_{k_1}(x)) p_{k_{N-1}}(\tau_{k_{N-2}} \circ \cdots \circ \tau_{k_1}(x)) \ldots p_{k_1}(x).$$

For $\omega \in \{1,2,\ldots,K\}^N$, define

$$\begin{aligned} T_\omega(x) &= T^N(x), \\ p_\omega &= p_{k_N}(\tau_{k_{N-1}} \circ \cdots \circ \tau_{k_1}(x)) p_{k_{N-1}}(\tau_{k_{N-2}} \circ \cdots \circ \tau_{k_1}(x)) \ldots p_{k_1}(x), \\ g_\omega &= \frac{p_\omega}{|T_\omega'(x)|}, W_N = \max_{L\in \mathscr{J}^{(N)}} \sum_{\omega\in\{1,2,\ldots,K\}^N} V_L g_\omega. \end{aligned}$$

Based on Lemma 2.1, Bahsoun and Góra [13] have proved the following Lemma for the iterates of $P_T$:

**Lemma 2.2.** *Let $T$ be a random map satisfying conditions of Lemma 2.1 and $N$ be a positive integer such that $A_N = 3\alpha^N + W_N < 1$. Then*

$$V_{[0,1]} P_T^N f \leq A_N V_{[0,1]} f + B_N \| f \|_1 \tag{2.5}$$

*where* $B_N = \beta_N \left(2\alpha^N + W_N\right), \beta_N = \max_{L\in\mathscr{J}^{(N)}} \frac{1}{\lambda(L)}$.

In the following Theorem (Theorem 2.3), Bahsoun and Góra proved the existence of invariant measures for position dependent random maps. The proof of this Theorem is based on the above Lemmas (Lemma 2.1 and Lemma 2.2) which is proved in [13].

**Theorem 2.3.** *[13] Consider the position dependent random map $T = \{\tau_1, \tau_2, \ldots, \tau_K; p_1(x), p_2(x), \ldots, p_K\}$. Assume that the random map $T$ satisfies conditions of Lemma 2.1. Then, $T$ possesses an invariant measure which is absolutely continuous with respect to Lebesgue measure. Moreover, the operator $P_T$ is quasi-compact in $BV(I)$.*

### 2.2 Ulam's Method for Position Dependent Random Maps

In [1] Góra and Boyarsky described Ulam's method for position dependent random maps. Moreover, they proved the convergence of Ulam's method. For the convenience of readers, we review the Ulam's method for position dependent random maps. Let $T=\{\tau_1(x),\tau_2(x),\dots,\tau_K(x);p_1(x),p_2(x),\dots,p_K(x)\}$ be a position dependent random map and the random map $T$ satisfies conditions of Theorem 2.3. Then, by the Theorem 2.3, the random map $T$ has an absolutely continuous invariant measure. We also assume that the random map has a unique acim $\mu^*$ with density function $f^*$. In the following we describe Ulam's method for $T$.

Consider the partition $\mathscr{P}^{(N)}=\{J_1,J_2,\dots,J_N\}$ of $[0,1]$ into $N$ subintervals such that $\max_{J_i\in\mathscr{P}^{(N)}}\lambda(J_i)$ goes to 0 as $N\to\infty$. For each $1\le k\le K$, construct the matrix

$$M_k^{(N)}=\left(\frac{\lambda\left(\tau_k^{-1}(J_j)\cap J_i\right)}{\lambda(J_i)}\right)_{1\le i,j\le N}.$$

Let $L^{(N)}$ be the set of functions $f$ in $L^1([0,1],\lambda)$ such that $f$ is constant on elements of the partition $\mathscr{P}^{(N)}$. Any $f\in L^{(n)}$ can be treated as a vector: vector $f=[f_1,f_2,\dots,f_N]$ corresponds to the function $f=\sum_{i=1}^N f_i\chi_{J_i}$. Let $Q^{(N)}$ be the isometric projection of $L^1$ onto $L^{(N)}$:

$$Q^{(N)}(f)=\sum_{i=1}^{N}\left(\frac{1}{\lambda(J_i)}\int_{J_i}fd\lambda\right)\chi_{J_i}=\left[\frac{1}{\lambda(J_1)}\int_{J_1}fd\lambda,\dots,\frac{1}{\lambda(J_N)}\int_{J_N}fd\lambda\right].$$

Let $p_k^{(N)}=Q^{(N)}p_k=\left[p_{k,1}^{(n)},p_{k,2}^{(N)},\dots,p_{k,N}^{(n)}\right]$. Let $f=[f_1,f_2,\dots,f_N]\in L^{(N)}$. Let the subscript **c** denotes the transpose of a matrix. We define the operator $P_T^{(N)}:L^{(N)}\to L^{(N)}$ by

$$P_T^{(n)}f \quad = \quad \sum_{k=1}^{K}\left(\mathbb{M}_k^{(n)}\right)^{\mathrm{c}}\mathrm{diag}\left(\left[p_{k,1}^{(N)}f_1,p_{k,2}^{(N)}f_2,\dots,p_{k,N}^{(N)}f_N\right]\right) \tag{2.6}$$

as a finite dimensional approximation to the operator $P_T$. Ulam's matrix with respect to the partition $\mathscr{P}^{(N)}$ is

$$\mathbb{M}^{*(N)}_{\mathscr{P}^{(N)}}=\sum_{k=1}^{K}\left(\mathbb{M}_k^{(N)}\right)^{\mathrm{c}}\mathrm{diag}\left[p_{k,1}^{(N)},p_{k,2}^{(N)},\dots,p_{k,N}^{(N)}\right]. \tag{2.7}$$

The following theorem is proved in[1] (see Theorem 3 in [1]).

