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On Vectorial Moment of the Darboux Vector

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Abstract

In this paper we define a new curve denoted by (c*). It is well known that any regular curve can be written by means of Frenet vectors and also via the vectorial moments. In a space we know a regular curve moves around an instantaneous rotation vector called as the Darboux vector. In this study we are interested in a curve plotted by the vectorial moment of the unit Darboux vector. The curve on which we worked generated by the vectorial moment of the unit Darboux vector satisfying the following condition that the curve is created by the vectorial moment of the unit Darboux vector whose components are of the Frenet vectors of a regular curve in Euclidean 3-space. We use c* to denote the vectorial moment vector of the unit Darboux vector and also c to denote the unit Darboux vector. We show that the new curve (c*) doesn't form a constant width curve pairs with the main curve. Then we calculate the Frenet apparatus of the regular curve (c*), drawn by the vectorial moment vector of c*. Also we point out that this new curve (c*) can be expressed as a linear combination of Frenet vectors. Further we assert that the principle normal and binormal of the curve (c*) doesn't form a constant width curve pairs with the main curve. Finally we draw a conclusion and compute the Frenet apparatus of the curve (c*) when the main curve is supposed to be an helix.

Keywords: *Darboux vector, vectorial moment, constant width curve pair, regular curve.*
2010 Mathematics Subject Classification: *53A04*

1. Introduction

We know that lots of studies have been made on curves in differential geometry, [1, 2, 3, 4, 5, 6]. Frenet frame is up to now examined in different spaces and so many characterizations are obtained. It is also demonstrated that any regular curve can be written by means of Frenet vectors and vectorial moment of a regular curve in some resent papers. In the present work we focus on the curve plotted by the vectorial moment c^* of the unit Darboux vector c.

2. Preliminaries

Let $\alpha : I \subseteq R \to E^3$ be a unit speed regular curve.Then the Frenet apparatus and Frenet formulas of the curve (α) are given, respectively as , $(see$ [10, 11]$)$

$$\begin{cases} T(s) = \alpha'(s) \quad , \quad N(s) = \frac{\alpha'(s)}{\|\alpha''(s)\|} \quad , \quad B(s) = T(s) \wedge N(s) \quad , \\ \\ \kappa(s) = \|\alpha''(s)\| \quad , \quad \tau(s) = \frac{det(\alpha'(s), \alpha''(s), \alpha'''(s))}{(\|\alpha'(t) \wedge \alpha''(t)\|)^2} \, , \end{cases} \tag{2.1}$$

$$T' = \kappa N, \quad N' = -\kappa T + \tau B, \quad B' = -\tau N. \tag{2.2}$$

In Euclidean 3-space a regular curve $\alpha(s)$ depending on the Frenet vectors moves around the axis of Darboux vector W and is given by , $(see$ [9]$)$

$$W = \tau T + \kappa B. \tag{2.3}$$

It follows that the unit Darboux vector c is defined as

$$c = \sin\theta T + \cos\theta B \tag{2.4}$$

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where θ is the angle between Darboux vector W and binormal vector B of the curve α, given by , (*see* [9])

$$\sin\theta = \frac{\tau}{\sqrt{\kappa^2+\tau^2}}\,, \quad \cos\theta = \frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\,. \tag{2.5}$$

Let $\alpha(s)$ be a regular curve and $\vec{x^*}$ be a vectorial moment vector of the vector $\vec{x}$ lying on this curve. Then the vector $\vec{x^*}$ is given as, (*see* [12])

$$\vec{x^*} = \vec{\alpha}\wedge\vec{x}. \tag{2.6}$$

According to above statement we can give the vectorial moment vectors of the Frenet vectors as follows , (*see* [12])

$$T^* = \alpha\wedge T,\ N^* = \alpha\wedge N,\ B^* = \alpha\wedge B.$$

In the Euclidean R^n-space let (α) and (α^*) be two unit speed curves from C^n-class. Let us consider two corresponding points of these curves, say $\alpha(s)=p$ and $\alpha^*(s)=p^*$, such that the tangent vectors of the curves are parallel and having opposite directions with constant distance at these points. Then the curves, defined here, constant-width curve pairs and denoted by $\{\alpha,\alpha^*\}$, (*see* [7]).

