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Anti-Invariant Semi-Riemannian Submersions from Indefinite Almost Contact Metric Manifolds

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Abstract

In this paper, we study an anti-invariant semi-Riemmannian submersions from indefinite almost contact metric manifolds. We obtain, the necessary and sufficient conditions for the characteristics vector filed to be vertical and horizontal. aMoreover, we find the conditions of integrability and hormonicness of this submersion map. Finally, we furnish an example of an anti-invariant semi-Riemannian submersion from indefinite almost contact metric manifold which is indefinite trans-Sasakian manifolds in the present paper.

Keywords: *semi-Riemannian submersion; Anti-invaraint submersion; indefinite trans-Sasakian manifolds.*
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1. Introduction

In 1966, the theory of semi-Riemannian submersions between semi-Riemannian manifolds was introduced by O'Neill [3, 4] and Gray [1] in 1967. Watson [2] study Riemannian submersions between almost Hermitian submersions. It is well known that Riemannian submersions are related with physics and have their applications in Kaluza-Klein theory ([14, 25, 26]) Yang-Mills theory ([2, 13]) the theory of supergravity and superstring theories [26]. Afterwords, Sahin introduced anti-invariant and semi-invariant Riemmanin submersion from almost Hermitian manifolds onto Riemannian manifolds. (see [5, 6, 7, 19]). Also, anti-invariant Riemannian submersions extensively studied by several authors (see [16, 17, 28]). In [8], Chinea defined almost contact Riemannian submersion between almost contact metric manifolds. In [12], Lee studied the vertical and horizontal distribution are ϕ-invariant. Moreover, the characteristic vector field ξ is horizontal. We note that only ϕ-holomorphic submersions have been consider on an almost contact manifolds [21]. Note that notion of anti-invariant submersions was generalized the notion conformal anti-invariant submersions [16]. In fact, anti-invariant Riemannian and Lagrangian submersions have been studying in different kinds of structures such as (see [11, 16, 17]). Recently, in 2018, Siddiqi and Akyol study the some properties of anti-invariant $\xi^{\perp}$-submersions from almost hyperbolic contact manifolds [15, 18]. In [20] Fagahfouri and Mashmouli study anti-invariant semi-Riemannian submersions.
In 1980, Oubina [23] introduced the notion of an indefinite trans-Sasakian manifold, of type (α,β) [10] with indefinite metric play significant role in Physics. Indefinite Sasakian manifold is an important kind of indefinite trans-Sasakian manifold with $\alpha=1$ and $\beta=1$. Indefinite cosymplectic manifold is another kind of indefinite trans-Sasakian manifold such that $\alpha=\beta=0$. Therefore, motivated by the above studies in this paper, we studied anti-invariant semi-Riemannian submersions from indefinite trans-Sasakian manifolds.

2. Semi-Riemannian submersion

In this section, we give necessary background for Semi-Riemannian submersions [9].

Let (M,g) and (N,g_N) be semi-Riemannian manifolds, where $dim(M)>dim(N)$. A surjective map $\pi:(M,g)\to(N,g_N)$ is called a *semi-Riemannian submersion* [3] if:

(S1) π has maximal rank, and

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(S2) π_*, restricted to $(ker\pi_*)^{\perp}$, is a linear isometry.
Under this case, for each $y \in N$, $\pi^{-1}(y)$ is a k-dimensional submanifold of M called *a fiber*, where $k = dim(M) - dim(N)$. A vector field on M is called *vertical* (resp. *horizontal*) if it is always tangent (resp. orthogonal) to fibers. A vector field X on M is called *basic* if X is horizontal and π-related to a vector field X_* on N, i.e., $\pi_* X_x = X_{*\pi(x)}$ for all $x \in M$. As usual, we denote by $\mathscr{V}$ and $\mathscr{H}$ the projections on the vertical distribution $ker\pi_*$ and the horizontal distribution $(ker\pi_*)^{\perp}$, respectively. The geometry of semi-Riemannian submersions is characterized by O'Neill's [3] tensors $\mathscr{T}$ and $\mathscr{A}$, defined as follows:

$$\mathscr{T}_E F = \mathscr{V}\nabla_{\mathscr{V}E}\mathscr{H}F + \mathscr{H}\nabla_{\mathscr{V}E}\mathscr{V}F, \tag{2.1}$$

$$\mathscr{A}_E F = \mathscr{V}\nabla_{\mathscr{H}E}\mathscr{H}F + \mathscr{H}\nabla_{\mathscr{H}E}\mathscr{V}F \tag{2.2}$$

for any vector fields E and F on M, where ∇ is the Levi-Civita connection of g. It is easy to see that $\mathscr{T}_E$ and $\mathscr{A}_E$ are skew-symmetric operators on the tangent bundle of M reversing the vertical and the horizontal distributions. We summarize the properties of the tensor fields $\mathscr{T}$ and $\mathscr{A}$. Let U, W be vertical and X, Y be horizontal vector fields on M, then we have

$$\mathscr{T}_U V = \mathscr{T}_V U, \tag{2.3}$$

$$\mathscr{A}_X Y = -\mathscr{A}_Y X = \frac{1}{2}\mathscr{V}[X,Y]. \tag{2.4}$$

On the other hand, from (2.1) and (2.2), we obtain

$$\nabla_V W = \mathscr{T}_V W + \hat{\nabla}_V W, \tag{2.5}$$

$$\nabla_V X = \mathscr{T}_V X + \mathscr{H}\nabla_V X, \tag{2.6}$$

$$\nabla_X V = \mathscr{A}_X V + \mathscr{V}\nabla_X V, \tag{2.7}$$

$$\nabla_X Y = \mathscr{H}\nabla_X Y + \mathscr{A}_X Y, \tag{2.8}$$

where $\hat{\nabla}_V W = \mathscr{V}\nabla_V W$ and $\mathscr{H}\nabla_W X = \mathscr{A}_X W$, if ξ is basic. It is not difficult to observe that $\mathscr{T}$ acts on the fibers as the second fundamental form while $\mathscr{A}$ acts on the horizontal distribution and measures of the obstruction to the integrability of this distribution. For details on semi-Riemannian submersions, we refer to O'Neill's paper [1] and to [21].
Finally, we recall the notion of the second fundamental form of a map between semi-Riemannian manifolds. Let (M, g) and (N, g_N) be semi-Riemannian manifolds and $\varphi : (M, g) \to (N, g_N)$ be a smooth map. Then the second fundamental form of ϕ is given by

$$(\nabla\phi_*)(E,F) = \nabla^{\phi}_E \phi_* F - \phi_*(E,F) \tag{2.9}$$

for $E, F \in TM$, where ∇^{φ} is the pull back connection and we denote for convenience by ∇ the Riemannian connections of the metrics g and g_N [1].It is known that the second fundamental form is symmetric. If ϕ is semi-Riemannian submersion [9] it can be easily prove that

$$(\nabla\phi_*)(E,F) = 0 \tag{2.10}$$

for $E, F \in \Gamma((kerF_*)^{\perp})$. A smooth map $\phi : (M, g_M) \to (N, g_N)$ is said to be harmonic [24] if $trace(\nabla\phi_*) = 0$. On the other hand, the tension field of ϕ is the section $\tau(\phi)$ of $\Gamma(\phi^{-1}TN)$ defined by

$$\tau(\phi) = div\phi_* = \sum_{i=1}^{m}(\nabla\phi_*)(e_i, e_i), \tag{2.11}$$

where $\{e_1, ...e_m\}$ is the orthonormal frame on M. Then it follows that ϕ is harmonic if and only if $\tau(\phi) = 0$, for details, [24].