**Theorem 2.4.** *Let $\alpha$ be sufficiently large where $\alpha$ is in Theorem 1 in [1]. Let $f_N^*$ be is a normalized fixed point of $P_T^{(N)}, N=1,2,\dots$. Then the sequence $\{f_N^*\}_{N=1}^\infty$ is pre-compact in $L^1$. Any limit point $f^*$ of the sequence $\{f_N^*\}_{N=1}^\infty$ is a fixed point of $P_T$.*

## 3. A General Algorithm for Finite Dimensional Approximation of the Frobenius-Perron Operator for Position Dependent Random Maps

Let $T=\{\tau_1,\tau_2,\dots,\tau_K;p_1(x),p_2(x),\dots,p_K(x)\}$ be a position dependent random map which satisfies the following assumptions:

there exists $A=3\alpha+\max\limits_{1\le i\le q}\sum_{k=1}^{K}V_{J_i}g_k<1$ and $B=2\beta\alpha+\beta\max\limits_{1\le i\le q}\sum_{k=1}^{K}V_{J_i}g_k>0$ with $\beta=\max\limits_{1\le i\le q}\frac{1}{\lambda(I_i)}$ such that $\forall f\in BV([0,1])$.

$$V_{[0,1]}P_Tf\le AV_{[0,1]}f+B\parallel f\parallel_1 . \tag{3.1}$$

We also assume that $T$ has a unique acim $\mu^*$ with density $f^*$.

Note that the invariant density $f^*$ of the unique acim $\mu^*$ is the fixed point of the Frobenius-Perron operator $P_T$. In the following we describe a general approximation algorithm for $f^*$. Our general algorithm is a generalization of the algorithm in [12] for single deterministic map to an algorithm for position dependent random maps.

For each $k=1,2,\dots,K$, let $U_{\tau_k}:L^\infty([0,1])\to L^\infty([0,1])$ be the Koopman operator of $\tau_k$ defined by

$$(U_{\tau_k}g)(x)=g(\tau_k(x)). \tag{3.2}$$

Note that each $U_{\tau_k}$ is the dual of the Frobenius-Perron operator $P_{\tau_k}$ of $\tau_k$.

**Definition 3.1.** *A sequence* $\{\phi_n\}_{n=1}^{\infty}$ *of functions in* $L^{\infty}([0,1])$ *is said to be a complete sequence if for any* $f \in L^1(0,1)$ *with* $\int_0^1 \phi_n(x) f(x) d\lambda(x) = 0,\ n = 1,2,\cdots$ *implies* $f = 0$.

**Proposition 3.2.** *Let* $T = \{\tau_1, \tau_2, \cdots, \tau_K; p_1(x), p_2(x), \ldots, p_K\}$ *be a position dependent random map which has a unique acim* $\mu^*$ *with density* $f^*$. *Let* $P_T$ *be the Frobenius-Perron operator of the random map* $T$. *Let* $\{\phi_n\}_{n=1}^{\infty}$ *be a complete sequence of functions. Then,* $f^*$ *is a fixed point of* $P_T$ *if and only if*

$$\int_I \left[\phi_n(x) - \sum_{k=1}^{K} p_k(x)\phi_n(\tau_k(x))\right] f^*(x) d\lambda(x) = 0,\ n = 1,2,\cdots. \tag{3.3}$$

*Proof.* Suppose that $f^*$ the unique invariant density of the random maps $T$. In other words,

$$(P_T f^*)(x) = f^*(x). \tag{3.4}$$

Then for $n = 1,2,\cdots$,

$$\begin{aligned}
\int_I f^*(x)\phi_n(x) d\lambda(x) &= \int_I (P_T f^*)(x)\phi_n(x) d\lambda(x) \\
&= \int_I \sum_{k=1}^{K} (P_{\tau_k}(p_k f^*))(x)\phi_n(x) d\lambda(x) \\
&= \sum_{k=1}^{K} \int_I (P_{\tau_k}(p_k f^*))(x)\phi_n(x) d\lambda(x) \\
&= \sum_{k=1}^{K} \int_I (p_k f^*)(x) U_{\tau_k}(\phi_n(x)) d\lambda(x) \\
&= \int_I f^*(x) \left[\sum_{k=1}^{K} p_k(x)\phi_n(\tau_k(x))\right] d\lambda(x).
\end{aligned}$$

Thus,

$$\int_I \left[\phi_n(x) - \sum_{k=1}^{K} p_k(x)\phi_n(\tau_k(x))\right] f^*(x) d\lambda(x) = 0,\ n = 1,2,\cdots.$$

Conversely, suppose that $f^*$ satisfies (3.3), that is,

$$\int_I \phi_n(x) f^*(x) d\lambda(x) = \int_I f^*(x) \sum_{k=1}^{K} p_k(x)\phi_n(\tau_k(x)) d\lambda(x).$$

Now,

$$\begin{aligned}
\int_I f^*(x)\phi_n(x) d\lambda(x) &= \int_I f^*(x) \sum_{k=1}^{K} p_k(x)\phi_n(\tau_k(x)) d\lambda(x) \\
&= \int_I f^*(x) \sum_{k=1}^{K} p_k(x) U_{\tau_k}(\phi_n(x)) d\lambda(x) \\
&= \sum_{k=1}^{K} \int_I f^*(x) p_k(x) U_{\tau_k}(\phi_n(x)) d\lambda(x) \\
&= \sum_{k=1}^{K} \int_I (P_{\tau_k}(p_k f^*))(x)\phi_n(x) d\lambda(x) \\
&= \int_I \sum_{k=1}^{K} (P_{\tau_k}(p_k f^*))(x)\phi_n(x) d\lambda(x) \\
&= \int_I (P_T f^*)(x)\phi_n(x) d\lambda(x).
\end{aligned}$$

