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Applications of the (k,s,h)-Riemann-Liouville and (k,h)-Hadamard Fractional Operators on Inequalities

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Abstract

This paper deals with some results of fractional inequalities involving two recent recent integral operators: the $(k,s,h)-$Riemann-Liouville integral and the $(k,h)-$Hadamard fractional operator. We generalize some classical integral inequalities as well as some other fractional inequalities. By simple arguments, we establish a relation between the two considered operators that allows us to establish several integral results.

Keywords: *$(k,s,h)-$Riemann-Liouville fractional integral, $(k,h)-$Hadamard fractional operator, Korkine identity, Gruss inequality.*
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1. Introduction

Let us consider the weighted Korkine's identity (see [9]):

$$\int_a^b p(x)\int_a^b p(x)\Phi(x)\Psi(x)dx-\int_a^b p(x)\Phi(x)dx\int_a^b p(x)\Psi(x)dx=\frac{1}{2}\int_a^b\int_a^b p(t)p(\tau)\left[\Phi(t)-\Phi(\tau)\right]\left[\Psi(t)-\Psi(\tau)\right]dtd\tau, \tag{1.1}$$

where p is a positive integrable function and Φ and Ψ are two real-valued integrable functions which are synchronous on $[a,b]$: that is:

$$(\Phi(t)-\Phi(\tau))(\Psi(t)-\Psi(\tau))\geq 0. \tag{1.2}$$

Also, we consider the weighted Gruss inequality (see [12]):

$$\int_a^b p(x)\int_a^b p(x)\Phi(x)\Psi(x)dx-\int_a^b p(x)\Phi(x)dx\int_a^b p(x)\Psi(x)dx\leq\frac{(M-m)(N-n)}{4}\left(\int_a^b p(t)dx\right)^2, \tag{1.3}$$

where p,Φ and Ψ are three integrable functions on $[a,b]$, satisfying $\int_a^b p(t)dt>0$ and

$$m\leq\Phi(u)\leq M, n\leq\Psi(u)\leq N; m,M,n,N\in\mathbb{R}, u\in[a,b]. \tag{1.4}$$

Recently, in [2] M. Bezziou et al. introduced a new class of fractional operators which is called the $(k,s,h)-$Riemann-Liouville fractional integrals with respect to a given function h:

$${}_k^sJ_{a,h}^{\alpha}(f(t))=\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^t\left(h^{s+1}(t)-h^{s+1}(\tau)\right)^{\frac{\alpha}{k}-1}h^s(\tau)h'(\tau)f(\tau)d\tau, \tag{1.5}$$

where $\Gamma_k(\alpha)=\int_0^\infty t^{\alpha-1}e^{-\frac{t^k}{k}}dt, \alpha>0, k>0, s\in\mathbb{R}-\{-1\}.$
The same authors introduced another class of fractional operators. It is called $(k,h)-$Hadamard fractional with respect to h and it is given by:

$${}_kI_{a,h}^{\alpha}(f(t))=\frac{1}{k\Gamma_k(\alpha)}\int_a^t\left(\log\frac{h(t)}{h(\tau)}\right)^{\frac{\alpha}{k}-1}\frac{h'(\tau)}{h(\tau)}f(\tau)d\tau, \tag{1.6}$$

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where $k>0$.

Many researchers have been concerned with the functional (1.1) and the inequality (1.3). For more details, we refer the reader to [1, 3, 4, 5, 7, 11] and the references therein.

The main purpose of this paper is to establish some new inequalities by using the above classes of operators. We generalize some results already published in the papers [6, 12].

2. Main Results

We begin by proving the following theorem:

Theorem 2.1. *Let f be an integrable positive function on $[a,b]$ and let h be a measurable, increasing and positive function on $(a,b]$, with $h\in C^1([a,b])$. Then for all $\alpha>0$, the following inequalities are valid:*

$$ {}_kI^{\alpha}_{a,h}[f(t)]\ {}_kI^{\alpha}_{\alpha,h}[t^rf(t)]-{}_kI^{\alpha}_{\alpha,h}[tf(t)]\ {}_kI^{\alpha}_{\alpha,h}\left[t^{r-1}f(t)\right]\leq\left[\frac{\left(\log\frac{h(t)}{h(a)}\right)^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\ {}_kI^{\alpha}_{a,h}(t^r)-\ {}_kI^{\alpha}_{a,h}(t)\ {}_kI^{\alpha}_{a,h}(t^{r-1})\right]\|f\|^2_{\infty}, \tag{2.1}$$

where $f\in L_{\infty}[a,b]$,
and

$$ {}_kI^{\alpha}_{a,h}[f(t)]\ {}_kI^{\alpha}_{\alpha,h}[t^rf(t)]-{}_kI^{\alpha}_{\alpha,h}[tf(t)]\ {}_kI^{\alpha}_{\alpha,h}\left[t^{r-1}f(t)\right]\leq\frac{(t-a)\left(t^{r-1}-a^{r-1}\right)}{2}\left[{}_kI^{\alpha}_{a,h}(f(t)\right]^2. \tag{2.2}$$

Proof. We consider the quantity

$$ {}^{s}_{k}H_{\alpha,h}(t,\tau)=\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\left(h^{s+1}(t)-h^{s+1}(\tau)\right)^{\frac{\alpha}{k}-1}h^s(\tau)h'(\tau)p(\tau), \tag{2.3}$$

where $\tau\in(a,t), t\in(a,b]$ and $p:[a,b]\to\mathbb{R}^+$ is a continuous function.