3. Indefinite Trans-Sasakian Manifolds

Let M be an $(2n+1)$- dimensional indefinite almost contact metric manifold [23] with an indefinite almost contact metric structure $(\phi, \xi, \eta, g, \varepsilon)$, where ϕ is a $(1,1)$ tensor field, ξ is a vector field, η is a 1-form and g_M is a compatible indefinite Riemannian metric on M such that

$$\phi^2 = -I + \eta\otimes\xi, \quad \phi\xi = 0, \quad \eta\circ\phi = 0, \quad \eta(\xi) = 1, \tag{3.1}$$

where I denotes the identity tensor.
The indefinite almost contact structure is said to be normal if $N + d\eta\otimes\xi = 0$, where N is the Nijenhuis tensor. Suppose that a indefinite metric tensor g is given in M and satisfies the condition.

$$g(\phi X, \phi Y) = g(X,Y) - \varepsilon\eta(X)\eta(Y), \ \ \varepsilon g(X,\xi) = \eta(X) \tag{3.2}$$

$$g(X,\phi Y) = -g(\phi X,Y), \tag{3.3}$$

for all X,Y on M, where ε is 1 or -1 a according as ξ is space like or timelike vector file and rank ϕ is $\phi = 2n$.
An indefinite almost contact metric structure $(\phi,\xi,\eta,g,\varepsilon)$ on M is called indefinite trans-Sasakian manifold if [23] .

$$(\nabla_X\phi)Y = \alpha(g(X,Y)\xi - \varepsilon\eta(Y)X) + \beta(g(\phi X,Y)\xi - \varepsilon\eta(Y)\phi X) \tag{3.4}$$

for all X, Y tangent to M, α and β are smooth functions on M and we say that the indefinite trans-Sasakian structure of type (α,β). Now from (3.3) it follows that

$$\nabla_X\xi = -\varepsilon\{\alpha(\phi X) + \beta(X - \eta(X)\xi)\}, \tag{3.5}$$

$$(\nabla_X\eta)Y = -\alpha g(\phi X,Y) + \beta[g(X,Y) - \varepsilon\eta(X)\eta(Y)], \tag{3.6}$$

where ∇ is the Riemannian connection of Levi-Civita covariant differentiation.
For an indefinite Trans-Sasakian manifold M the following relations holds [23]:

$$\begin{aligned} R(\xi,X)Y &= (\alpha^2-\beta^2)(\eta(Y)X - \eta(X)Y) + 2\alpha\beta(\eta(Y)\phi X - \eta(X)\phi Y) \\ &\quad + \varepsilon(Y\alpha)\phi X - (X\alpha)\phi Y + (Y\beta)\phi^2 X - (X\beta)\phi^2 Y. \end{aligned} \tag{3.7}$$

$$S(X,\xi) = (2m(\alpha^2-\beta^2) - \xi\beta)\eta(X) - \varepsilon(2m-1)X\beta - (\phi X)\alpha. \tag{3.8}$$

4. Anti-invariant semi-Riemannian submersions

Definition 4.1. *. Let $M(\phi,\xi,\eta,g_M,\varepsilon)$ be a an indefnite trans Sasakian manifold and (N,g_N) be a sem-Riemannian manifold. A semi-Riemannian submersion $F : M(\phi,\xi,\eta,g_M,\varepsilon) \longrightarrow (N,g_N)$ is called anti-invariant semi-Riemannian submersion if $kerF_*$ is anti-invariant with respect to ϕ, i.e. $\phi(kerF_*) \subseteq (kerF_*)^\perp$.*

Let $F : M(\phi,\xi,\eta,g_M) \longrightarrow (N,g_N)$ be an anti-invariant semi-Riemannian submersion from an indefinite trans-Sasakian manifold $M(\phi,\xi,\eta,g_M)$ to a semi-Riemannian manifold (N,g_N). First of all from Definition 4.1, we have $\phi(kerF_*) \cap (kerF_*)^\perp \neq 0$. We denote the complementary orthogonal distribution to $\phi(kerF_*)$ in $(kerF_*)^\perp$ by μ. Then we have

$$(kerF_*)^\perp = \phi(kerF_*)\oplus\mu. \tag{4.1}$$

5. Anti-invariant submersion admitting vertical structure vector field

In this section, we will study anti-invariant submersion from an indefinite trans-Sasakian manifold onto a Riemannian manifold such that the characteristic vector field ξ is vertical.
It is easy to see that μ is an invariant distribution of $(kerF_*{}^\perp)$, under the endomorphism ϕ. Thus, for $X \in \Gamma((kerF_*{}^\perp))$, we write

$$\phi X = BX + CX, \tag{5.1}$$

where $BX \in \Gamma((kerF_*)$ and $CX \in \Gamma(\mu)$. On the other hand, since $F_*(kerF_*{}^\perp) = TN$ and F is a Riemannian submersion, using (4.2) we derive $g_N(F_*\phi V, F_*CX) = 0$, for every $X \in \Gamma((kerF_*{}^\perp))$ and $V \in \Gamma(kerF_*)$, which implies that

$$TN = F_*(\phi(kerF_*)\oplus F_*(\mu). \tag{5.2}$$

Theorem 5.1. *Let $M(\phi,\xi,\eta,g_M,\varepsilon)$ be an indefinite trans-Sasakian manifold of dimension $2m+1$ and $(N,g_N$ is a semi-Riemannian manifold of dimension n. Let $F : M(\phi,\xi,\eta,g_M) \longrightarrow (N,g_N)$ be an anti-invariant semi-Riemannian submersion such that $(\phi(kerF_*) = (kerF_*{}^\perp)$. Then the characteristic vector field ξ is vertical and $m = n$.*

Proof. By assumption $(\phi(kerF_*) = (kerF_*{}^\perp)$, for any $U \in (kerF_*$, we have $g_M(\xi,\phi U) = -g_M(\phi\xi,U) = 0$, which shows that the structure vector field is vertical. Now we suppose that $U_1,...,U_{k-1},\xi = U_k$ be an orthonormal frame of $(kerF_*)$, where $k = 2m-n+1$. Since $(\phi(kerF_*) = (kerF_*{}^\perp)$, $U_1,...,U_{k-1},\xi = U_k$ from an orthonormal frame of $\Gamma((kerF_*{}^\perp))$. So, by help of (4.3) we obtain $k = n+1$ which implies that $m = n$. □

Theorem 5.2. *Let $M(\phi,\xi,\eta,g_M,\varepsilon)$ be an indefinite trans Sasakian manifold of dimension $2m+1$ and $(N,g_N$ is a semi-Riemannian manifold of dimension n. Let $F : M(\phi,\xi,\eta,g_M,\varepsilon) \longrightarrow (N,g_N)$ be an anti-invariant semi-Riemannian submersion. Then the fibers are not totally umbilical.*

Proof. Using (2.5) and (3.5) we obtain

$$\mathcal{T}_U\xi = -\varepsilon\alpha\phi U + \varepsilon\beta\phi^2 U \tag{5.3}$$

for any $U \in \Gamma((kerF_*$. If the fibers are totally umbilical, then we have $\mathcal{T}_U V = g_M(U,V)H$ for any vertical vector fields U,V where H is the mean curvature vector field of any fiber. Since $\mathcal{T}_\xi\xi = 0$, we have $H = 0$, which shows that fibers are minimal. Hence the fibers are totally geodesic, which is a contradiction to the fact that $\mathcal{T}_U\xi = -\varepsilon\alpha\phi U \neq 0$. □

From (3.1) and (4.2) we have following Lemma.