Thus,

$$\int_I (f^*(x) - (P_T f^*)(x))\phi_n(x)d\lambda(x) = 0,\ n = 1,2,\cdots.$$

From Definition 3.1, $f^*(x) - (P_T f^*)(x) = 0$. This proves that

$$(P_T f^*)(x) = f^*(x).$$

□

Thus, the fixed point problem (3.4) of the Frobenius-Perron operator $P_T$ for the position dependent random map T is equivalent to homogeneous moment problem (3.3). We propose the following general algorithm for computing fixed point of $P_T$.

**General Algorithm**: Consider two complete sequences of functions $\phi_n$ and $\psi_n$. Let $N$ be a positive integer. Construct the $N \times N$ matrix $A = (a_{ij})_{1 \le i,j \le N}$ given by

$$a_{ij} = \int_0^1 \left( \phi_i(x) - \sum_{k=1}^{K} p_k(x)\phi_i(\tau_k(x)) \right) \psi_j(x)d\lambda(x), i,j = 1,2,\ldots,N. \tag{3.5}$$

Solve the homogeneous linear system of equation $Av = 0$ for nonzero $v = (v_1, v_2, \ldots, v_N)$ with $\| \sum_{i=1}^N v_i \psi_i \|_{L^1} = 1$. Then, $f_N = \sum_{i=1}^N v_i \psi_i$ is a normalized approximation of the fixed point $f^*$ of $P_T$.

**Lemma 3.3.** *$Av = 0$ has a nontrivial solution $v$.*

*Proof.* For a nonzero vector $\eta = (\eta_1, \eta_2, \ldots, \eta_N)$, the constant function $g(x) = 1$ can be written as $g(x) = 1 = \sum_{i=1}^N \eta_i \phi_i$. Moreover, for each $k = 1,2,\ldots,K$, $U_{\tau_k} 1(x) = 1(\tau_k(x)) = 1$. For each $j = 1,2,\ldots,N$,

$$\begin{aligned}
\sum_{i=1}^{N} a_{ij}\eta_i &= \sum_{i=1}^{N} \eta_i \int_0^1 \left( \phi_i(x) - \sum_{k=1}^{K} p_k(x)\phi_i(\tau_k(x)) \right) \psi_j(x)d\lambda(x) \\
&= \int_0^1 \left( \sum_{i=1}^{N} \eta_i\phi_i(x) - \sum_{k=1}^{K} p_k(x)U_{\tau_k}(\sum_{i=1}^{N} \eta_i\phi_i(x)) \right) \psi_j(x)d\lambda(x) \\
&= \int_0^1 \left( 1 - \sum_{k=1}^{K} p_k(x)U_{\tau_k}(1(x)) \right) \psi_j(x)d\lambda(x) \\
&= \int_0^1 \left( 1 - \sum_{k=1}^{K} p_k(x)1 \right) \psi_j(x)d\lambda(x) \\
&= \int_0^1 (1-1)\, \psi_j(x)d\lambda(x) \\
&= 0.
\end{aligned}$$

Thus, $A^c\eta = 0$, where $A^c$ is the transpose of $A$. Thus, $A$ is singular. □

**Remark 3.4.** *The main purpose of the above general algorithm is to find a normalized function $f \in$ span $\{\psi_1, \psi_2, \ldots, \psi_{\rm N}\}$ such that*

$$\int_I \left[ \phi_n(x) - \sum_{k=1}^{K} p_k(x)\phi_n(\tau_k(x)) \right] f(x)d\lambda(x) = 0,\ n = 1,2,\cdots.$$

Let $N$ be a positive integer. Divide the interval $I = [0,1]$ into $N$ subintervals $J_i = [\frac{i-1}{N}, \frac{i}{N}], i = 1,2,\ldots,N$. Let $\lambda$ be the Lebesgue measure on $I$. For each $j = 1,2,\ldots,N$, let $\chi_{J_i}$ be the characteristic function on $J_i$. As before, Let $L^{(N)}$ be the subspace of $L^1([0,1])$ consisting of functions which are piecewise constant on the subinterval $J_i, i = 1,2,\ldots,N$.
For each $i = 1,2,\ldots,N$, let

$$\psi := \mathbf{1}_i = N\chi_{J_i},\ \ \phi_i = \chi_{J_i}.$$

Then, $\{\psi_i\}_{i=1}^N$ is a density basis of $L^{(N)}$. Thus, $f = \sum_{i=1}^N v_i \psi_i$ is a density if and only if $v \geq 0$ and $\| v \|_1 = \sum_{i=1}^N |v_i| = 1$. In the following, we show that $(i, j)$ element of the matrix $A$ in the above general algorithm is the $(i, j)$ element of the Ulam's matrix described in the previous section.