3. Vectorial Moment of a Darboux Vector

In this section we calculate the Frenet apparatus of the regular curve drawn by the vectorial moment vector of c^* where c^* is the vectorial moment of the unit Darboux vector of a regular curve α. As a linear combination of Frenet vectors, we can express the curve α in the following way , (*see* [8])

$$\alpha(s) = f(s)T(s)+g(s)N(s)+h(s)B(s) \tag{3.1}$$

with the derivatives of f,g,h

$$f^{'}(s) = 1+g(s)\kappa(s), \quad g^{'}(s) = h(s)\tau(s)-f(s)\kappa(s), \quad h^{'}(s) = -g(s)\tau(s)\,. \tag{3.2}$$

Theorem 3.1. *Let $\alpha=\alpha(s)$ be a regular curve in E^3 with the unit Darboux vector $\vec{c}$.Then the Frenet apparatus of the curve (c^*) generated by the vectorial moment vector $\vec{c^*}$ of the unit Darboux vector $\vec{c}$ are given as*

$$T_{c^*} = -mg\sin\theta T + muN - mg\cos\theta B,$$

$$\begin{aligned} N_{c^*} = m\gamma\Big\{&\big(u(Z-(g\theta^{'})^2\cos\theta\sqrt{\kappa^2+\tau^2})+g^3(\theta^{'})^4\cos\theta\big)T \\ &+\big(g\theta^{'}(u(g\theta^{'})^{'}-gu^{'}\theta^{'})\big)N \\ &-\big(u(X+(g\theta^{'})^2\sin\theta\sqrt{\kappa^2+\tau^2})+g^3(\theta^{'})^4\sin\theta\big)B\Big\}, \end{aligned}$$

$$B_{c^*} = \gamma XT+\gamma YN+\gamma ZB,$$

$$\kappa_{c^*} = m^3\sqrt{X^2+Y^2+Z^2},$$

$$\begin{aligned} \tau_{c^*} = \gamma^2\Big\{&(u^{''}-u(\kappa^2+\tau^2)-g(\theta^{'})^2\sqrt{\kappa^2+\tau^2})(gu\theta^{'}\sqrt{\kappa^2+\tau^2}+g^2(\theta^{'})^3) \\ &+u(\theta^{'}u+2\tau u^{'}-(g\theta^{'}\cos\theta)^{''})(\kappa u+(g\theta^{'}\sin\theta)^{'})-u(\kappa^{'}u+2\kappa u^{'}+(g\theta^{'}\sin\theta)^{''}) \\ &-gu^{'}\theta^{'}(u\kappa^{'}\cos\theta+u\tau^{'}\sin\theta+2u^{'}\sqrt{\kappa^2+\tau^2})+2g^{'}(\theta^{'})^2-g\theta^{'}\theta^{''}\Big\} \end{aligned}$$

with the coefficients A, u, m, γ, X, Y and Z

$$A = h\cos\theta+f\sin\theta\,,\ u = A\theta^{'}-\cos\theta\,,\ m = \frac{1}{\sqrt{(g\theta^{'})^2+u^2}}\,,\ \gamma = \frac{1}{\sqrt{X^2+Y^2+Z^2}}\,,$$

$$X = \tau u^2+gu^{'}\theta^{'}\cos\theta-u(g\theta^{'}\cos\theta)^{'},\ Y = g^2(\theta^{'})^3+gu\theta^{'}\sqrt{\kappa^2+\tau^2}\ \ \textit{and}$$

$$Z = \kappa u^2+u(g\theta^{'}\sin\theta)^{'}-gu^{'}\theta^{'}\sin\theta\,.$$

Proof. From equ.(2.6), vectorial moment $\vec{c^*}$ can be evaluated as

$$\vec{c^*} = \vec{\alpha}\wedge\vec{c},$$

$$\vec{c^*} = (f(s)T(s)+g(s)N(s)+h(s)B(s))\wedge(\sin\theta T+\cos\theta B)\,, \tag{3.3}$$

$$\vec{c^*} = g(s)\cos\theta T(s)+(h(s)\sin\theta-f(s)\cos\theta)N(s)-g(s)\sin\theta B(s)\,.$$