From (1.1) and (1.5), we can write

$$\begin{aligned}
&\frac{1}{2}\int_a^t\int_a^t {}^{s}_{k}H_{\alpha,h}(t,\tau)\ {}^{s}_{k}H_{\alpha,h}(t,\rho)\left(\Phi(t)-\Phi(\tau)\right)\left(\Psi(t)-\Psi(\rho)\right)d\tau d\rho \\
=\ &\frac{(s+1)^{2\left(1-\frac{\alpha}{k}\right)}}{2k^2\Gamma_k^2(\alpha)}\int_a^t\int_a^t\left(h^{s+1}(t)-h^{s+1}(\tau)\right)^{\frac{\alpha}{k}-1}\left(h^{s+1}(t)-h^{s+1}(\rho)\right)^{\frac{\alpha}{k}-1} \\
&\times h'(\tau)h'(\rho)h^s(\tau)h^s(\rho)p(\tau)p(\rho)\left(\Phi(t)-\Phi(\tau)\right)\left(\Psi(t)-\Psi(\rho)\right)d\tau d\rho \\
=\ &{}^{s}_{k}J^{\alpha}_{a,h}p(t)\ {}^{s}_{k}J^{\alpha}_{a,h}(p\Phi\Psi)(t)-\ {}^{s}_{k}J^{\alpha}_{a,h}(p\Phi)(t)\ {}^{s}_{k}J^{\alpha}_{a,h}(p\Psi)(t).
\end{aligned} \tag{2.4}$$

In (2.4), replacing $p(t)=f(t)$, $\Phi(t)=t$ and $\Psi(t)=t^{r-1}, t\in[a,b]$, we have

$$\begin{aligned}
&\lim_{s\longrightarrow -1^+}\frac{(s+1)^{2\left(1-\frac{\alpha}{k}\right)}}{2k^2\Gamma_k^2(\alpha)}\int_a^t\int_a^t\left(h^{s+1}(t)-h^{s+1}(\tau)\right)^{\frac{\alpha}{k}-1}h^s(\tau)h'(\tau) \\
&\left(h^{s+1}(t)-h^{s+1}(\rho)\right)^{\frac{\alpha}{k}-1}h^s(\rho)h'(\rho)(\tau-\rho)\left(\tau^{r-1}-\rho^{r-1}\right) \\
&\times f(\tau)f(\rho)d\tau d\rho \\
=\ &\frac{1}{2k^2\Gamma_k^2(\alpha)}\int_a^t\int_a^t\lim_{s\longrightarrow -1^+}\left[\left(\frac{h^{s+1}(t)-h^{s+1}(\tau)}{s+1}\right)^{\frac{\alpha}{k}-1}\right. \\
&\times\left.\left(\frac{h^{s+1}(t)-h^{s+1}(\rho)}{s+1}\right)^{\frac{\alpha}{k}-1}h^s(\tau)h^s(\rho)\right]h'(\tau)h'(\rho) \\
&\times(\tau-\rho)\left(\tau^{r-1}-\rho^{r-1}\right)f(\tau)f(\rho)d\tau d\rho \\
=\ &\frac{1}{2k^2\Gamma_k^2(\alpha)}\int_a^t\int_a^t\left(\log\frac{h(t)}{h(\tau)}\right)^{\frac{\alpha}{k}-1}\left(\log\frac{h(t)}{h(\rho)}\right)^{\frac{\alpha}{k}-1} \\
&\times\frac{h'(\tau)h'(\rho)}{h(\tau)h(\rho)}(\tau-\rho)\left(\tau^{r-1}-\rho^{r-1}\right)f(\tau)f(\rho)d\tau d\rho \\
=\ &{}_kI^{\alpha}_{a,h}[f(t)]\ {}_kI^{\alpha}_{a,h}[t^rf(t)]-\ {}_kI^{\alpha}_{a,h}[tf(t)]\ {}_kI^{\alpha}_{a,h}\left[t^{r-1}f(t)\right]
\end{aligned} \tag{2.5}$$

Since $f\in L_{\infty}[a,b]$, then we have

$$\begin{aligned}
&\frac{(s+1)^{2(1-\frac{\alpha}{k})}}{2k^2\Gamma_k^2(\alpha)}\int_a^t\int_a^t\left(h^{s+1}(t)-h^{s+1}(\tau)\right)^{\frac{\alpha}{k}-1}\left(h^{s+1}(t)-h^{s+1}(\rho)\right)^{\frac{\alpha}{k}-1}\\
&\times h'(\tau)h'(\rho)h^s(\tau)h^s(\rho)(\tau-\rho)\left(\tau^{r-1}-\rho^{r-1}\right)f(\tau)f(\rho)d\tau d\rho \qquad (2.6)\\
\leq\ & \sup_{(\tau,\rho)\in[a,b]^2}|f(\tau)f(\rho)|\frac{(s+1)^{2(1-\frac{\alpha}{k})}}{2k^2\Gamma_k^2(\alpha)}\int_a^t\int_a^t\left(h^{s+1}(t)-h^{s+1}(\tau)\right)^{\frac{\alpha}{k}-1}\\
&\times h'(\tau)h'(\rho)h^s(\tau)h^s(\rho)(\tau-\rho)\left(\tau^{r-1}-\rho^{r-1}\right)d\tau d\rho\\
\leq\ & \|f\|_\infty^2\left[{}_k^sJ_{a,h}^{\alpha}(1)\,{}_k^sJ_{a,h}^{\alpha}(t^r)-{}_k^sJ_{a,h}^{\alpha}(t)\,{}_k^sJ_{a,h}^{\alpha}(t^{r-1})\right].
\end{aligned}$$