$$\begin{aligned}
a_{ij} &= \int_0^1 \left( \phi_i(x) - \sum_{k=1}^K p_k(x)\phi_i(\tau_k(x)) \right) \psi_j(x) d\lambda(x) \\
&= \int_0^1 \left( \chi_{J_i}(x) - \sum_{k=1}^K p_k(x)\chi_{J_i}(\tau_k(x)) \right) \mathbf{1}_j(x) d\lambda(x) \\
&= \int_0^1 \chi_{J_i}(x)\mathbf{1}_j(x)\lambda(x) - \int_0^1 \sum_{k=1}^K p_k(x)\chi_{J_i}(\tau_k(x))\mathbf{1}_j(x) d\lambda(x) \\
&= \int_0^1 \chi_{J_i}(x)\mathbf{1}_j(x)\lambda(x) - \sum_{k=1}^K \int_0^1 p_k(x)\chi_{J_i}(\tau_k(x))\mathbf{1}_j(x) d\lambda(x) \\
&= N \int_{I_i \cap I_j} d\lambda(x) - \sum_{k=1}^K \int_0^1 p_k(x)\chi_{\tau_k^{-1}(J_i)}(x)\mathbf{1}_j(x) d\lambda(x) \\
&= \delta_{ij} - \sum_{k=1}^K \frac{\lambda(J_j \cap \tau_k^{-1}(J_i))}{\lambda(I_j)} \cdot p_{k,j}^{(N)},
\end{aligned}$$

where $p_{k,j}^{(N)}$ is the restriction of $Q^{(N)}(p_k(x))$ on $I_j$ for the isometric projection $Q^{(N)}$ of $L^1$ into $L^{(N)}$ defined in the previous section. Hence $Av = 0$ if and only if $v^c = v^c \mathbb{M}_N$, where $v^c$ is the transpose of $v$ and

$$\mathbb{M}_N = (m_{ij}), \;\; m_{ij} = \sum_{k=1}^K \frac{\lambda(J_j \cap \tau_k^{-1}(J_j))}{\lambda(I_i)} \cdot p_{k,i}^{(N)}. \tag{3.6}$$

$\mathbb{M}_N$ in Equation (3.6) is exactly the Ulam's matrix $\mathbb{M}_{\mathscr{P}^{(N)}}^{*(N)}$ for position dependent random maps $T$ described in Equation (2.7).

## 4. Monte Carlo and Quasi Monte Carlo approach to Ulam's Method for Position Dependent Random Maps

In this section, we present a generalization of Monte Carlo and Quasi Monte Carlo approach to Ulam's method described in [12] and [15] of single deterministic maps to Monte Carlo and Quasi Monte Carlo approach to Ulam's method for position dependent random maps.

### 4.1 Monte Carlo-Ulam approach to Ulam's method for position dependent random maps

Recall from Section 2.2, Ulam's matrix $\mathbb{M}_{\mathscr{P}^{(N)}}^{*(N)}$ for a position dependent random map

$$T = \{\tau_1(x), \tau_2(x), \dots, \tau_K(x); p_1(x), p_2(x), \dots, p_K(x)\}$$

with respect to the partition $\mathscr{P}^{(N)}$ is given by

$$\mathbb{M}_{\mathscr{P}^{(N)}}^{*(N)} = \sum_{k=1}^K \left( \mathbb{M}_k^{(N)} \right)^{\mathrm{c}} \operatorname{diag}\left[ p_{k,1}^{(N)}, p_{k,2}^{(N)}, \dots, p_{k,N}^{(N)} \right], \tag{4.1}$$

where for each $k = 1, 2, \dots, K$,

$$M_k^{(N)} = \left( \frac{\lambda\left(\tau_k^{-1}(J_j) \cap J_i\right)}{\lambda(J_i)} \right)_{1 \leq i,j \leq N}.$$

Computation of $\mathbb{M}_{\mathscr{P}^{(N)}}^{*(N)}$ involves computations of $K$ matrices $M_k^{(N)} = \left( \frac{\lambda\left(\tau_k^{-1}(J_j) \cap J_i\right)}{\lambda(J_i)} \right)_{1 \leq i,j \leq N}$ where inverse images of sets (intervals) under $\tau_k$ are necessary to compute. If $\tau_k, k = 1, 2, \dots K$ has a complicated formula, then in many cases inverse images of $tau_k$ are difficult to obtain and the computation of the Ulam's matrix becomes complicated and challenging. The Monte Carlo approach to Ulam's method simplifies the above difficulties and makes the numerical method more efficient. The Monte Carlo approach to Ulam's method allows us to approximate the entries of the matrices $M_k^{(N)}, k = 1, 2, \dots, K$. In the following we describe the Monte Carlo approach to Ulam's method:

1. Choose $N$ (a positive integer) and and consider the partition $\{J_1, J_2, \ldots, J_N\}$ of subintervals of equal lengths, where $J_i = [x_{i-1}, x_i], h = \lambda(J_i) = \frac{1}{N}, j = 1, 2, \ldots, N$.
2. for each $k = 1, 2, K$ do
   (a) Choose $M$ ($M$ is a positive integer, same $M$ for each $k$);
   (b) for $i = 1, 2, \ldots, N$ do
      i. Choose $M$ points $\{z_{i,1}, z_{i,2}, \ldots, z_{i,M}\}$ randomly from the interval $J_i$ with uniform distribution.
      ii. for $j = 1, 2, \ldots, N$ do
         A. Let $q_{ij}$ be the number of points $\{\tau_k(z_{i,1}), \tau_k(z_{i,2}), \ldots, \tau_k(z_{i,M})\}$ in $J_j$
         B. Let $\frac{q_{ij}}{M}$ be an approximation of the $(i, j)$-th entry of the matrix $M_k^{(N)}$
   (c) Compute $\left[p_{k,1}^{(N)}, p_{k,2}^{(N)}, \ldots, p_{k,N}^{(N)}\right]$
3. Compute the Ulam's matrix $\mathbb{M}^{*(N)}_{\mathscr{P}^{(N)}} = \sum_{k=1}^{K} \left(\mathbb{M}_k^{(N)}\right)^{c} \operatorname{diag}\left[p_{k,1}^{(N)}, p_{k,2}^{(N)}, \ldots, p_{k,N}^{(N)}\right]$.
4. Compute a eigenvector $v$ (a normalized eigenvecto) of $\mathbb{M}^{*(N)}_{\mathscr{P}^{(N)}}$ with eigen value 1.
5. Compute $f^{(N)} = \sum_{i=1}^{N} v_i \cdot \chi_{J_i}(x)$ as an approximation of the actual density function $f^*$ of the absolutely continuous invariant measure $\mu^*$ for the position dependent random map $T = \{\tau_1(x), \tau_2(x), \ldots, \tau_K(x); p_1(x), p_2(x), \ldots, p_K(x)\}$.