If we take the derivative of c^* and put into the equ.(2.2) and (3.2) we get

$$(c^*)' = -g\theta' \sin\theta T + (\theta'(h\cos\theta + f\sin\theta) - \cos\theta)N - g\theta' \cos\theta B \tag{3.4}$$

If we suppose $A = h\cos\theta + f\sin\theta$ we have

$$(c^*)' = -g\theta' \sin\theta T + (A\theta' - \cos\theta)N - g\theta' \cos\theta B.$$

After processing norm, we figure out

$$\begin{aligned}\|(c^*)'\| &= \sqrt{(-g\theta' \sin\theta)^2 + (A\theta' - \cos\theta)^2 + (-g\theta' \cos\theta)^2}\\ &= \sqrt{(g\theta')^2 + (A\theta' - \cos\theta)^2}.\end{aligned}$$

It follows that the tangent vector T_{c^*} of the curve (c^*) is

$$T_{c^*} = \frac{(c^*)'}{\|(c^*)'\|} = \frac{-g\theta' \sin\theta T + (A\theta' - \cos\theta)N - g\theta' \cos\theta B}{\sqrt{(g\theta')^2 + (A\theta' - \cos\theta)^2}}.$$

If we substitute

$$m = \frac{1}{\sqrt{(g\theta')^2 + (A\theta' - \cos\theta)^2}} \tag{3.5}$$

into above equality then we obtain the tangent vector T_{c^*} as

$$T_{c^*} = -mg\theta' \sin\theta T + m(A\theta' - \cos\theta)N - mg\theta' \cos\theta \tag{3.6}$$

Now we again take the derivative of $(c^*)'$ from (3.4) we have

$$(c^*)'' = [\kappa(\cos\theta - A\theta') - (g\theta' \sin\theta)']T + (A\theta' - \cos\theta)'N + [\tau(A\theta' - \cos\theta) - (g\theta' \cos\theta)']B \tag{3.7}$$

If we substitute u for the expression $A\theta - \cos\theta$, that is,

$$u = A\theta' - \cos\theta \tag{3.8}$$

then equations (3.4) and (3.7) can be rewritten as

$$\begin{aligned}(c^*)' &= -g\theta' \sin\theta T + uN - g\theta' \cos\theta B\\ (c^*)'' &= (-\kappa u - (g\theta' \sin\theta)')T + u'N + (\tau u - (g\theta' \cos\theta)')B\end{aligned}$$

when we apply the vectorial product of $(c^*)'$ and $(c^*)''$ we get

$$\begin{aligned}(c^*)' \wedge (c^*)'' = \Big(&\tau u^2 + gu'\theta' \cos\theta - u(g\theta' \cos\theta)', g^2(\theta')^3 + gu\theta'\sqrt{\kappa^2+\tau^2},\\ &\kappa u^2 + u(g\theta' \sin\theta)' - gu'\theta' \sin\theta\Big).\end{aligned}$$

When we take the norm of this product we have

$$\|(c^*)' \wedge (c^*)''\| = \left\{(\tau u^2 + gu'\theta' \cos\theta - u(g\theta' \cos\theta)')^2 + \left(g^2(\theta')^3 + gu\theta'\sqrt{\kappa^2+\tau^2}\right)^2 + (\kappa u^2 + u(g\theta' \sin\theta)' - gu'\theta' \sin\theta)^2\right\}^{1/2}.$$