This implies that

$$\begin{aligned}
&\sup_{(\tau,\rho)\in[a,b]^2}\frac{|f(\tau)f(\rho)|}{2k^2\Gamma_k^2(\alpha)}\int_a^t\int_a^t\lim_{s\longrightarrow -1^+}\left[\left(\frac{h^{s+1}(t)-h^{s+1}(\tau)}{s+1}\right)^{\frac{\alpha}{k}-1}\right.\\
&\left.\times\, h^s(\tau)h^s(\rho)h'(\tau)h'(\rho)(\tau-\rho)\left(\tau^{r-1}-\rho^{r-1}\right)\right]d\tau d\rho \qquad (2.7)\\
=\ & \|f\|_\infty^2\left[{}_kI_{a,h}^{\alpha}(1)\,{}_kI_{a,h}^{\alpha}(t^r)-{}_kI_{a,h}^{\alpha}(t)\,{}_kI_{a,h}^{\alpha}(t^{r-1})\right].
\end{aligned}$$

Hence, by (2.5), (2.6) and (2.7), we get (2.1).
To obtain (2.2) we observe that from (2.5), we get

$$\begin{aligned}
&{}_kI_{a,h}^{\alpha}[f(t)]\,{}_kI_{a,h}^{\alpha}[t^rf(t)]-{}_kI_{a,h}^{\alpha}[tf(t)]\,{}_kI_{a,h}^{\alpha}\left[t^{r-1}f(t)\right]\\
\leq\ & \sup_{(\tau,\rho)\in[a,t]^2}\frac{(\tau-\rho)\left(\tau^{r-1}-\rho^{r-1}\right)}{2k^2\Gamma_k^2(\alpha)}\int_a^t\int_a^t\lim_{s\longrightarrow -1^+}\left[\left(\frac{h^{s+1}(t)-h^{s+1}(\tau)}{s+1}\right)^{\frac{\alpha}{k}-1}\right.\\
&\left.\times\left(\frac{h^{s+1}(t)-h^{s+1}(\rho)}{s+1}\right)^{\frac{\alpha}{k}-1}h'(\tau)h'(\rho)h^s(\tau)h^s(\rho)f(\tau)f(\rho)\right]d\tau d\rho \qquad (2.8)\\
\leq\ & \frac{(t-a)\left(t^{r-1}-a^{r-1}\right)}{2k^2\Gamma_k^2(\alpha)}\int_a^t\int_a^t\left(\log\frac{h(t)}{h(\tau)}\right)^{\frac{\alpha}{k}-1}\left(\log\frac{h(t)}{h(\rho)}\right)^{\frac{\alpha}{k}-1}\frac{h'(\tau)h'(\rho)}{h(\tau)h(\rho)}\\
&f(\tau)f(\rho)]d\tau d\rho.
\end{aligned}$$

Thanks to (2.8), we end the proof. □

Remark 2.2. *1- If we take $\alpha=k=1,h(x)=e^x$ and $t=b$ in (2.1), we obtain the second part of Theorem 5 in [9].*
2- If we take $\alpha=k=1,h(x)=e^x$ and $t=b$ in the (2.2), we obtain the first part of Theorem 5 of [9].

Theorem 2.3. *Let f be an integrable positive function on $[a,b]$ and let h be a measurable, increasing and positive function on $(a,b]$ with $h\in C^1([a,b])$. Then for all $\alpha>0$, the following inequalities hold:*

$$ {}_kI_{a,h}^{\alpha}[f(t)]\,{}_kI_{\alpha,h}^{\alpha}[t^rf(t)]-\left({}_kI_{\alpha,h}^{\alpha}\left[t^{r-1}f(t)\right]\right)^2\leq\left[\frac{\left(\log\frac{h(t)}{h(a)}\right)^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\,{}_kI_{a,h}^{\alpha}(t^r)-\left({}_kI_{a,h}^{\alpha}(t^{r-1})\right)^2\right]\|f\|_\infty^2, \qquad (2.9)$$

where $f\in L_{\infty}[a,b]$,
and

$$ {}_kI_{a,h}^{\alpha}[f(t)]\,{}_kI_{\alpha,h}^{\alpha}[t^rf(t)]-\left({}_kI_{\alpha,h}^{\alpha}\left[t^{r-1}f(t)\right]\right)^2\leq\frac{\left(t^{r-1}-a^{r-1}\right)^2}{2}\left[{}_kI_{a,h}^{\alpha}(f(t)\right]^2. \qquad (2.10)$$

Proof. Applying Theorem 1 for $\Phi(x)=\Psi(x)=t^{r-1}$, we obtain (2.9) and (2.10). □

Theorem 2.4. *Let f be an integrable positive function on $[a,b]$ and let h be a measurable, increasing and positive function on $(a,b]$ with $h\in C^1([a,b])$. Then for all $t\in(a,b]$ and $\alpha>0,\beta>0$ the following inequalities hold:*