Note that the computation of $i$-th row of each matrix $M_k^{(N)}$ is independent of the computation of other rows. Therefore, for each $k = 1, 2, \ldots N$ one can use $p$ processors to calculate $l$ rows (here, $N = pl$).

### 4.2 Quasi Monte Carlo-Ulam Parallel Algorithm for Position Dependent Random Maps

In a Monte Carlo approach to Ulam's methods, $M$ points in each interval $J_i, i = 1, 2, \ldots, N$ are randomly chosen with uniform distribution. In a Quasi Monte Carlo method $M$ points $\{z_{i,1}, z_{i,2}, \ldots, z_{i,M}\}$ are chosen deterministically as follows:

$$z_{i,m} = x_{i-1} + \frac{m}{M} h, m = 1, 2, \ldots, N.$$

All other steps are similar to Monte Carlo Method in Section 4.1. This type of deterministic selections makes the numerical method more efficient as we will see the next section with examples.

## 5. Numerical Examples

In this section, we consider position dependent random maps $T$ satisfying conditions of Theorem 2.3 with unique invariant density $f^*$ and we apply Monte Carlo method and Quasi Monte Carlo approaches to Ulam's method described in the previous section. Moreover, we find the $L^1$ norms $\| f^* - f_N \|_1$, for some $N \geq 1$ where $f_N$ is an approximation of $f^*$. Monte Carlo and Quasi Monte Carlo approach to Ulam's method can be applied to any position dependent map satisfying conditions of Theorem 2.3. However, first we consider a simple position dependent random map $T$, where the density $f^*$ of the invariant measure $\mu^*$ is known. In the first example, the component maps of the position dependent random map $T$ are piecewise linear and Markov and the probabilities are position dependent piecewise constants. The main reason for the consideration of a such a simple position dependent random map is that the actual density is known in this case and we can compare our numerically approximate densities with the actual density. In the second example, we consider a position dependent random map where the component maps are non-Markov and the actual density is not known.

**Example 5.1.** *Consider the position dependent random map $T = \{\tau_1(x), \tau_2(x); p_1(x), p_2(x)\}$ where $\tau_1, \tau_2 : [0,1] \to [0,1]$ are defined by*

$$\tau_1(x) = \begin{cases} 3x + \frac{1}{4}, & 0 \leq x < \frac{1}{4}, \\ 3x - \frac{3}{4}, & \frac{1}{4} \leq x < \frac{1}{2}, \\ 4x - 2, & \frac{1}{2} \leq x < \frac{3}{4}, \\ 4x - 3, & \frac{3}{4} \leq x \leq 1 \end{cases}$$

$$\tau_2(x) = \begin{cases} 4x, & 0 \le x < \frac{1}{4}, \\ 4x-1, & \frac{1}{4} \le x < \frac{1}{2}, \\ 3x-\frac{3}{2}, & \frac{1}{2} \le x < \frac{3}{4}, \\ 3x-\frac{9}{4}, & \frac{3}{4} \le x \le 1, \end{cases}$$

*and the position dependent probabilities* $p_1, p_2 : [0,1] \to [0,1]$ *are defined by*

$$p_1(x) = \begin{cases} \frac{1}{4}, & 0 \le x < \frac{1}{4}, \\ \frac{1}{4}, & \frac{1}{4} \le x < \frac{1}{2}, \\ \frac{3}{4}, & \frac{1}{2} \le x < \frac{3}{4}, \\ \frac{3}{4}, & \frac{3}{4} \le x \le 1 \end{cases}$$

*and*

$$p_2(x) = \begin{cases} \frac{3}{4}, & 0 \le x < \frac{1}{4}, \\ \frac{3}{4}, & \frac{1}{4} \le x < \frac{1}{2}, \\ \frac{1}{4}, & \frac{1}{2} \le x < \frac{3}{4}, \\ \frac{1}{4}, & \frac{3}{4} \le x \le 1 \end{cases}$$

*If* $x \in [0, \frac{1}{4})$, *then* $\sum_{k=1}^{2} g_k(x) = \sum_{k=1}^{2} \frac{p_k(x)}{|\tau_k'(x)|} = \frac{\frac{1}{4}}{3} + \frac{\frac{3}{4}}{4} = \frac{13}{48} < 1.$
*If* $x \in [\frac{1}{4}, \frac{1}{2})$, *then* $\sum_{k=1}^{2} g_k(x) = \sum_{k=1}^{2} \frac{p_k(x)}{|\tau_k'(x)|} = \frac{\frac{1}{4}}{3} + \frac{\frac{3}{4}}{4} = \frac{13}{48} < 1.$
*If* $x \in [\frac{1}{2}, \frac{3}{4})$, *then* $\sum_{k=1}^{2} g_k(x) = \sum_{k=1}^{2} \frac{p_k(x)}{|\tau_k'(x)|} = \frac{\frac{3}{4}}{4} + \frac{\frac{1}{4}}{3} = \frac{13}{48} < 1.$
*If* $x \in [\frac{1}{2}, \frac{3}{4})$, *then* $\sum_{k=1}^{2} g_k(x) = \sum_{k=1}^{2} \frac{p_k(x)}{|\tau_k'(x)|} = \frac{\frac{3}{4}}{4} + \frac{\frac{1}{4}}{3} = \frac{13}{48} < 1.$