We may apply some compressions as

$$\begin{aligned}X &= \tau u^2 + gu'\theta' \cos\theta - u(g\theta' \cos\theta)',\\ Y &= g^2(\theta')^3 + gu\theta'\sqrt{\kappa^2+\tau^2},\\ Z &= \kappa u^2 + u(g\theta' \sin\theta)' - gu'\theta' \sin\theta.\end{aligned}$$

Now we can write binormal vector B_{c^*} as

$$B_{c^*} = \frac{(c^*)' \wedge (c^*)''}{\|(c^*)' \wedge (c^*)''\|} = \frac{XT + YN + ZB}{\sqrt{X^2+Y^2+Z^2}}.$$

If we substitute γ for

$$\gamma = \frac{1}{\sqrt{X^2+Y^2+Z^2}} \tag{3.9}$$

then the binormal vector B_{c^*} can be written as

$$B_{c^*} = \gamma XT + \gamma YN + \gamma ZB. \tag{3.10}$$

From (3.6) and (3.10) we can figure out the normal vector of the curve (c^*) in the following way

$$N_{c^*} = B_{c^*} \wedge T_{c^*}$$

$$N_{c^*} = m\gamma\Big\{\big(u(Z-(g\theta^{'})^2\cos\theta\sqrt{\kappa^2+\tau^2}) + g^3(\theta^{'})^4\cos\theta\big)T + \big(g\theta^{'}(u(g\theta^{'})^{'} - gu^{'}\theta^{'})\big)N - \big(u(X+(g\theta^{'})^2\sin\theta\sqrt{\kappa^2+\tau^2}) + g^3(\theta^{'})^4\sin\theta\big)B\Big\}.$$

By the same process from (3.5) and (3.9) we can determine the curvature κ_{c^*} of the curve (c^*) as

$$\kappa_{c^*} = m^3\{X^2+Y^2+Z^2\}^{1/2}.$$

By making use of the third derivative of (c^*), we take the derivative of (3.7),

$$(c^*)^{'''} = -(\kappa^{'}u + 2\kappa u^{'} + (g\theta^{'}\sin\theta)^{''})T + (u^{''} - u(\kappa^2+\tau^2) - g(\theta^{'})^2\sqrt{\kappa^2+\tau^2})N + (\tau^{'}u + 2\tau u^{'} - (g\theta^{'}\cos\theta)^{''})B \tag{3.11}$$

It remains only to find the torsion τ_{c^*} of the curve (c^*). We know that

$$\tau_{c^*} = \frac{det((c^*)^{'},(c^*)^{''},(c^*)^{'''})}{\|(c^*)^{'}\wedge(c^*)^{'}\|^2}$$

From equalities (3.4), (3.7), (3.11) and (3.9) we can write

$$\begin{aligned} det((c^*)^{'},(c^*)^{''},(c^*)^{'''}) = &\Big[-gu^{'}\theta^{'}\big(u(\cos\theta\kappa^{'}+\sin\theta\tau^{'}) + 2u^{'}\sqrt{\kappa^2+\tau^2}\big) + 2g^{'}(\theta^{'})^2 - g\theta^{'}\theta^{''}\Big] \\ &+ \Big[\big(u^{''} - u(\kappa^2+\tau^2) - g(\theta^{'})^2\sqrt{\kappa^2+\tau^2}\big)\big(g\theta^{'}u\sqrt{\kappa^2+\tau^2} + g^2(\theta^{'})^3\big)\Big] \\ &+ u\Big[\big(\tau^{'}u + 2\tau u^{'} - (g\theta^{'}\sin\theta)^{''}\big)\big(\kappa u + (g\theta^{'}\sin\theta)^{'}\big) \\ &- \big(\kappa^{'}u + 2\kappa u^{'} + (g\theta^{'}\cos\theta)^{''}\big)\big(\tau u + (g\theta^{'}\cos\theta)^{'}\big)\Big] \end{aligned}$$