$$\begin{aligned}
&{}_kI_{a,h}^{\alpha}[f(t)]\,{}_kI_{\alpha,h}^{\beta}[t^rf(t)]+{}_kI_{a,h}^{\beta}[f(t)]\,{}_kI_{\alpha,h}^{\alpha}[t^rf(t)]\\
&-{}_kI_{\alpha,h}^{\alpha}[tf(t)]\,{}_kI_{\alpha,h}^{\beta}\left[t^{r-1}f(t)\right]-{}_kI_{\alpha,h}^{\beta}[tf(t)]\,{}_kI_{\alpha,h}^{\alpha}\left[t^{r-1}f(t)\right]\\
\leq\ & \left[\frac{\left(\log\frac{h(t)}{h(a)}\right)^{\frac{\beta}{k}}}{\Gamma_k(\beta+k)}\,{}_kI_{a,h}^{\alpha}(t^r)-{}_kI_{a,h}^{\alpha}(t)\,{}_kI_{a,h}^{\beta}(t^{r-1})\right. \qquad (2.11)\\
&\left.-\,{}_kI_{a,h}^{\alpha}(t^{r-1})\,{}_kI_{a,h}^{\beta}(t)-\frac{\left(\log\frac{h(t)}{h(a)}\right)^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\,{}_kI_{a,h}^{\beta}(t^r)\right]\|f\|_\infty^2,
\end{aligned}$$

where $f\in L_{\infty}[a,b]$,
and

$$\begin{aligned}
& {}_{k}I_{a,h}^{\alpha}[f(t)]\ {}_{k}I_{\alpha,h}^{\beta}[t^{r}f(t)]+{}_{k}I_{a,h}^{\beta}[f(t)]\ {}_{k}I_{\alpha,h}^{\alpha}[t^{r}f(t)] \\
& -{}_{k}I_{\alpha,h}^{\alpha}[tf(t)]\ {}_{k}I_{\alpha,h}^{\beta}\left[t^{r-1}f(t)\right]-{}_{k}I_{\alpha,h}^{\beta}[tf(t)]\ {}_{k}I_{\alpha,h}^{\alpha}\left[t^{r-1}f(t)\right] \\
\leq\ & (t-a)\left(t^{r-1}-a^{r-1}\right)\ {}_{k}I_{a,h}^{\alpha}[f(t)]\ {}_{k}I_{a,h}^{\beta}[f(t)]. \qquad (2.12)
\end{aligned}$$

Proof. We consider the quantities

$$\begin{cases}
{}_{k}^{s}H_{\alpha,h}(t,\tau)=\frac{(s+1)^{1-\frac{\alpha}{k}}\left(h^{s+1}(t)-h^{s+1}(\tau)\right)^{\frac{\alpha}{k}-1}}{k\Gamma_{k}(\alpha)}h'(\tau)h^{s}(\tau)p(\tau), \\
\text{and} \\
{}_{k}^{s}H_{\beta,h}(t,\rho)=\frac{(s+1)^{1-\frac{\beta}{k}}\left(h^{s+1}(t)-h^{s+1}(\rho)\right)^{\frac{\beta}{k}-1}}{k\Gamma_{k}(\beta)}h'(\rho)h^{s}(\rho)p(\rho),
\end{cases} \qquad (2.13)$$

where $\tau\in(a,t),\rho\in(a,t),t\in(a,b]$ and $p:[a,b]\to\mathbb{R}^{+}$ is a continuous function.
From (1.1) and (2.13), we can write

$$\begin{aligned}
& \int_{a}^{t}\int_{a}^{t}{}_{k}^{s}H_{\alpha,h}(t,\tau)\ {}_{k}^{s}H_{\beta,h}(t,\rho)(\Phi(t)-\Phi(\tau))(\Psi(t)-\Psi(\rho))\,d\tau ds \qquad (2.14)\\
=\ & \frac{(s+1)^{2-\frac{\alpha+\beta}{k}}}{2k^{2}\Gamma_{k}(\alpha)\Gamma_{k}(\beta)}\int_{a}^{t}\int_{a}^{t}\left(h^{s+1}(t)-h^{s+1}(\tau)\right)^{\frac{\alpha}{k}-1}\left(h^{s+1}(t)-h^{s+1}(\rho)\right)^{\frac{\beta}{k}-1} \\
& \times h'(\tau)h'(\rho)h^{s}(\tau)h^{s}(\rho)p(\tau)p(\rho)(\Phi(t)-\Phi(\tau))(\Psi(t)-\Psi(\rho))\,d\tau d\rho \\
=\ & {}_{k}^{s}J_{a,h}^{\alpha}p(t)\ {}_{k}^{s}J_{a,h}^{\beta}(p\Phi\Psi)(t)+{}_{k}^{s}J_{a,h}^{\beta}p(t)\ {}_{k}^{s}J_{a,h}^{\alpha}(p\Phi\Psi)(t) \\
& -{}_{k}^{s}J_{a,h}^{\alpha}(p\Phi)(t)\ {}_{k}^{s}J_{a,h}^{\beta}(p\Psi)(t)-{}_{k}^{s}J_{a,h}^{\beta}(p\Phi)(t)\ {}_{k}^{s}J_{a,h}^{\alpha}(p\Psi)(t).
\end{aligned}$$

Taking $p(t)=f(t)$, $\Phi(t)=t$ and $\Psi(t)=t^{r-1}, t\in[a,b]$ in (2.14), we have