Lemma 5.3. *Let F be an anti-invariant semi-Riemannian submersion from an indefinite trans-Sasakian manifold $M(\phi,\xi,\eta,g_M,\varepsilon)$ to a semi-Riemannian manifold (N,g_N). Then we have*

$BCX = 0,$

$\phi BX + C^2X = -X,$

for any $X \in \Gamma((kerF_*{}^{\perp}))$.

Proof. Using (3.4) one can easily obtain

$$\nabla_X Y = -\phi\nabla_X\phi Y + \varepsilon\alpha(g(Y,\phi X))\xi + \varepsilon\beta(g(\phi Y\phi X))\xi \quad (5.4)$$

for any $X,Y \in \Gamma((kerF_*{}^{\perp}))$. □

Lemma 5.4. *Let F be an anti-invariant semi-Riemannian submersion from an indefinite trans-Sasakian manifold $M(\phi,\xi,\eta,g_M,\varepsilon)$ to a semi-Riemannian manifold (N,g_N). Then we have*

$$CX = -\frac{\varepsilon}{\alpha}\mathscr{A}_X\xi, \quad (5.5)$$

$$g_M(\mathscr{A}_X\xi,\phi U) = 0, \quad (5.6)$$

$$g_M(\nabla_Y\mathscr{A}_X\xi,\phi U) = -g_M(\mathscr{A}_X\xi,\phi\mathscr{A}_Y U) + \varepsilon\alpha\eta(U)g_M(\mathscr{A}_X\xi,Y) \quad (5.7)$$

$$+\varepsilon\beta\eta(U)g_M(\mathscr{A}_X\xi,\phi X)$$

$$g_M(X,\mathscr{A}_Y\xi) = -\varepsilon g_M(Y,\mathscr{A}_X\xi) \quad (5.8)$$

for $X,Y \in \Gamma((kerF_*{}^{\perp}))$ and $U \in \Gamma((kerF_*)$.

Proof. By virtue of (2.7) and (3.5) we have (4.6).
For $X \in \Gamma((kerF_*{}^{\perp}))$ and $U \in \Gamma((kerF_*)$, by virtue of (3.2), (4.2)and (4.6) we get

$$g_M(\mathscr{A}_X\xi,\phi U) = -\varepsilon g_M(\alpha\phi X - \alpha BX,\phi U) \quad (5.9)$$

$$= -\varepsilon\alpha g_M(X,U) + \varepsilon\alpha\eta(X)\eta(U) - \varepsilon\alpha g_M(\phi BX,U).$$

Since $\phi BX \in \Gamma((kerF_*{}^{\perp}))$ and $\xi \in \Gamma((kerF_*)$, (4.10) implies (4.7).
Now from (4.7) we get

$$g_M(\nabla_Y\mathscr{A}_X\xi,\phi U) = -g_M(\mathscr{A}_X\xi,\nabla_Y\phi U)$$

for $X,Y \in \Gamma((kerF_*)^{\perp})$ and $U \in \Gamma((kerF_*)$. Then using (2.7) and (3.4) we have

$$g_M(\nabla_Y\mathscr{A}_X\xi,\phi U) = -g_M(\mathscr{A}_X\xi,\phi\mathscr{A}_Y U) - g_M(\mathscr{A}_X\xi,\phi(\mathscr{V}\nabla_Y U))$$

$$+\varepsilon\alpha\eta(U)g_M(\mathscr{A}_X\xi,Y) + \varepsilon\beta\eta(U)g_M(\mathscr{A}_X\xi,\phi X).$$

Since $\phi(\mathscr{V}\nabla_Y U) \in \Gamma(kerF_*) = \Gamma((kerF_*)^{\perp})$, we obtain (4.8).
Using (2.11), we obtain directly (4.9) □

Now, we study the integrability of the distribution $(kerF_*)^{\perp}$ and then we investigate the geometry of leaves of $(kerF_*)$ and $(kerF_*)^{\perp}$. We note it is known that the distribution $(kerF_*)$ is integrable.

Theorem 5.5. *Let F be an anti-invariant semi-Riemannian submersion from an indefinite trans-Sasakian manifold $M(\phi,\xi,\eta,g_M,\varepsilon)$ to a semi-Riemannian manifold (N,g_N). Then the following assertions are equivalent to each other;*

1. *$(kerF_*)^{\perp}$ is integrable,*
2. $g_N((\nabla F_*)(Y,BX),F_*\phi V) = g_N((\nabla F_*)(X,BY),F_*\phi V) + \frac{\varepsilon}{\alpha}g_M(\mathscr{A}_X\xi,\phi\mathscr{A}_Y V) - \frac{\varepsilon}{\alpha}g_M(\mathscr{A}_Y\xi,\phi\mathscr{A}_X V)$
3. $g_M(\mathscr{A}_X BY - \mathscr{A}_Y BX,\phi V) = \frac{\varepsilon}{\alpha}g_M(\mathscr{A}_X\xi,\phi\mathscr{A}_Y V) - \frac{\varepsilon}{\alpha}g_M(\mathscr{A}_Y\xi,\phi\mathscr{A}_X V)$

for $X,Y \in \Gamma((kerF_*)^{\perp})$ and $V \in \Gamma((kerF_*)$.

Proof. Using (4.5) for $X,Y \in \Gamma((kerF_*)^{\perp})$ and $U \in \Gamma((kerF_*)$, we get

$$g_M([X,Y],V) = g_M(\nabla_X Y,V) - g_M(\nabla_Y X,V)$$
$$= g_M(\nabla_X \phi Y,\phi V) - g_M(\nabla_Y \phi X,\phi V)$$
$$+2\varepsilon\alpha(g_M(\phi X,Y))g_M(V,\xi) + 2\varepsilon\beta(g_M(\phi X,\phi Y))g_M(V,\xi).$$

Then from (4.2) we have

$$g_M([X,Y],V) = g_M(\nabla_X BY,\phi V) - \frac{\varepsilon}{\alpha} g_M(\nabla_X \mathscr{A}_Y \xi,\phi V)$$
$$- g_M(\nabla_Y BX,\phi V) + \frac{\varepsilon}{\alpha} g_M(\nabla_Y \mathscr{A}_{\mathscr{X}} \xi,\phi V)$$
$$+2\varepsilon\alpha(g_M(\phi X,Y))g_M(V,\xi) + 2\varepsilon\beta(g_M(\phi X,\phi Y))g_M(V,\xi).$$

Using (2.2), (2.7) and if we take into account that F is a semi-Riemannian submersion, we obtain

$$g_M([X,Y],V) = g_N(F_*\nabla_X BY,F_*\phi V) - \frac{\varepsilon}{\alpha} g_M(\nabla_X \mathscr{A}_{\mathscr{Y}} \xi,\phi V)$$
$$- g_N(F_*\nabla_Y BX,F_*\phi V) + \frac{\varepsilon}{\alpha} g_M(\nabla_Y \mathscr{A}_{\mathscr{X}} \xi,\phi V)$$
$$-2(g_M(\mathscr{A}_X \xi,Y))g_M(V,\xi) + 2\varepsilon\frac{\beta}{\alpha^2}(g_M(\mathscr{A}_X \xi,\mathscr{A}_Y \xi)g_M(V,\xi).$$