*Moreover,*

$A = 3\alpha + \max_{1 \le i \le q} \sum_{k=1}^{K} V_{J_i} g_k = 3 \cdot \frac{13}{48} + 0 = \frac{39}{48} < 1$. *Here,* $B = 2\beta\alpha + \beta \ \max_{1 \le i \le q} \sum_{k=1}^{K} V_{J_i} g_k > 0$ *with* $\beta = \max_{1 \le i \le q} \frac{1}{\lambda(J_i)}$. *Thus, the random map* $T$ *satisfies condition of Theorem 2.3.*

*From the Lasota–Yorke result ([9]), both* $\tau_1$ *and* $\tau_2$ *has acim. Moreover,* $\tau_1$ *and* $\tau_2$ *are piecewise linear, expanding and Markov. The Frobenius-Perron matrix* $P_{\tau_1}$ *of* $\tau_1$ *is the transpose of* $M_{\tau_1}$ *where*

$$M_{\tau_1} = \begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}.$$

*The matrix representation of the Frobenius–Perron operator* $P_{\tau_2}$ *is the transpose of* $M_{\tau_2}$ *where*

$$M_{\tau_2} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix}.$$

*It is easy to show that both* $\tau_1$ *and* $\tau_2$ *have unique acim. Thus, the random map* $T = \{\tau_1(x), \tau_2(x); p_1(x), p_2(x)\}$ *also has a unique acim (see Proposition 1 in [1]). The matrix representation of the Frobenius–Perron operator* $P_T f = \sum_{k=1}^{2} P_{\tau_k}(p_k f)(x)$ *is the transpose of the matrix* $M_T$, *where*

$$M_T = \begin{bmatrix} \frac{3}{16} & \frac{13}{16} & \frac{13}{16} & \frac{13}{16} \\ \frac{13}{16} & \frac{13}{16} & \frac{13}{16} & \frac{3}{16} \\ \frac{13}{16} & \frac{13}{16} & \frac{13}{16} & \frac{3}{16} \\ \frac{13}{16} & \frac{13}{16} & \frac{13}{16} & \frac{13}{16} \end{bmatrix}.$$

*The normalized density* $f^*$ *of the unique acim of the random map* $T$ *is the left eigenvector of the matrix* $M_T$ *associated with the eigenvalue* 1 *(after adding the normalizing condition). In fact,* $f^* = \left[1, \frac{13}{12}, \frac{13}{12}, \frac{5}{6}\right]$.

**Monte Carlo approach to Ulam's method***: In Figure 5.1 (a) and 5.1 (b) we have plotted the actual density and approximate density for Monte Carlo approach to Ulam's method.*

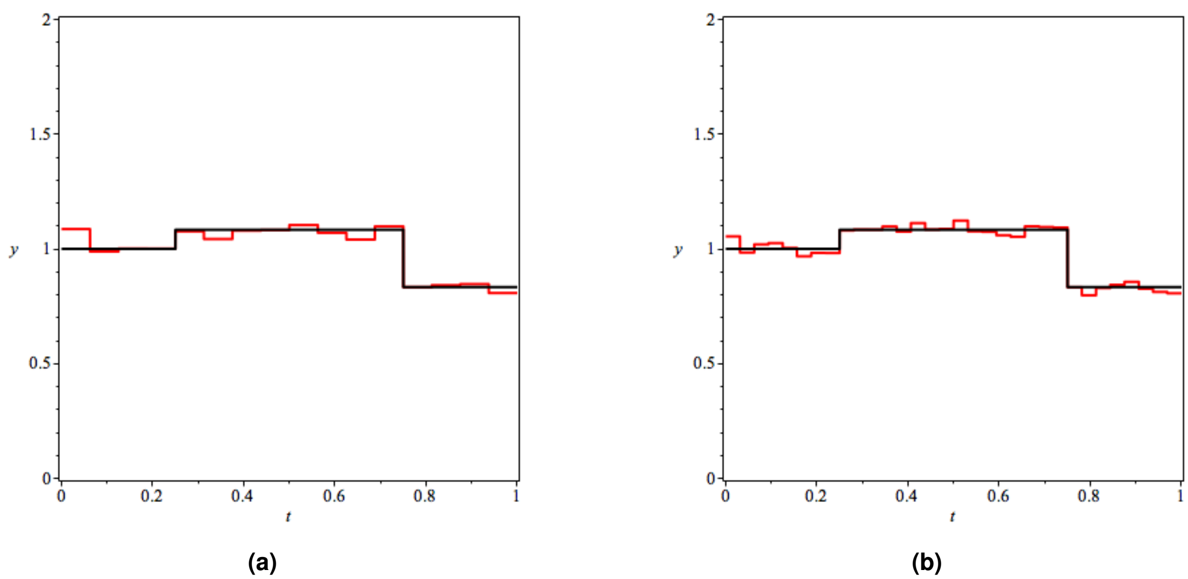
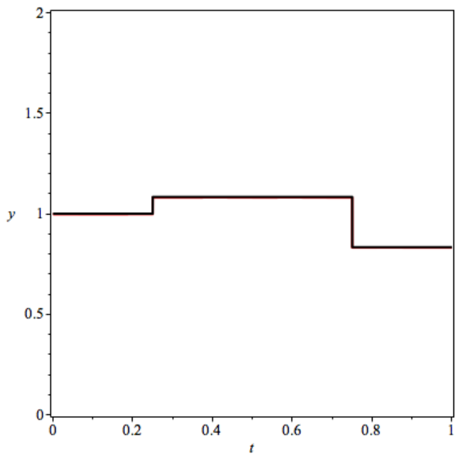