$$\begin{aligned}
& \lim_{s\longrightarrow-1^{+}}\frac{1}{2k^{2}\Gamma_{k}(\alpha)\Gamma_{k}(\beta)}\int_{a}^{t}\int_{a}^{t}\left(\frac{h^{s+1}(t)-h^{s+1}(\tau)}{s+1}\right)^{\frac{\alpha}{k}-1} \\
& \times\left(\frac{h^{s+1}(t)-h^{s+1}(\rho)}{s+1}\right)^{\frac{\beta}{k}-1}h'(\tau)h'(\rho)(\tau-\rho) \\
& \times\left(\tau^{r-1}-\rho^{r-1}\right)f(\tau)f(\rho)\,d\tau d\rho \qquad (2.15)\\
=\ & \frac{1}{2k^{2}\Gamma_{k}(\alpha)\Gamma_{k}(\beta)}\int_{a}^{t}\int_{a}^{t}\lim_{s\longrightarrow-1^{+}}\left[\left(\frac{h^{s+1}(t)-h^{s+1}(\tau)}{s+1}\right)^{\frac{\alpha}{k}-1}\right. \\
& \times\left(\frac{h^{s+1}(t)-h^{s+1}(\rho)}{s+1}\right)^{\frac{\beta}{k}-1}h'(\tau)h'(\rho)h^{s}(\tau)h^{s}(\rho) \\
& \left.\times(\tau-\rho)\left(\tau^{r-1}-\rho^{r-1}\right)f(\tau)f(\rho)\right]d\tau d\rho \\
=\ & {}_{k}I_{a,h}^{\alpha}[t^{r}f(t)]\ {}_{k}I_{a,h}^{\beta}[f(t)]-{}_{k}I_{a,h}^{\alpha}[tf(t)]\ {}_{k}I_{a,h}^{\beta}\left[t^{r-1}f(t)\right] \\
& -{}_{k}I_{a,h}^{\alpha}\left[t^{r-1}f(t)\right]_{k}J_{a,h}^{\beta}[tf(t)]\ +{}_{k}I_{a,h}^{\alpha}[f(t)]\ {}_{k}I_{a,h}^{\beta}[t^{r}f(t)].
\end{aligned}$$

The fact that $f\in L_{\infty}[a,b]$ allows us to observe that

$$\begin{aligned}
& {}_{k}I_{a,h}^{\alpha}[t^{r}f(t)]\ {}_{k}I_{a,h}^{\beta}[f(t)]-{}_{k}I_{a,h}^{\alpha}[tf(t)]\ {}_{k}I_{a,h}^{\beta}\left[t^{r-1}f(t)\right] \qquad (2.16)\\
& -{}_{k}I_{a,h}^{\alpha}\left[t^{r-1}f(t)\right]_{k}J_{a,h}^{\beta}[tf(t)]\ +{}_{k}I_{a,h}^{\alpha}[f(t)]\ {}_{k}I_{a,h}^{\beta}[t^{r}f(t)] \\
\leq\ & \sup_{(\tau,\rho)\in[a,b]^{2}}\frac{|f(\tau)f(\rho)|}{2k^{2}\Gamma_{k}(\alpha)\Gamma_{k}(\beta)}\int_{a}^{t}\int_{a}^{t}\lim_{s\longrightarrow-1^{+}}\left[\left(\frac{h^{s+1}(t)-h^{s+1}(\tau)}{s+1}\right)^{\frac{\alpha}{k}-1}\right. \\
& \left.\times\left(\frac{h^{s+1}(t)-h^{s+1}(\rho)}{s+1}\right)^{\frac{\beta}{k}-1}h'(\tau)h'(\rho)(\tau-\rho)\left(\tau^{r-1}-\rho^{r-1}\right)\right]d\tau d\rho \\
\leq\ & \|f\|_{\infty}^{2}\left[{}_{k}I_{a,h}^{\beta}(1)\ {}_{k}I_{a,h}^{\alpha}(t^{r})-{}_{k}I_{a,h}^{\alpha}(t)\,{}_{k}I_{a,h}^{\beta}(t^{r-1})\right. \\
& \left.-{}_{k}I_{a,h}^{\alpha}(t^{r-1})\,{}_{k}I_{a,h}^{\beta}(t)-{}_{k}I_{a,h}^{\alpha}(1)\ {}_{k}I_{a,h}^{\beta}(t^{r})\right].
\end{aligned}$$

Hence, by (2.15) and (2.16), we get (2.11).

On the other hand, we have

$$\begin{aligned}
& {}_kI^{\alpha}_{a,h}[f(t)]\ {}_kI^{\beta}_{\alpha,h}[t^r f(t)] + {}_kI^{\beta}_{a,h}[f(t)]\ {}_kI^{\alpha}_{\alpha,h}[t^r f(t)] \\
& -{}_kI^{\alpha}_{\alpha,h}[t f(t)]\ {}_kI^{\beta}_{\alpha,h}\left[t^{r-1} f(t)\right] - {}_kI^{\beta}_{\alpha,h}[t f(t)]\ {}_kI^{\alpha}_{\alpha,h}\left[t^{r-1} f(t)\right] \\
\leq & \sup_{(\tau,\rho)\in[a,t]^2} \frac{\left|(\tau-\rho)\left(\tau^{r-1}-\rho^{r-1}\right)\right|}{2k^2\Gamma_k(\alpha)\Gamma_k(\beta)} \int_a^t \int_a^t \lim_{s\longrightarrow -1^+}\left[\left(\frac{h^{s+1}(t)-h^{s+1}(\tau)}{s+1}\right)^{\frac{\alpha}{k}-1}\right. \\
& \left.\times\left(\frac{h^{s+1}(t)-h^{s+1}(\rho)}{s+1}\right)^{\frac{\beta}{k}-1} h'(\tau)h'(\rho)h^s(\tau)h^s(\rho)f(\tau)f(\rho)\right] d\tau d\rho \qquad (2.17) \\
\leq & \frac{(t-a)\left(t^{r-1}-a^{r-1}\right)}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)} \int_a^t \int_a^t \left(\log\frac{h(t)}{h(\tau)}\right)^{\frac{\alpha}{k}-1} \left(\log\frac{h(t)}{h(\rho)}\right)^{\frac{\beta}{k}-1} \\
& \times \frac{h'(\tau)h'(\rho)}{h(\tau)h(\rho)} f(\tau)f(\rho)\,d\tau d\rho.
\end{aligned}$$