Thus from (2.12) and (4.8) we have

$$g_M([X,Y],V) = g_N(-(\nabla F_*)(X,BY) + (\nabla F_*)(Y,BX),F_*\phi V)$$
$$+\frac{\varepsilon}{\alpha} g_M(\mathscr{A}_Y \xi,\phi \mathscr{A}_X V) - \frac{\varepsilon}{\alpha} g_M(\mathscr{A}_X \xi,\phi \mathscr{A}_Y V)$$

which proves $(i) \Leftrightarrow (ii)$. On other hand using (2.12) we get

$$(\nabla F_*)(Y,BX) - (\nabla F_*)(X,BY) = -F_*(\nabla_Y BX - \nabla_X BY).$$

Then (2.7) implies that

$$(\nabla F_*)(Y,BX) - (\nabla F_*)(X,BY) = -F_*(\mathscr{A}_Y BX - \mathscr{A}_X BY).$$

From (2.2) $\mathscr{A}_Y BX - \mathscr{A}_X BY \in \Gamma((kerF_*)^{\perp})$, this shows that $(ii) \Leftrightarrow (iii)$ □

Hence we have the following Lemma:

Lemma 5.6. *Let $F : M(\phi,\xi,\eta,g_M,\varepsilon) \longrightarrow (N,g_N)$ be an anti-invariant semi-Riemannian submersion such that $\phi(kerF_*) = (kerF_*)^{\perp}$, where $M(\phi,\xi,\eta,g_M)$ is an indefinite trans-Sasakian manifold and (N,g_N) is a semi-Riemannian manifold . Then following assertions are equivalent to each other;*

1. *$(kerF_*)^{\perp}$ is integrable,*
2. *$(\nabla F_*)(Y,\phi X),F_*\phi V) = (\nabla F_*)(X,\phi Y)$*
3. *$\mathscr{A}_X \phi Y = \mathscr{A}_Y \phi X$.*

Theorem 5.7. *Let F be an anti-invariant semi-Riemannian submersion from indefinite trans-Sasakian $M(\phi,\xi,\eta,g_M,\varepsilon)$ to a semi-Riemannian manifold (N,g_N). Then the following assertions are equivalent to each other;*

1. *$(kerF_*)^{\perp}$ define a totally geodesic foliation on M.*
2. *$g_M(\mathscr{A}_X BY,\phi V) = -\frac{\varepsilon}{\alpha} g_M(\mathscr{A}_Y \xi,\phi \mathscr{A}_X V)$*
3. *$g_N((\nabla F_*)(X,\phi Y),F_*\phi V) = \varepsilon\alpha g_M(\mathscr{A}_Y \xi,X)\eta(V) - \frac{\beta}{\alpha^2} g_M(\mathscr{A}_Y \xi,\mathscr{A}_X \xi)\eta(V)$*

for $X,Y \in \Gamma((kerF_*{}^{\perp}))$ and $V \in \Gamma((kerF_*)$.

Proof. From (2.7), (4.2), (4.5) and (4.8) we obtain

$$g_M(\nabla_X Y,V) = g_M(\mathscr{A}_X BY,\phi V) + \frac{\varepsilon}{\alpha} g_M(\mathscr{A}_Y \xi,\phi \mathscr{A}_X V) \tag{5.10}$$
$$-\alpha\eta(V)(g_M(\mathscr{A}_Y \xi,X) + g_M(\mathscr{A}_X \xi,y))$$

for $X,Y \in \Gamma((kerF_*)^{\perp})$ and $V \in \Gamma((kerF_*)$. Using (4.9) in (4.11) we get

$$g_M(\nabla_X Y,V) = g_M(\mathscr{A}_X BY,\phi V) + \frac{\varepsilon}{\alpha} g_M(\mathscr{A}_Y \xi,\phi \mathscr{A}_X V)$$

The last equation shows that (1) $\Leftrightarrow$ (2).
For $X,Y\in\Gamma((kerF_*)^\perp)$ and $V\in\Gamma((kerF_*)$,

$$g_M(\mathscr{A}_XBY,\phi V)=-\frac{\varepsilon}{\alpha}g_M(\mathscr{A}_Y\xi,\phi\mathscr{A}_XV)$$
$$=g_M(\nabla_X\mathscr{A}_Y\xi,\phi V)-\varepsilon\alpha\eta(V)g_M(\mathscr{A}_Y\xi,X)-\varepsilon\beta\eta(V)g_M(\mathscr{A}_Y\xi,\phi X)$$

$$=-g_M(\nabla_X\phi Y,\phi V)+g_M(\nabla_XBY,\phi V)-\varepsilon\alpha\eta(V)g_M(X,\mathscr{A}_Y\xi)-\varepsilon\beta\eta(V)g_M(\mathscr{A}_Y\xi,\phi X). \tag{5.11}$$

Since differential F_* preserves the lengths of horizontal vectors the relation (4.12) forms

$$g_M(\mathscr{A}_XBY,\phi V)=g_N(F_*\nabla_X\phi Y,F_*\phi V)-g_M(\nabla_XBY,\phi V) \tag{5.12}$$
$$-\varepsilon\alpha g_M(\mathscr{A}_Y\xi,X)\eta(V)-\varepsilon\frac{\beta}{\alpha^2}g_M(\mathscr{A}_Y\xi,\mathscr{A}_X\xi)\eta(V).$$

Using (4.5), (3.2), (2.12) and (2.13) in (4.13) respectively, we obtain

$$g_M(\mathscr{A}_XBY,\phi V)=g_N(-(\nabla F_*)(X,\phi Y),F_*\phi V)$$
$$-\varepsilon\alpha g_M(\mathscr{A}_Y\xi,X)\eta(V)-\varepsilon\frac{\beta}{\alpha^2}g_M(\mathscr{A}_Y\xi,\mathscr{A}_X\xi)\eta(V)$$

which tells that (2) $\Leftrightarrow$ (3).

□

Lemma 5.8. *Let $F:M(\phi,\xi,\eta,g_M,\varepsilon)\longrightarrow(N,g_N)$ be an anti-invariant semi-Riemannian submersion such that $\phi(kerF_*)=(kerF_*)^\perp$, where $M(\phi,\xi,\eta,g_M)$ is an indefinite trans-Sasakian manifold and (N,g_N) is a semi-Riemannian manifold . Then following assertions are equivalent to each other:*

1. *$(kerF_*)^\perp$ defines a totally geodesic folition on M.*
2. *$\mathscr{A}_X\phi Y=0$.*
3. *$(\nabla F_*)(X,\phi Y)=0$ for $X,Y\in\Gamma((kerF_*)^\perp)$ and $V\in\Gamma((kerF_*)$.*

We note that a differentiable map F between two semi-Riemannian manifolds is called totally geodesic if $\nabla F_*=0$. Using Theorem 4.2 one can easily prove that the fibers are not totally geodesic. Hence we have the following Theorem.

Theorem 5.9. *Let $F:M(\phi,\xi,\eta,g_M,\varepsilon)\longrightarrow(N,g_N)$ be an anti-invariant semi-Riemannian submersion where $M(\phi,\xi,\eta,g_M,\varepsilon)$ is an indefinite trans-Sasakian manifold and (N,g_N) is a semi-Riemannian manifold . Then F is not totally geodesic map.*

Finally, we give a necessary and sufficient condition for an anti-invariant semi-Riemannian submersion such that $\phi(kerF_*)=(kerF_*)^\perp$ to be harmonic.