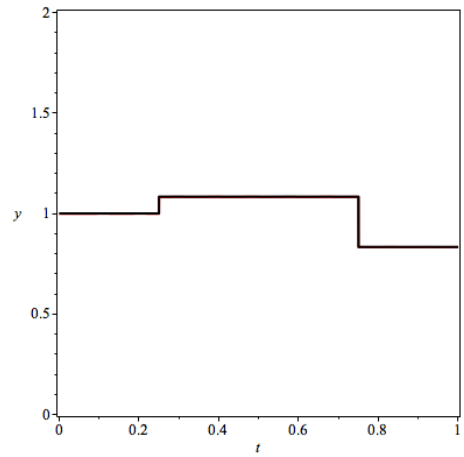

**Figure 5.1.** Monte Carlo approach to Ulam's method for the random map $T$: Figure 5.1 (a) the graph of the approximate density function $f_{16}$ (Monte Carlo -Ulam's method with $N = 16, K = 1000$:red curve) and the actual density function $f^*$ (black curve); Figure 5.1 (b) the graph of the approximate density function $f_{32}$ (Monte Carlo -Ulam's method with $N = 32, K = 1000$:red curve) and the actual density function $f^*$ (black curve);

*The* $L^1-$*norm* $\| f_N - f^* \|_1$ *is measured (with Maple 15) to estimate the convergence of the approximate density* $f_N$ *to the actual density* $f^*$ *for our Monte Carlo approach to Ulam's method.*

| $N$ | $\parallel f_N - f^* \parallel_1$ |
|---|---|
| *16* | 0.01815903102 |
| *32* | 0.01630702912 |

**Quasi Monte Carlo approach to Ulam's method**: *In Figure 5.2 (a) and 5.2 (b) we have plotted the actual density and approximate density for Quasi Monte Carlo-Ulam's method.*




**(a)** **(b)**

**Figure 5.2.** Quasi Monte Carlo approach to Ulam's method for the random map $T$: Figure 2 (a)the graph of the approximate density function $f_{16}$ (Quasi Monte Carlo -Ulam's method with $N = 16, K = 1000$:red curve) and the actual density function $f^*$ (black curve); Figure 2 (b) Figure 1 (a)the graph of the approximate density function $f_{32}$ (Quasi Monte Carlo approach to Ulam's method with $N = 32, K = 1000$: red curve) and the actual density function $f^*$ (black curve);

*The $L^1-$norm $\parallel f_N - f^* \parallel_1$ is measured (with Maple 18) to estimate the convergence of the approximate density $f_N$ to the actual density $f^*$ for our Quasi Monte Carlo- Ulam's method.*

| $N$ | $\parallel f_N - f^* \parallel_1$ |
|---|---|
| *16* | 0.002504794762 |
| *32* | 0.001252884246 |

.

**Example 5.2.** *We consider the position dependent random map $T = \{\tau_1(x), \tau_2(x); p_1(x), p_2(x)\}$ where $\tau_1, \tau_2 : [0,1] \to [0,1]$ are defined by (see Example 5.2 of [5] for this random map)*

$$\tau_1(x) = \begin{cases} 2x, & 0 \le x < \frac{1}{2}, \\ \frac{5}{4}x + \frac{1}{10}, & \frac{1}{2} \le x \le \frac{2}{3}, \\ \frac{3}{4}x + \frac{1}{4}, & \frac{2}{3} < x \le 1, \end{cases}$$

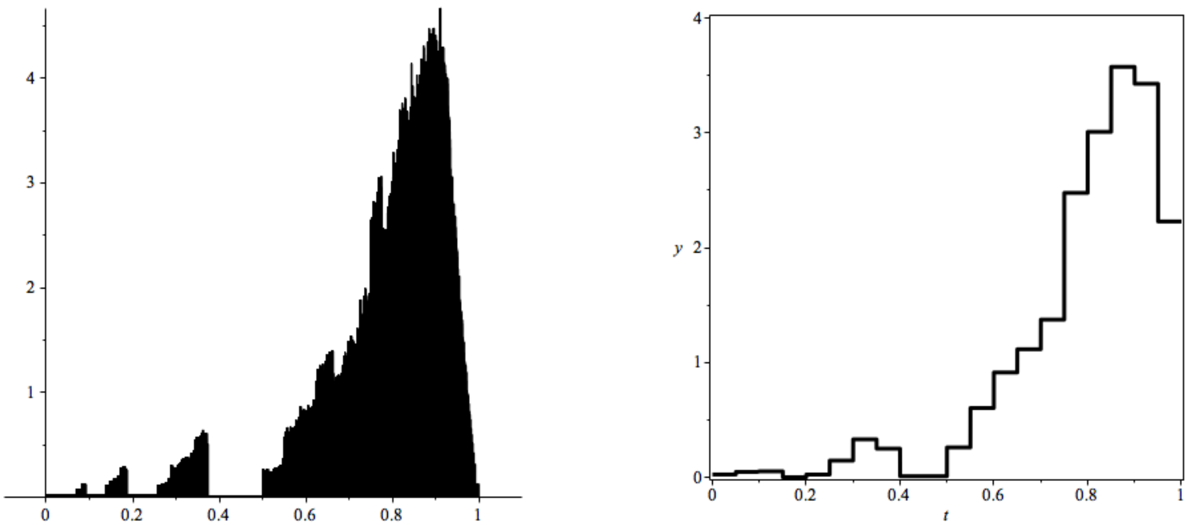
$$\tau_2(x) = \begin{cases} \frac{1}{2}x, & 0 \le x < \frac{1}{2}, \\ \frac{3}{4}x - \frac{1}{8}, & \frac{1}{2} \le x \le \frac{2}{3}, \\ \frac{3}{2}x - \frac{1}{2}, & \frac{2}{3} < x \le 1, \end{cases}$$