Thanks to (2.15) and (2.17), we obtain (2.12). □

Remark 2.5. *If we take $\alpha=\beta$ in Theorem 4, we obtain Theorem 1.*

Theorem 2.6. *Let Φ and Ψ be two integrable functions on $[a,b]$ satisfying the condition (1.4) , p be a positive function on $[a,b]$ and h be a measurable, increasing, positive function on $(a,b]$, with $h\in C^1([a,b])$. Then, we have:*

$$\begin{aligned}
& \left|{}_kI^{\alpha}_{a,h}[p(t)]\ {}_kI^{\alpha}[p\Phi\Psi(t)] - {}_kI^{\alpha}_{a,h}[p\Phi(t)]\,{}_kI^{\alpha}_{a,h}[p\Psi(t)]\right| \qquad (2.18) \\
\leq & \frac{(M-m)(N-n)}{4}\left({}_kI^{\alpha}_{a,h}[p(t)]\right)^2,
\end{aligned}$$

where $\alpha>0, k>0$.

We need the following lemma:

Lemma 2.7. *Let φ be an integrable function on $[a,b]$ satisfying the condition (1.4) on $[a,b]$, let p be a positive function on $[a,b]$ and let h be a measurable, increasing, positive function on $(a,b]$ with $h\in C^1([a,b])$. Then, we have*

$$\begin{aligned}
& {}_kI^{\alpha}_{a,h}[p(t)]\ {}_kI^{\alpha}_{a,h}\left[p\varphi^2(t)\right] - \left({}_kI^{\alpha}_{a,h}[p\varphi(t)]\right)^2 \qquad (2.19) \\
= & \left(M\left({}_kI^{\alpha}_{a,h}[p(t)]\right) - {}_kI^{\alpha}_{a,h}[p\varphi(t)]\right)\left({}_kI^{\alpha}_{a,h}[p\varphi(t)] - m\left({}_kI^{\alpha}_{a,h}[p(t)]\right)\right) \\
& -\left({}_kI^{\alpha}_{a,h}[p(t)]\right)\left({}_kI^{\alpha}_{a,h}[(M-\varphi(t))(\varphi(t)-m)p(t)]\right).
\end{aligned}$$

Proof. Let φ be an integrable function on $[a,b]$ satisfying the condition (1.4) on $[a,b]$. For any $\tau,\rho\in[a,b]$, we have the following identity

$$\begin{aligned}
& p(\rho)p(\tau)\varphi^2(\tau) + p(\tau)p(\rho)\varphi^2(\rho) - 2p(\tau)\varphi(\tau)p(\rho)\varphi(\rho) \\
= & (Mp(\rho)-p(\rho)\varphi(\rho))(p(\tau)\varphi(\tau)-mp(\tau)) \\
& +(Mp(\tau)-p(\tau)\varphi(\tau))(p(\rho)\varphi(\rho)-mp(\rho)) \qquad (2.20) \\
& -p(\rho)p(\tau)(M-\varphi(\tau))(\varphi(\tau)-m) \\
& -p(\tau)p(\rho)(M-\varphi(\rho))(\varphi(\rho)-m).
\end{aligned}$$

We consider the quantity (2.3). Then, multiplying (2.20) by ${}^s_kH_{\alpha,h}(t,\tau)\times{}^s_kH_{\alpha,h}(t,\rho), (\tau,\rho)\in(a,t)^2, s\in\mathbb{R}-\{-1\}$ and integrating with respect to τ and ρ over $(a,t)^2$, we get

$$\frac{1}{2k^2\Gamma_k^2(\alpha)}\left\{\int_a^t\int_a^t \lim_{s\longrightarrow -1^+}\left[\left(\frac{h^{s+1}(t)-h^{s+1}(\tau)}{s+1}\right)^{\frac{\alpha}{k}-1} h^s(\tau)h'(\tau)(p(\tau)\varphi(\tau)-mp(\tau))\right.\right.$$
$$\left.\times\left(\frac{h^{s+1}(t)-h^{s+1}(\rho)}{s+1}\right)^{\frac{\alpha}{k}-1} h^s(\rho)h'(\rho)(Mp(\rho)-p(\rho)\varphi(\rho))\right]d\tau d\rho$$
$$+\int_a^t\int_a^t \lim_{s\longrightarrow -1^+}\left[\left(\frac{h^{s+1}(t)-h^{s+1}(\tau)}{s+1}\right)^{\frac{\alpha}{k}-1} h^s(\tau)h'(\tau)(Mp(\tau)-p(\tau)\varphi(\tau))\right.$$
$$\left.\times\left(\frac{h^{s+1}(t)-h^{s+1}(\rho)}{s+1}\right)^{\frac{\alpha}{k}-1} h^s(\rho)h'(\rho)(p(\rho)\varphi(\rho)-mp(\rho))\right]d\tau d\rho$$
$$-\int_a^t\int_a^t \lim_{s\longrightarrow -1^+}\left[\left(\frac{h^{s+1}(t)-h^{s+1}(\tau)}{s+1}\right)^{\frac{\alpha}{k}-1} h^s(\tau)h'(\tau)(M-\varphi(\tau))(\varphi(\tau)-m)\right.$$
$$\left.\times\left(\frac{h^{s+1}(t)-h^{s+1}(\rho)}{s+1}\right)^{\frac{\alpha}{k}-1} h^s(\rho)h'(\rho)p(\rho)p(\tau)\right]d\tau d\rho$$
$$-\int_a^t\int_a^t \lim_{s\longrightarrow -1^+}\left[\left(\frac{h^{s+1}(t)-h^{s+1}(\tau)}{s+1}\right)^{\frac{\alpha}{k}-1} h^s(\tau)h'(\tau)p(\tau)\right.$$
$$\left.\left.\times\left(\frac{h^{s+1}(t)-h^{s+1}(\rho)}{s+1}\right)^{\frac{\alpha}{k}-1} h^s(\rho)h'(\rho)p(\rho)(M-\varphi(\rho))(\varphi(\rho)-m)\right]d\tau d\rho\right\}$$