Theorem 5.10. *Let $F:M(\phi,\xi,\eta,g_M,\varepsilon)\longrightarrow(N,g_N)$ be an anti-invariant semi-Riemannian submersion such that $\phi(kerF_*)=(kerF_*)^\perp$, where $M(\phi,\xi,\eta,g_M,\varepsilon)$ is an indefinite trans-Sasakian manifold and (N,g_N) is a semi-Riemannian manifold . Then F is harmonic if and only if $Trace\phi\mathscr{T}_V=0$ for $V\in\Gamma(kerF_*)$.*

Proof. From [24] we know that F is harmonic if and only if F has minimal fibers. Thus F is harmonic if and only if $\sum_{i=1}^k\mathscr{T}_{e_i}e_i=0$, where $k=2m+1-n$ is dimension of $kerF_*$. On the other hand, from (2.5), (2.6) and (3.4) we get

$$\mathscr{T}_V\phi W=\phi\mathscr{T}_VW+\varepsilon\alpha(-\eta(W)V+g(V,W)\xi)+\varepsilon\beta(-\eta(W)\phi V+g(\phi V,W)\xi) \tag{5.13}$$

for any $W,V\in\Gamma((kerF_*)$. Using (4.14), we get

$$\sum_{i=1}^k g_M(\mathscr{T}_{e_i}\phi e_i,V)=-\sum_{i=1}^k g_M(\mathscr{T}_{e_i}e_i,\phi V)+\varepsilon\alpha(n-1)\eta(V)+\varepsilon\beta(\sum_{i=1}^k g_M(\phi e_i,e_i)\eta(V))-\sum_{i=1}^k g_M(e_i,\phi V))$$

for any $V\in\Gamma((kerF_*)$. (2.10) implies that

$$\sum_{i=1}^k g_M(\phi e_i,\mathscr{T}_{e_i}V)=-\sum_{i=1}^k g_M(\mathscr{T}_{e_i}e_i,\phi V)$$

$$+\varepsilon\alpha(n-1)\eta(V)+\beta(\sum_{i=1}^k g_M(\phi e_i,e_i)\eta(V))-\sum_{i=1}^k g_M(\phi e_i,V))$$

Then, using (2.3) we have

$$\sum_{i=1}^k g_M(\phi e_i,\mathscr{T}_Ve_i)=-\sum_{i=1}^k g_M(\mathscr{T}_{e_i}e_i,\phi V)$$

$$+\varepsilon\alpha(n-1)\eta(V)+\varepsilon\beta(\sum_{i=1}^k g_M(\phi e_i,e_i)\eta(V))-\sum_{i=1}^k g_M(\phi e_i,V)).$$

Hence, proof comes from (3.2).

□

6. Anti-invariant submersion admitting horizontal structure vector field

In this section, we will study anti-invariant submersion from an indefinite trans-Sasakian manifold onto a semi-Riemannian manifold such that the characteristic vector field ξ is horizontal. Using (4.1), we have $\mu = \phi\mu \oplus \{\xi\}$. For any horizontal vector field X we put

$$\phi X = BX + CX, \tag{6.1}$$

where $BX \in \Gamma(kerF_*)$ and $CX \in \Gamma(\mu)$.
Now we suppose that V is vertical and X is horizontal vector field. Using above relation and (3.2) we obtain

$$g_M(\phi V, CX) = 0.$$

From this last relation we have $g_M(F_*\phi V, F_*CX) = 0$ which implies that

$$TN = F_*(\phi kerF_*)) \oplus F_*(\mu). \tag{6.2}$$

Theorem 6.1. *Let $M(\phi,\xi,\eta,g_M,\varepsilon)$ be an indefinite trans-Sasakian manifold of dimension $2m+1$ and (N,g_N) is a semi-Riemannian manifold of dimension n. Let $F : M(\phi,\xi,\eta,g_M) \longrightarrow (N,g_N)$ be an anti-invariant semi-Riemannian submersion such that $(\phi(kerF_*) = (kerF_*{}^{\perp}) \oplus \{\xi\}$. Then $m+1 = n$.*

Proof. We assume that $U_1,...,U_k$ be an orthonormal frame of $(kerF_*)$, where $k = 2m-n+1$. Since $(\phi(kerF_*) = (kerF_*{}^{\perp}) \oplus \{\xi\}$, $\phi U_1,...,\phi U_k,\xi$ from an orthonormal frame of $\Gamma((kerF_*{}^{\perp}))$. So, by help of (4.3) we obtain $k = n-1$ which implies that $m+1 = n$. □

From(3.1) and (4.16) we obtain following Lemma.

Lemma 6.2. *Let F be an anti-invariant semi-Riemannian submersion from an indefinite trans-Sasakian manifold $M(\phi,\xi,\eta,g_M,\varepsilon)$ to a Riemannian manifold (N,g_N). Then we have*

$$BCX = 0,$$

$$\phi^2 X = \phi BX + C^2 X =,$$

for any $X \in \Gamma((kerF_*{}^{\perp}))$.

Proof. Using (3.4) one can esily obtain

$$\nabla_X Y = -\phi\nabla_X \phi Y + \eta(\nabla_X Y)\xi + \varepsilon\alpha\eta(Y)\phi X + \varepsilon\beta\eta(Y)X - \varepsilon\beta\eta(Y)\eta(X)\xi \tag{6.3}$$

for any $X,Y \in \Gamma((kerF_*{}^{\perp}))$. □

Lemma 6.3. *Let F be an anti-invariant semi-Riemannian submersion from an indefinite trans-Sasakian manifold $M(\phi,\xi,\eta,g_M,\varepsilon)$ to a semi-Riemannian manifold (N,g_N). Then we have*

$$BX = -\frac{\varepsilon}{\alpha}\mathscr{A}_X\xi, \tag{6.4}$$

$$\mathscr{T}_U\xi = \varepsilon\beta U, \tag{6.5}$$

$$g_M(\mathscr{A}_X\xi, \phi U) = 0, \tag{6.6}$$

$$g_M(\nabla_Y\mathscr{A}_X\xi, \phi U) = -g_M(\mathscr{A}_X\xi, \phi\mathscr{A}_Y U) - \varepsilon\beta\eta(U)g_M(\mathscr{A}_X\xi, \phi Y) \tag{6.7}$$

$$g_M(\nabla_X CY, \phi U) = -g_M(CY, \phi\mathscr{A}_X U) - \varepsilon\beta\eta(U)gM(CY\phi X) \tag{6.8}$$

for $X,Y \in \Gamma((kerF_*{}^{\perp}))$ and $U \in \Gamma((kerF_*)$.

Proof. By virtue of (2.8), (3.5) and (4.15) we have (4.18). Using (2.6) and (3.6) we obtain (4.19). Since $\mathscr{A}_X\xi$ is vertical and ϕU is horizontal for $X \in \Gamma((kerF_*{}^{\perp}))$ and $U \in \Gamma((kerF_*$, we have (4.20). Now using (4.20) we get

$$g_M(\nabla_Y\mathscr{A}_X\xi, \phi U) = -g_M(\mathscr{A}_X\xi, \nabla_Y\phi U)$$

for $X,Y \in \Gamma((kerF_*{}^{\perp}))$ and $U \in \Gamma((kerF_*)$. Then using (2.7) and (3.4) we have

$$g_M(\nabla_Y\mathscr{A}_X\xi, \phi U) = -g_M(\mathscr{A}_X\xi, \phi\mathscr{A}_Y U) - g_M(\mathscr{A}_X\xi, \phi(\mathscr{V}\nabla_Y U))$$

$$+\varepsilon\beta g_M(\mathscr{A}_X\xi, \xi)g_M(\phi Y, U) - \varepsilon\beta\eta(U)g_M(\mathscr{A}_X\xi, \phi Y).$$

Since $\phi(\mathscr{V}\nabla_Y U) \in \Gamma(kerF_*{}^{\perp})$, we obtain (4.21).
From (4.1) we get

$$g_M(CY, \phi U) = 0$$

$$0 = g_M(\nabla_X CY, \phi U) + g_M(CY, \nabla_X \phi U)$$

$$= g_M(\nabla_X CY, \phi U) + g_M(CY, \phi \nabla_X U)$$

$$g_M(\nabla_X CY, \phi U) = g_M(CY, \phi(\mathscr{A}_X U)) - \varepsilon\beta\eta(U) g_M(CY, \phi X).$$

Hence we obtain (4.22).