and

$$\|(c^*)^{'}\wedge(c^*)^{'}\|^2 = \frac{1}{\gamma^2}.$$

Considering the equalities above together we obtain

$$\begin{aligned} \tau_{c^*} = \gamma^2\Big\{&\big(u^{''} - u(\kappa^2+\tau^2) - g(\theta^{'})^2\sqrt{\kappa^2+\tau^2}\big)\big(g\theta^{'}u\sqrt{\kappa^2+\tau^2} + g^2(\theta^{'})^3\big) \\ &+ u\big(\tau^{'}u + 2\tau u^{'} - (g\theta^{'}\sin\theta)^{''}\big)\big(\kappa u + (g\theta^{'}\sin\theta)^{'}\big) \\ &- u\big(\kappa^{'}u + 2\kappa u^{'} + (g\theta^{'}\cos\theta)^{''}\big)\big(\tau u + (g\theta^{'}\cos\theta)^{'}\big) \\ &- gu^{'}\theta^{'}\big(u(\cos\theta\kappa^{'}+\sin\theta\tau^{'}) + 2u^{'}\sqrt{\kappa^2+\tau^2}\big) + 2g^{'}(\theta^{'})^2 - g\theta^{'}\theta^{''}\Big\}, \end{aligned}$$

so the proof is completed. □

Theorem 3.2. *Let (α) be a regular curve with the normal vector N.Then we claim that the curve (N^*) generated by the vectorial moment vector N^* of the normal vector N doesn't form a constant width curve pairs with the main curve (α).*

Proof. By making use of equ.(3.1) we can write the vectorial moment vector N^* as

$$N^* = \alpha\wedge N = -h(s)T(s) + f(s)B(s). \tag{3.12}$$

From figure(1) we have

$$\overrightarrow{\alpha N^*} = m_1T + m_2N + m_3B$$

and it follows

$$\begin{aligned} \vec{N^*} - \vec{\alpha} &= m_1T + m_2N + m_3B \\ -hT + fB - fT - gN - hB &= m_1T + m_2N + m_3B \end{aligned}$$

From the equalities given above it is obvious that

$$m_1 = -(h+f),\ \ m_2 = -g,\ \ m_3 = -(h-f).$$

If we put these values into (3.12) we get

$$N^* = \alpha - (h+f)T - gN - (h-f)B. \tag{3.13}$$

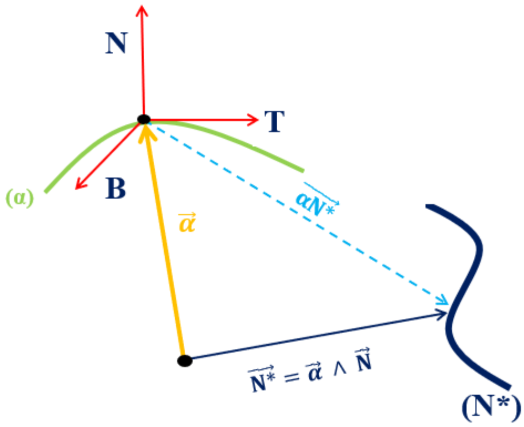

Figure 3.1: curve generated by N^*

We see that (3.12) and (3.13)make sense if and only if $N^* \in Sp\{T,B\}$. Hence we obtain $g = 0$ and $\tau = 0$. Since we have $g = 0$ and $\tau = 0$ then the equalities (3.1) and (3.13) transform into the followings

$$\alpha = fT + hB \quad ,$$
$$N^* = \alpha - (h+f)T - (h-f)B$$

and we find out

$$f = s + c_1, \quad f = 0, \quad h = c_2.$$

Finally we obtain $f = 0$ and $f = s + c_1$, that is, we get a contradiction. Therefore the curves (α) and (N^*) don't belong to the constant width curve pairs and this completes the proof. □