*and the position dependent probabilities $p_1, p_2 : [0,1] \to [0,1]$ are defined by*

$$p_1(x) = \begin{cases} 0.8, & 0 \le x < \frac{1}{2}, \\ 0.725, & \frac{1}{2} \le x \le \frac{2}{3}, \\ 0.4, & \frac{2}{3} < x \le 1, \end{cases}$$

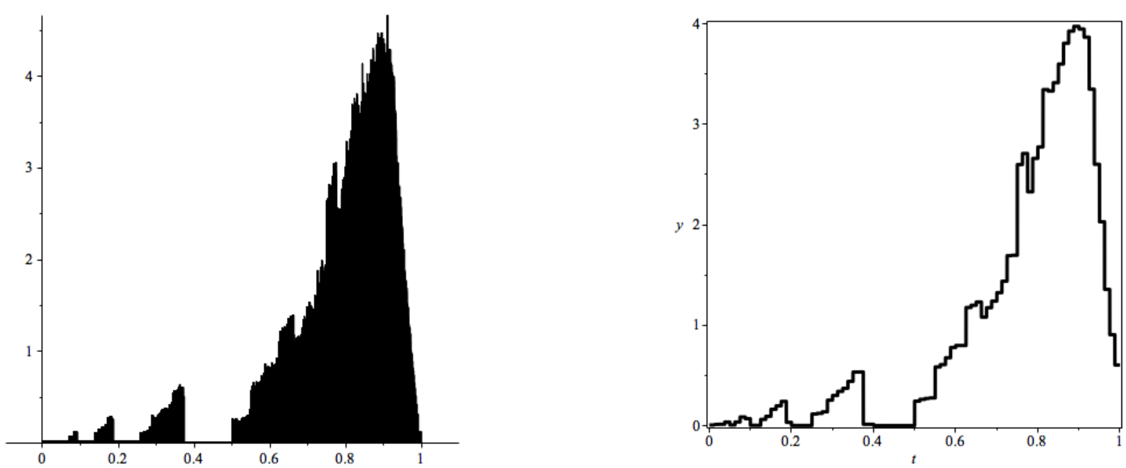
*and* $p_2(x) = 1 - p_1(x)$.

*It can be easily shown that the random map $T$ satisfies condition of Theorem 2.3. Thus, $T$ has an acim. Unfortunately, we do not know the actual density of the acim. Góra and Boyarsky [1] presented a Markov approximation of the random map $T$ then they presented the density of the Markov random maps. Note that the density obtained from a Markov approximation of the random maps is only an approximate density. In Figure 5.3 and Figure 5.4 we have presented histogram and approximate densities via Monte Carlo approach to Ulam's and Quasi Monte Carlo approach to Ulam's method.*




**(a)** **(b)**

**Figure 5.3.** Histigram and Monte Carlo approach to Ulam's method: Figure 5.3 (a) the histogram of the density function of the random map $T$ with 500,000 points on the trajectory of the random map $T$ with 1000 subintervals for $[0,1]$.; Figure 5.3 (b) Monte Carlo approach to Ulam's method for the random map $T$: The graph of the approximate density function $f_{20}$ with $K = 1000$.




**(a)** **(b)**

**Figure 5.4.** Histigram and Quasi Monte Carlo approach to Ulam's method: Figure 5.4 (a) the histogram of the density function of the random map $T$ with 500,000 points on the trajectory of the random map $T$ with 1000 subintervals for $[0,1]$.; Figure 5.4 (b) Quasi Monte Carlo approach to Ulam's method for the random map $T$: The graph of the approximate density function $f_{80}$ with $K = 1000$.

## 6. Conclusion

In this paper, we study numerical computations of invariant measures for position dependent random maps. First, we present the Frobenius-Perron operator and the existence of invariant measures for position dependent random maps. We present the Ulam's method for the computation of invariant measures for position dependent random maps. A general algorithm for approximating fixed points of the Frobenius-Perron operator for position dependent random maps is presented. Then we present the Monte Carlo and the Quasi Monte Carlo approach to Ulam's method for the computation of invariant measures for position dependent random maps. Finally, we present two examples of position dependent random maps along with the numerical computations of invariant measures using the Monte Carlo and the Quasi Monte Carlo approach to Ulam's method. In the first example, we present $L^1$ norm errors between the numerical approximation of the density of the invariant measure and analytical density of invariant measures for the random map. The numerical examples show that the Monte Carlo approach and the Quasi Monte Carlo approach to Ulam's method are useful tools for the computation of invariant measures for position dependent random maps. Our numerical schemes are generalizations of numerical schemes described in [12] and [15] of single deterministic maps to numerical schemes for position dependent random maps. In future, we plan on studying the speed of convergence of the Monte Carlo approach and the Quasi Monte Carlo approach to Ulam's method.

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