□

We now study the integrability of the distribution $(KerF_*)^{\perp}$ and then we investigate the geometry of leaves of $(KerF_*)$ and $(KerF_*)^{\perp}$.

Theorem 6.4. *Let F be an anti-invariant semi-Riemannian submersion from an indefinite trans-Sasakian manifold $M(\phi, \xi, \eta, g_M, \varepsilon)$ to a semi-Riemannian manifold (N, g_N). Then the following assertions are equivalent to each other;*

1. $(kerF_*)^{\perp}$ *is integrable,*
2. $g_N((\nabla F_*)(Y, \mathscr{A}_X\xi), F_*\phi V) = g_N((\nabla F_*)(X, \mathscr{A}_X\xi), F_*\phi V) + g_M(CX, \phi\mathscr{A}_Y V)$
 $-g_M(CY, \phi\mathscr{A}_X V) + \varepsilon\alpha(g_M(\mathscr{A}_Y\xi, V)\eta(Y) - g_M(\mathscr{A}_Y\xi, V))\eta(X)$
 $+\varepsilon\beta(g_M(\mathscr{A}_X\xi, \phi V)\eta(Y) - g_M(\mathscr{A}_Y\xi, \phi V)\eta(X)$
 $+\varepsilon\beta((g_M(\mathscr{A}_Y\xi, \phi Y) - g_M(\phi\mathscr{A}_Y\xi, \phi X))\eta(V)$
3. $g_M(\mathscr{A}_X\mathscr{A}_Y\xi - \mathscr{A}_Y\mathscr{A}_X\xi - \phi V) = g_M(CX, \phi\mathscr{A}_Y V) - g_M(CY, \phi\mathscr{A}_X V)$
 $+\varepsilon\alpha(g_M(\mathscr{A}_Y\xi, V)\eta(Y) - g_M(\mathscr{A}_Y\xi, V))\eta(X)$
 $+\varepsilon\beta(g_M(\mathscr{A}_X\xi, \phi V)\eta(Y) - g_M(\mathscr{A}_Y\xi, \phi V)\eta(X)$
 $+\varepsilon\beta((g_M(\mathscr{A}_Y\xi, \phi Y) - g_M(\phi\mathscr{A}_Y\xi, \phi X))\eta(V)$

for $X, Y \in \Gamma((kerF_*)^{\perp})$ and $V \in \Gamma((kerF_*)$.

Proof. From (4.15), (4.17) and (4.18) we have

$$g_M(\nabla_X Y, V) = g_M(\nabla_X CY, \phi V) - g_M(\nabla_X \mathscr{A}_Y\xi, \phi V) - \varepsilon\alpha(g_M(\mathscr{A}_Y\xi, V)\eta(Y) + \varepsilon\beta(g_M(\mathscr{A}_X\xi, \phi V)\eta(Y) \tag{6.9}$$

for $X, Y \in \Gamma((kerF_*)^{\perp})$ and $V \in \Gamma((kerF_*)$. Using (4.21) in (4.23) we obtain

$$g_M(\nabla_X Y, V) = g_M(\nabla_X CY, \phi V) - g_M(\mathscr{A}_Y\xi, \phi\mathscr{A}_X V) - \varepsilon\alpha(g_M(\mathscr{A}_Y\xi, V)\eta(Y)$$

$$+\varepsilon\beta(g_M(\mathscr{A}_X\xi, \phi V)\eta(Y) + \varepsilon\beta(g_M(\mathscr{A}_Y\xi, \phi X)\eta(V)$$

By help (4.21) and (4.22), the last relation becomes

$$g_M(\nabla_X Y, V) = g_M(CY, \phi\mathscr{A}_X V) - g_M(\nabla_X \mathscr{A}_Y\xi, \phi V) - \varepsilon\alpha(g_M(\mathscr{A}_Y\xi, V)\eta(Y)$$

$$+\varepsilon\beta(g_M(\mathscr{A}_X\xi, \phi V)\eta(Y) + \varepsilon\beta(g_M(\mathscr{A}_Y\xi, \phi X)\eta(V)$$

Interchanging the role of X and Y, we get

$$g_M(\nabla_Y X, V) = g_M(CX, \phi\mathscr{A}_Y V) - g_M(\nabla_Y \mathscr{A}_X\xi, \phi V) - \varepsilon\alpha(g_M(\mathscr{A}_X\xi, V)\eta(X)$$

$$+\varepsilon\beta(g_M(\mathscr{A}_Y\xi, \phi V)\eta(X) + \varepsilon\beta(g_M(\mathscr{A}_X\xi, \phi Y)\eta(V)$$

so that combining the above two relations, we have

$$g_M([X,Y]), V) = g_M(\nabla_Y \mathscr{A}_X\xi, \phi V) - g_M(\nabla_X \mathscr{A}_Y\xi, \phi V) + g_M(CX, \phi\mathscr{A}_Y V) - g_M(CY, \phi\mathscr{A}_X V)$$

$$+\varepsilon\alpha(g_M(\mathscr{A}_X\xi, V)\eta(Y) - \varepsilon\alpha(g_M(\mathscr{A}_Y\xi, V)\eta(Y)$$

$$+\varepsilon\beta(g_M(\mathscr{A}_X\xi, \phi V)\eta(Y) - \varepsilon\beta(g_M(\mathscr{A}_Y\xi, \phi V)\eta(X)$$

$$+\varepsilon\beta(g_M(\mathscr{A}_X\xi, \phi Y)\eta(V) - \varepsilon\beta(g_M(\mathscr{A}_Y\xi, \phi X)\eta(V).$$

Since differential F_* preserves the length of horizontal vectors we obtain

$$g_M([X,Y]), V) = g_N(F_*\nabla_Y \mathscr{A}_X\xi, F_*\phi V) - g_N(F_*\nabla_X \mathscr{A}_Y\xi, F_*\phi V) + g_M(CX, \phi\mathscr{A}_Y V) - g_M(CY, \phi\mathscr{A}_X V)$$

$$+\alpha(g_M(\mathscr{A}_X\xi, V)\eta(Y) - \varepsilon\alpha(g_M(\mathscr{A}_Y\xi, V)\eta(Y)$$

$$+\varepsilon\beta(g_M(\mathscr{A}_X\xi, \phi V)\eta(Y) - \varepsilon\beta(g_M(\mathscr{A}_Y\xi, \phi V)\eta(X)$$

$$+\varepsilon\beta(g_M(\mathscr{A}_X\xi, \phi Y)\eta(V) - \varepsilon\beta(g_M(\mathscr{A}_Y\xi, \phi X)\eta(V).$$