Theorem 3.3. *Let (α) be a regular curve with the binormal vector B. Then we claim that the curve (B^*) generated by the vectorial moment vector B^* of the binormal vector B doesn't form a constant width curve pairs with the main curve (α).*

Proof. By making use of equ.(3.1) we may write the vectorial moment vector B^* as

$$B^* = \alpha \wedge B = g(s)T(s) - f(s)N(s)\,. \tag{3.14}$$

From figure (2) we have

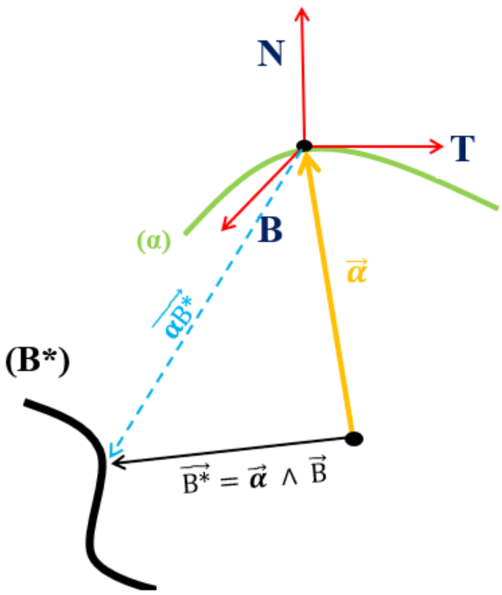

Figure 3.2: curve generated by B^*

$$\overrightarrow{\alpha B^*} = m_1 T + m_2 N + m_3 B$$

and it follows that

$$\vec{B^*}-\vec{\alpha}=m_1T+m_2N+m_3B,$$
$$gT-fN-fT-gN-hB=m_1T+m_2N+m_3B.$$

From the above equalities it is clear that

$m_1=g-f,\ m_2=-(g+f),\ m_3=-h.$

If we put these values into (3.14) we get

$$B^*=\alpha+(g-f)T-(g+f)N-hB \tag{3.15}$$

From equalities (3.14) and (3.15) we obtain $h=0$ and $\tau=0$. Hence the vectors α and B^* can be written as

$$\alpha=fT+gN \quad ,$$
$$B^*=\alpha+(g-f)T-(g+f)N$$

and it gives us

$f^{'}=1+g\kappa \quad and \quad g^{'}=-f\kappa.$

We have two cases to investigate, that is, $g=0$ or $g\neq 0$.

Case 1. If $g=0$, then we have

$$\begin{aligned} g=0 &\Rightarrow f\kappa=0, \kappa\neq 0\\ &\Rightarrow f=0\\ &\Rightarrow f^{'}=1+g\kappa\\ &\Rightarrow f^{'}=1\\ &\Rightarrow f=s+c,\ c\in R \end{aligned}$$

It follows that $f=0$ and $f=s+c$, that is, we get a contradiction.
Case 2. If $g\neq 0$, then we have

$$\begin{aligned} g\neq 0 &\Rightarrow \kappa=\frac{-g}{f}\\ &\Rightarrow \kappa=\frac{f^{'}-1}{g} \end{aligned}$$

It follows that

$gg^{'}+ff^{'}=1.$

On the other hand we also have

$$\begin{aligned} g\neq 0 &\Rightarrow g=\frac{f^{'}-1}{\kappa}\\ &\Rightarrow g=-f\kappa \end{aligned}$$

After processing some straightforward computations, it is seen that f' is a constant while g' is not. Therefore the distance between the curves (α) and (B^*) is

$d(\alpha,B^*)=\|\overrightarrow{\alpha B^*}\|^2=2g^2+2f^2\neq const.$

We deduce from this expression that the curves (α) and (B^*) don't belong to the constant-width curve pairs.
It is worth noting that Theorem 3.2 and Theorem 3.3 are mentioned in one of the papers entitled "Vectorial moments of curves in Euclidean 3-space"[12]. So the proof is completed. □