therefore,

$$\frac{1}{2k^2\Gamma_k^2(\alpha)}\left\{\int_a^t\int_a^t\left[\left(\log\frac{h(t)}{h(\tau)}\right)^{\frac{\alpha}{k}-1}\left(\log\frac{h(t)}{h(\rho)}\right)^{\frac{\alpha}{k}-1}\frac{h'(\tau)}{h(\tau)}\right.\right.$$
$$\left.\times\frac{h'(\rho)}{h(\rho)}(Mp(\rho)-p(\rho)\varphi(\rho))(p(\tau)\varphi(\tau)-mp(\tau))\right]d\tau d\rho$$
$$+\int_a^t\int_a^t\left[\left(\log\frac{h(t)}{h(\tau)}\right)^{\frac{\alpha}{k}-1}\left(\log\frac{h(t)}{h(\rho)}\right)^{\frac{\alpha}{k}-1}\frac{h'(\rho)}{h(\rho)}\right.$$
$$\left.\times\frac{h'(\tau)}{h(\tau)}(Mp(\tau)-p(\tau)\varphi(\tau))(p(\rho)\varphi(\rho)-mp(\rho))\right]d\tau d\rho$$
$$-\int_a^t\int_a^t\left[\left(\log\frac{h(t)}{h(\tau)}\right)^{\frac{\alpha}{k}-1}\left(\log\frac{h(t)}{h(\rho)}\right)^{\frac{\alpha}{k}-1}\frac{h'(\rho)}{h(\rho)}p(\rho)\right.$$
$$\left.\times\frac{h'(\tau)}{h(\tau)}p(\tau)(M-\varphi(\tau))(\varphi(\tau)-m)\right]d\tau d\rho$$
$$-\int_a^t\int_a^t\left[\left(\log\frac{h(t)}{h(\tau)}\right)^{\frac{\alpha}{k}-1}\left(\log\frac{h(t)}{h(\rho)}\right)^{\frac{\alpha}{k}-1}\frac{h'(\tau)}{h(\tau)}p(\tau)\right.$$
$$\left.\left.\times\frac{h'(\rho)}{h(\rho)}p(\rho)(M-\varphi(\rho))(\varphi(\rho)-m)\right]d\tau d\rho\right\}.$$

Which ends the proof of lemma 7. □

Proof. **of Theorem 6.** Let us define the functional

$$G(\tau,\rho)=(\Phi(\tau)-\Phi(\rho))(\Psi(\tau)-\Psi(\rho)),\tau,\rho\in(a,t),t\in(a,b], \tag{2.21}$$

where Φ and Ψ are two integrable functions on $[a,b]$ satisfying (1.4).
Now, multiplying (2.21) by ${}_k^sH_{\alpha,h}(t,\tau)\,{}_k^sH_{\alpha,h}(t,\rho),\tau,\rho\in(a,t)$, which gives (2.3) and integrating the resulting identity with respect to τ and ρ over $(a,t)^2$, we have

$$\begin{aligned}&\frac{1}{2}\int_a^t\int_a^t {}_k^sH_{\alpha,h}(t,\tau)\,{}_k^sH_{\alpha,h}(t,\rho)(\Phi(\tau)-\Phi(\rho))(\Psi(\tau)-\Psi(\rho))d\tau d\rho\\ =\;& {}_kI_{a,h}^{\alpha}[p(t)]\;{}_kI^{\alpha}[p\Phi\Psi(t)]-{}_kI_{a,h}^{\alpha}[p\Phi(t)]_kI_{a,h}^{\alpha}[p\Psi(t)].\end{aligned} \tag{2.22}$$

Using Cauchy Schwarz inequality, we get

$$\begin{aligned}&\left({}_kI_{a,h}^{\alpha}[p(t)]\;{}_kI^{\alpha}[p\Phi\Psi(t)]-{}_kI_{a,h}^{\alpha}[p\Phi(t)]_kI_{a,h}^{\alpha}[p\Psi(t)]\right)^2\\ \leq\;&\frac{1}{2}\int_a^t\int_a^t {}_k^sH_{\alpha,h}(t,\tau)\,{}_k^sH_{\alpha,h}(t,\rho)(\Phi(\tau)-\Phi(\rho))^2d\tau d\rho\\ &\times\frac{1}{2}\int_a^t\int_a^t {}_k^sH_{\alpha,h}(t,\tau)\,{}_k^sH_{\alpha,h}(t,\rho)(\Psi(\tau)-\Psi(\rho))^2d\tau d\rho,\end{aligned} \tag{2.23}$$

where

$$\begin{aligned}
&\frac{1}{2}\int_a^t\int_a^t {}_k^sH_{\alpha,h}(t,\tau)\ {}_k^sH_{\alpha,h}(t,\rho)\left(\Phi(\tau)-\Phi(\rho)\right)^2 d\tau d\rho \\
=\ &\left({}_k^sJ_{a,h}^{\alpha}[p(t)]\right)\ \left({}_k^sJ_{a,h}^{\alpha}\left[p\Phi^2(t)\right]\right)-\ \left({}_k^sJ_{a,h}^{\alpha}[p\Phi(t)]\right)^2,
\end{aligned} \tag{2.24}$$

and

$$\begin{aligned}
&\frac{1}{2}\int_a^t\int_a^t {}_k^sH_{\alpha,h}(t,\tau)\ {}_k^sH_{\alpha,h}(t,\rho)\left(\Psi(\tau)-\Psi(\rho)\right)^2 d\tau d\rho \\
=\ &\left({}_k^sJ_{a,h}^{\alpha}[p(t)]\right)\ \left({}_k^sJ_{a,h}^{\alpha}\left[p\Psi^2(t)\right]\right)-\ \left({}_k^sJ_{a,h}^{\alpha}[p\Psi(t)]\right)^2.
\end{aligned} \tag{2.25}$$