Using (2.12) we have

$$g_M([X,Y]), V) = g_N(-(\nabla F_*)(Y, \mathscr{A}_X\xi), F_*\phi V) - g_N(-(\nabla F_*)(X, \mathscr{A}_Y\xi), F_*\phi V)$$

$$+g_M(CX, \phi\mathscr{A}_Y V) - g_M(CY, \phi\mathscr{A}_X V)$$

$+\varepsilon\alpha(g_M(\mathscr{A}_X\xi,V)\eta(Y)-\varepsilon\alpha(g_M(\mathscr{A}_Y\xi,V)\eta(Y)$

$+\varepsilon\beta(g_M(\mathscr{A}_X\xi,\phi V)\eta(Y)-\varepsilon\beta(g_M(\mathscr{A}_Y\xi,\phi V)\eta(X)$

$+\varepsilon\beta(g_M(\mathscr{A}_X\xi,\phi Y)\eta(V)-\varepsilon\beta(g_M(\mathscr{A}_Y\xi,\phi X)\eta(V).$

which proves (1) $\Leftrightarrow$ (2).
On the other hand using (2.12) we get

$$(\nabla F_*)(Y,BX)-(\nabla F_*)(X,BY)=-F_*(\nabla_Y BX-\nabla_X BY). \tag{6.10}$$

Using (2.7) and (4.8) we obtain

$g_N(-F_*(\mathscr{A}_Y\mathscr{A}_X\xi-\mathscr{A}_X\mathscr{A}_Y\xi),F_*\phi V)=g_M(CX,\phi\mathscr{A}_Y V)-g_M(CY,\phi\mathscr{A}_X V)$

$+\varepsilon\alpha(g_M(\mathscr{A}_X\xi,V)\eta(Y)-\varepsilon\alpha(g_M(\mathscr{A}_Y\xi,V)\eta(X)$

$+\varepsilon\beta(g_M(\mathscr{A}_X\xi,\phi V)\eta(Y)-\varepsilon\beta(g_M(\mathscr{A}_Y\xi,\phi V)\eta(X)$

$+\varepsilon\beta(g_M(\mathscr{A}_X\xi,\phi Y)\eta(V)-\varepsilon\beta(g_M(\mathscr{A}_Y\xi,\phi X)\eta(V).$

which shows that (2) $\Leftrightarrow$ (3) □

Remark We assume that $(kerF_*)^\perp=\phi kerF_*\oplus\{\xi\}$. Using (4.15) one can prove that $CX=0$.

Theorem 6.5. *Let $M(\phi,\xi,\eta,g_M,\varepsilon)$ be an indefinite trans-Sasakian manifold of dimension $2m+1$ and (N,g_N) is a semi-Riemannian manifold of dimension n. Let $F:M(\phi,\xi,\eta,g_M,\varepsilon)\longrightarrow(N,g_N)$ be an anti-invariant semi-Riemannian submersion such that $(\phi(kerF_*)=(kerF_*{}^\perp)\oplus\{\xi\}$. Then $kerF_*{}^\perp)$ is not integrable.*

Proof. From (3.2) it follows that

$\phi(\nabla_X Y)=\nabla_X BY-\varepsilon(\alpha(g(X,Y)\xi-\eta(Y)X)-\varepsilon\beta(g(\phi Y,X)\xi-\eta(Y)\phi X)$

for $X,Y\in\Gamma((kerF_*)^\perp)$. Interchanging the role of X and Y, we get

$\phi(\nabla_Y X)=\nabla_Y BX-\varepsilon(\alpha(g(X,Y)\xi-\eta(X)Y)-\varepsilon\beta(g(\phi Y,X)\xi-\eta(X)\phi Y)$

so that combining the above two relations, we have

$\phi([X,Y])=\nabla_X BY-\nabla_Y BX+\varepsilon\alpha(\eta(Y)X-\eta(X)Y)+\varepsilon\beta(\eta(Y)\phi X-\eta(X)\phi Y).$

Using (2.7), (3.2), (4.18) and (3.4) one obtain

$\phi([X,Y])=\mathscr{A}_X BY-\mathscr{A}_Y BX+\mathscr{V}\nabla_X BY-\mathscr{V}\nabla_Y BX+\varepsilon\alpha(\eta(Y)X-\eta(X)Y)+\varepsilon\beta(\eta(Y)\phi X-\eta(X)\phi Y).$

If $((kerF_*)^\perp)$ is integrable we have

$\varepsilon\alpha(\eta(Y)X-\eta(X)Y)+\varepsilon\beta(\eta(Y)\phi X-\eta(X)\phi Y)=\mathscr{A}_X\mathscr{A}_Y\xi-\mathscr{A}_Y\mathscr{A}_X\xi$

On the other hand, we know that if $\mathscr{H}=((kerF_*)^\perp)$ is integrable then $\mathscr{A}=0$. Hence the last relation led to the contradiction with (3.4). □

From (2.8) and (3.6), we can give following Theorem.

Theorem 6.6. *Let $M(\phi,\xi,\eta,g_M,\varepsilon)$ be an indefinite trans-Sasakian manifold of dimension $2m+1$ and (N,g_N) is a semi-Riemannian manifold of dimension n. Let $F:M(\phi,\xi,\eta,g_M,\varepsilon)\longrightarrow(N,g_N)$ be an anti-invariant semi-Riemannian submersion such that $(\phi(kerF_*)\subset(kerF_*{}^\perp)$. Then $kerF_*{}^\perp)$ does not define a totally geodesic foliation on M.*

For the distribution $kerF_*$ we have;

Theorem 6.7. *Let F be an anti-invariant semi-Riemannian submersion from an indefinite trans-Sasakian $M(\phi,\xi,\eta,g_M,\varepsilon)$ to a semi-Riemannian manifold (N,g_N). Then the following assertions are equivalent to each other:*

1. *$(kerF_*)$ define a totally geodesic foliation on M.*
2. *$g_N((\nabla F_*)(V,\phi X),F_*\phi W)=0$ for $X\in\Gamma((kerF_*{}^\perp))$ and $V,W\in\Gamma((kerF_*)$.*
3. *$\mathscr{T}_V BX+\mathscr{A}_{CX}V\in\Gamma(\mu)$*

Proof. Since $g_M(W,X)=0$ we have $g_M(\nabla_V W,X)=0=g_M(W,\nabla_V X)=0$. From (3.2)and (4.15) we get

$g_M(\nabla_V W,X)=g_M(\phi W,\nabla_V BX)-g_M(\phi W,\nabla_V CX)$.

Using (2.5) and (2.6) we obtain $g_M(\nabla_V W,X)=g_M(\phi W,\nabla_V \phi X)$. Then semi-Riemannian submersion F (2.12) imply that

$g_M(\nabla_V W,X)=g_M(F_*\phi W,(\nabla F_*)(V\phi X))$

which is (1) $\Leftrightarrow$ (2). By direct calculation, we derive

$g_M(F_*\phi W,(\nabla F_*)(V\phi X))=-g_M(\phi W,\nabla_V \phi X)$.

Using (4.15) we have

$g_M(F_*\phi W,(\nabla F_*)(V\phi X))=-g_M(\phi W,\nabla_V BX+\nabla_V CX)$.

Hence we get

$g_M(F_*\phi W,(\nabla F_*)(V\phi X))=-g_M(\phi W,\nabla_V BX+[V,CX]+\nabla_{CX}V)$.

Since $[V,CX]\in\Gamma(kerF_*)$, using (2.5) and (2.7), we obtain

$g_M(F_*\phi W,(\nabla F_*)(V\phi X))=-g_M(\phi W,\mathscr{T}_V BX+\mathscr{A}_{CX}V)$.