Theorem 3.4. *Let (α) be a regular curve with the unit Darboux vector c.Then we assert that the curve (c^*) generated by the vectorial moment vector c^* of the unit Darboux c doesn't form a constant-width curve pairs with the main curve (α).*

Proof. By making use of equ.(3.3) we can write $\vec{c^*}$ as

$$\vec{c^*} \quad = \quad g(s)\cos\theta T(s)+(h(s)\sin\theta-f(s)\cos\theta)N(s)-g(s)\sin\theta B$$

From figure (3) we have

$\overrightarrow{\alpha c^*}=m_1T+m_2N+m_3B$

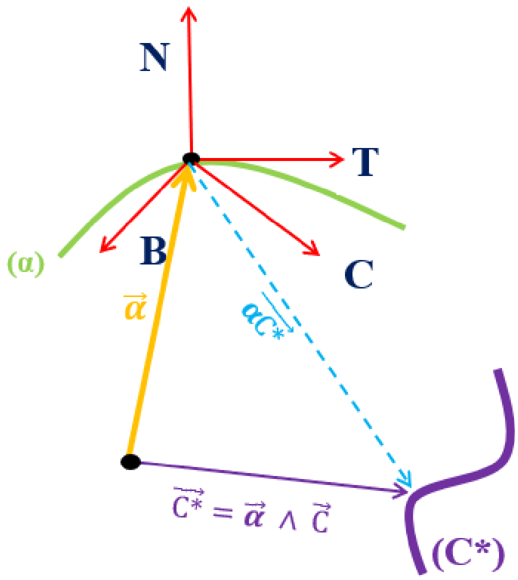

Figure 3.3: curve generated by c^*

and it follows that

$$m_1T + m_2N + m_3B = \vec{c^*} - \vec{\alpha}$$
$$m_1T + m_2N + m_3B = g\cos\theta T + (h\sin\theta - f\cos\theta)N - g\sin\theta B - fT - gN - hB$$

From the above equalities it is obvious that

$$m_1 = g\cos\theta - f, \; m_2 = h\sin\theta - f\cos\theta - g, \; m_3 = -g\sin\theta - h$$

Putting these values into (3.3) we get

$$c^* = \alpha + (g\cos\theta - f)T + (h\sin\theta - f\cos\theta - g)N - (g\sin\theta + h)B \quad .$$

Now let's look at the distance between the points $\alpha(s)$ and $c^*(s)$,

$$d(\alpha, c^*) = \|\overrightarrow{\alpha c^*}\| = \sqrt{2g^2 + f^2 + h^2 + (h\sin\theta - f\cos\theta)^2} \quad .$$

Definitely this distance can't be constant. Hence the curve pairs $\{\alpha, c^*\}$ don't form a constant-width curve pairs with the main curve (α).So the proof is completed. □

Corollary 3.5. *Let's consider that the main curve α be a general helix. Then we can draw a conclusion that the Frenet apparatus of the curve $c^*(s)$ are*

$$T_{c^*} = -N,$$

$$N_{c^*} = \frac{\kappa}{\sqrt{(\kappa^2+\tau^2)(\kappa+\tau)}}(-\kappa T + \tau B),$$

$$B_{c^*} = -\frac{\kappa}{\sqrt{(\kappa^2+\tau^2)(\kappa+\tau)}}(\tau T + \kappa B),$$

$$\kappa_{c^*} = \kappa,$$

$$\tau_{c^*} = \frac{\kappa^2+\tau^2}{\kappa^2+\kappa\tau}.$$

Proof. If we take curve $\alpha(s)$ as a general helix, then we have $\kappa/\tau = const$.
It follows that the angle θ between the vectors W and B of the curve $\alpha(s)$ is constant too.
In this case equalities (3.5), (3.8) and (3.9) can be rewritten as

$$u = -\cos\theta, \quad m = \frac{1}{\cos\theta}, \; \gamma = -\frac{1}{\cos\theta\sqrt{\kappa+\tau}}. \tag{3.16}$$

From equ.(2.5) and equ.(3.16) we can write

$$
\begin{aligned}
T_{c^*} &= muN = \frac{1}{\cos\theta} - \cos\theta N = -N, \\
N_{c^*} &= m\gamma\{u^3\kappa T - u^3\tau B\} = \frac{\kappa}{\sqrt{(\kappa^2+\tau^2)(\kappa+\tau)}}(-\kappa T + \tau B), \\
B_{c^*} &= \gamma\{u^2\tau T - u^2\kappa B\} = -\frac{\kappa}{\sqrt{(\kappa^2+\tau^2)(\kappa+\tau)}}(\tau T + \kappa B), \\
\kappa_{c^*} &= m^3\{(\tau u^2)^2 + (\kappa u^2)^2\}^{1/2} = \cos\theta\sqrt{\kappa^2\tau^2} = \kappa, \\
\tau_{c^*} &= \gamma^2\{-u\sqrt{\kappa^2+\tau^2} + (\tau' u^2)(\kappa u) - (\kappa' u^2)(\tau u)\} \\
&= \frac{1}{\cos\theta(\kappa+\tau)}(\sqrt{(\kappa^2+\tau^2)(\kappa+\tau)} + \cos^2\theta\left(\frac{\tau}{\kappa}\right)'\kappa^2) \\
&= \frac{\kappa^2+\tau^2}{\kappa(\kappa+\tau)}
\end{aligned}
$$

and this completes the proof. □

Example 3.6. *Let a regular curve $\alpha(t) = (3t^2, 4t^3, 3t^4)$ be given. Vectorial moment curves of the principal normal vector N^*, binormal vector B^* and unit Darboux vector c^* of the curve α are given, respectively, as*

$$
\begin{aligned}
N^* &= \frac{1}{9}(16t^9 + 5t^7 + 2t^{11}, -12t^{10} - 24t^8 - 6t^6, 3t^5 + 4t^9 - 8t^7) \\
B^* &= \frac{1}{3}(4t^5 + 6t^7, 6t^8 - 3t^4, -6t^5 - 8t^7) \\
c^* &= (6t^7 + 4t^5 - 2t^9, 9t^8 - 3t^4, -10t^7 - 6t^5)
\end{aligned}
$$

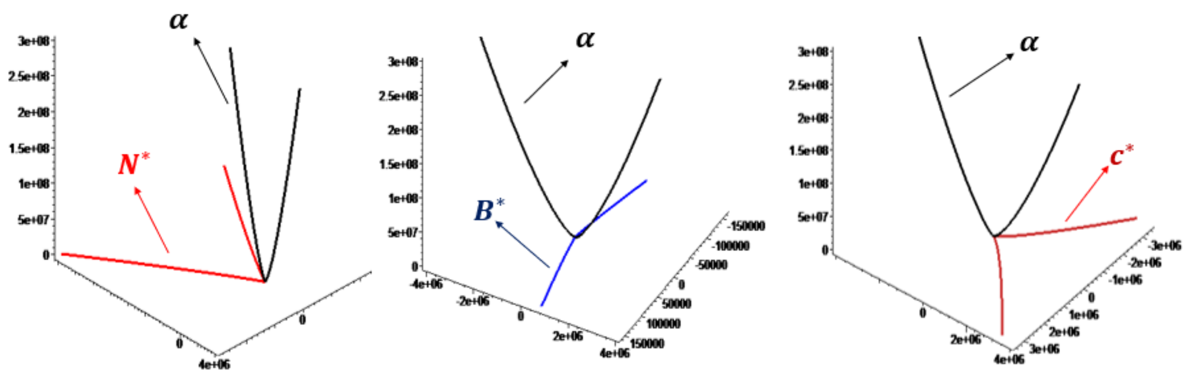

Figure 3.4: Note that the curve α and N^*, B^*, c^* do not form a constant width curve pairs

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