This shows (2) $\Leftrightarrow$ (3). □

Lemma 6.8. *Let $M(\phi,\xi,\eta,g_M,\varepsilon)$ be be an anti-invariant semi-Riemannian submersion such that $(\phi(kerF_*)^{\perp})=(kerF_*\oplus\{\xi\}$, where $M(\phi,\xi,\eta,g_M)$ is an indefinite trans-Sasakian manifold and (N,g_N) is a semi-Riemannian manifold. Then following assertions are equivalent to each other;*

1. *$(kerF_*)$ define a totally geodesic foliation on M.*
2. *$(\nabla F_*)(V,\phi X)=0$ for $X\in\Gamma((kerF_*{}^{\perp}))$ and $V,W\in\Gamma((kerF_*)$.*
3. *$\mathscr{T}_V\phi W=0$.*

Theorem 6.9. *Let $F:M(\phi,\xi,\eta,g_M,\varepsilon)\longrightarrow(N,g_N)$ be an anti-invariant semi-Riemannian submersion such that $(\phi(kerF_*)^{\perp})=(kerF_*\oplus\{\xi\}$, where $M(\phi,\xi,\eta,g_M,\varepsilon)$ is an indefinite trans Sasakian manifold and (N,g_N) is a semi-Riemannian manifold. Then F is totally geodesic map if and only if*

$$\mathscr{T}_V\phi V=0,\forall V,W\in\Gamma((kerF_*) \tag{6.11}$$

and

$$\mathscr{A}_X\phi W=0,\forall X\in\Gamma((kerF_*{}^{\perp}),\forall W\in\Gamma((kerF_*). \tag{6.12}$$

Proof. First of all, we recall that the second fundamental form of a semi-Riemannian submersion satisfies (2.13). For $W,V\in\Gamma((kerF_*)$, by using (2.6), (2.12) and (3.3) we get

$$(\nabla F_*)(W,V)=-F_*(\phi\mathscr{T}_W\phi V). \tag{6.13}$$

On the other hand by using (2.12) and (3.3) we have

$$(\nabla F_*)(X,W)=-F_*(\phi\nabla X_\phi W). \tag{6.14}$$

for $X\in\Gamma((kerF_*{}^{\perp})$. Then from (2.8), we obtain

$(\nabla F_*)(X,W)=F_*(\phi\mathscr{A}_X\phi W-\alpha g(W,\phi X)\xi-\beta g(\phi X,\phi W)\xi)$.

Since ϕ is non-singular, proof comes from (4.27), (4.28) and (2.13).

□

Finally, we give a necessary and sufficient condition for an anti-invariant semi-Riemannian submersion such that $((kerF_*)^{\perp})=\phi(kerF_*\oplus\{\xi\}$ to be harmonic.

Theorem 6.10. *Let $F:M(\phi,\xi,\eta,g_M,\varepsilon)\longrightarrow(N,g_N)$ be an anti-invariant semi-Riemannian submersion such that $(\phi(kerF_*)^{\perp})=(kerF_*\oplus\{\xi\}$, where $M(\phi,\xi,\eta,g_M)$ is an indefinite trans-Sasakian manifold and (N,g_N) is a semi-Riemannian manifold. Then F is harmonic if and only if $Trace\phi\mathscr{T}_V=0$ for $V\in\Gamma((kerF_*)$.*

Proof. From[ ] we know that F is harmonic if and only if F has minimal fibers. Thus F is harmonic if and only if $\sum_{i=1}^{k}\mathscr{T}_{e_i}e_i=0$, where k is dimension of $kerF_*$. On the other hand, from (2.5), (2.6) and (3.4) we get

$$\sum_{i=1}^{k}g_M(\mathscr{T}_{e_i}\phi e_i,V)=-\sum_{i=1}^{k}g_M(\mathscr{T}_{e_i}e_i,\phi V) \tag{6.15}$$

for any $V\in\Gamma((kerF_*)$. (2.10) implies that

$$\sum_{i=1}^{k}g_M(\phi e_i,\mathscr{T}_{e_i}V)=\sum_{i=1}^{k}g_M(\mathscr{T}_{e_i}e_i,\phi V)$$

Then, using (2.3) we have

$$\sum_{i=1}^{k}g_M(\phi e_i,\mathscr{T}_V e_i)=\sum_{i=1}^{k}g_M(\mathscr{T}_{e_i}e_i,\phi V)$$

Hence, proof comes from (3.2). □

Example 6.11. *Let $\overline{\mathbb{R}}^5$ be a five-dimensional Euclidean space given by*

$\overline{\mathbb{R}}^5 = \{(x,y,z,u,v) \in \mathbb{R}^5 \mid (x,y) \neq (0,0),\ (u,v) \neq (0,0)\ and z \neq 0\}.$

The vector fields

$$E_1 = 2(-\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}), E_2 = 2\frac{\partial}{\partial y}, E_3 = 2\frac{\partial}{\partial z}, E_4 = 2(-\frac{\partial}{\partial u} + v\frac{\partial}{\partial z}), E_5 = 2\frac{\partial}{\partial v} \quad .$$

are linearly independent at each point of $\overline{\mathbb{R}}^5$. Then, we can choose an indifinite trans-Sasakian structure $(\varphi,\xi,\eta,g,\varepsilon)$ on $\overline{\mathbb{R}}^5$ such as $\xi = E_3,\ \eta = \frac{\varepsilon}{2}dz,\ g$ is defined by $g(E_i,E_j) = \varepsilon\delta_i^j$ and φ is defined by as follows:

$$\varphi_\varepsilon E_1 = \varepsilon E_2,\ \varphi_\varepsilon E_2 = -\varepsilon E_1,\ \varphi_\varepsilon E_3 = 0,\ \varphi_\varepsilon E_4 = \varepsilon E_5,\ \varphi_\varepsilon E_5 = -\varepsilon E_4 \quad .$$

Indeed, $(\varphi,\xi,\eta,g,\varepsilon)$ is an indefinite trans-Sasakian structure on $\overline{\mathbb{R}}^5$ with $\alpha = -1$ and $\beta = 1$, and $\varepsilon = \pm 1$.

Now, we consider the map $\pi : (\overline{\mathbb{R}}^5,\varphi,\xi,\eta,g) \to (\mathbb{R}^3,g_3)$ defined by the following:

$$\pi(x,y,z,u,v) = \left(\frac{x-y}{\sqrt{2}}, \frac{u-v}{\sqrt{2}}, z\right) \quad ,$$

where g_3 is the Euclidean metric on $\mathbb{R}^3$. Then, the Jacobian matrix of π is as follows:

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since the rank of this matrix is equal to 3, the map π is a submersion. Secondly, we easily see that π satisfies the condition **S2)**. Therefore, π is a semi-Riemannian submersion. After some computations, we have

$$ker\pi_* = span\left\{V = \frac{E_1+E_2}{\sqrt{2}},\quad W = \frac{E_4+E5}{\sqrt{2}}\right\} \quad ,$$

and

$$ker\pi_*^\perp = span\left\{X = \frac{E_1-E_2}{\sqrt{2}},\quad Y = \frac{E_4-E_5}{\sqrt{2}},\quad \xi\right\} \quad .$$

In addition, we have $\varphi(V) = -X$ and $\varphi(W) = -Y$. Hence, we see that π is an anti-invariant submersion admitting horizontal Reeb vector field.

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