From Lemma 7 and thanks to the inequalities(2.22),(2.24) and (2.25), we have

$$\begin{aligned}
&\lim_{s\longrightarrow -1^+}\left[{}_k^sJ_{a,h}^{\alpha}p(t)\ {}_k^sJ_{a,h}^{\alpha}(p\Phi\Psi)(t)-\ {}_k^sJ_{a,h}^{\alpha}(p\Phi)(t)\ {}_k^sJ_{a,h}^{\alpha}(p\Psi)(t)\right]^2 \qquad (2.26)\\
\leq\ &\lim_{s\longrightarrow -1^+}\left[\left(M_k^sJ_{a,h}^{\alpha}[p(t)]-\ {}_k^sJ_{a,h}^{\alpha}[p\Phi(t)]\right)\left({}_k^sJ_{a,h}^{\alpha}[p\Phi(t)]-m_{kk}^sJ_{a,h}^{\alpha}[p(t)]\right)\right]\\
&\times\lim_{s\longrightarrow -1^+}\left[\left(N_k^sJ_{a,h}^{\alpha}[p(t)]-\ {}_k^sJ_{a,h}^{\alpha}[p\Psi(t)]\right)\left({}_k^sJ_{a,h}^{\alpha}[p\Psi(t)]-n_k^sJ_{a,h}^{\alpha}[p(t)]\right)\right].
\end{aligned}$$

Which implies that

$$\begin{aligned}
&\left[{}_kI_{a,h}^{\alpha}p(t)\ {}_kI_{a,h}^{\alpha}(p\Phi\Psi)(t)-\ {}_kI_{a,h}^{\alpha}(p\Phi)(t)\ {}_kI_{a,h}^{\alpha}(p\Psi)(t)\right]^2 \qquad (2.27)\\
\leq\ &\left[\left(M_kI_{a,h}^{\alpha}[p(t)]-\ {}_kI_{a,h}^{\alpha}[p\Phi(t)]\right)\left({}_kI_{a,h}^{\alpha}[p\Phi(t)]-m_kI_{a,h}^{\alpha}[p(t)]\right)\right]\\
&\left[\left(N_kI_{a,h}^{\alpha}[p(t)]-\ {}_kI_{a,h}^{\alpha}[p\Psi(t)]\right)\left({}_kI_{a,h}^{\alpha}[p\Psi(t)]-n_kI_{a,h}^{\alpha}[p(t)]\right)\right].
\end{aligned}$$

Thanks to the elementary inequality $4xy\leq(x+y)^2, x,y\in\mathbb{R}$, we can obtain

$$\begin{aligned}
&\left(M_kI_{a,h}^{\alpha}[p(t)]-\ {}_kI_{a,h}^{\alpha}[p\Phi(t)]\right)\left({}_kI_{a,h}^{\alpha}[p\Phi(t)]-m_kI_{a,h}^{\alpha}[p(t)]\right)\\
\leq\ &\left(\frac{M-m}{2}I_{a,h}^{\alpha}[p(t)]\right)^2
\end{aligned} \tag{2.28}$$

and

$$\begin{aligned}
&\left(N_kI_{a,h}^{\alpha}[p(t)]-\ {}_kI_{a,h}^{\alpha}[p\Phi(t)]\right)\left({}_kI_{a,h}^{\alpha}[p\Phi(t)]-n_kI_{a,h}^{\alpha}[p(t)]\right)\\
\leq\ &\left(\frac{N-m}{2}I_{a,h}^{\alpha}[p(t)]\right)^2.
\end{aligned} \tag{2.29}$$

Hence by (2.27),(2.28) and (2.29) we get (2.18). □

Remark 2.8. *1- Taking $k=\alpha=1$ and $h(x)=e^x$ in (2.18), we get (1.3).*
2- Taking $k=\alpha=1, s=0$ and $h(x)=x$ in (2.22), we get (1.1).

Theorem 2.9. *Let Φ be an integrable function on $[a,b]$ satisfying the condition (1.4), let p be a positive function on $[a,b]$ and let h be a measurable, increasing, positive function on $(a,b]$ and $h\in C^1([a,b])$. Then, we have*

$$\begin{aligned}
&\left|{}_kI_{a,h}^{\alpha}[p(t)]\ {}_kI^{\alpha}\left[p\Phi^2(t)\right]-\ \left({}_kI_{a,h}^{\alpha}[p\Phi(t)]\right)^2\right|\\
\leq\ &\frac{(M-m)^2}{4}\left({}_kI_{a,h}^{\alpha}[p(t)]\right)^2.
\end{aligned} \tag{2.30}$$

Proof. Applying Theorem 6 for $\Phi(x)=\Psi(x)$, we obtain (2.30). □

Now we use two real positive parameters to prove the resuts:

Theorem 2.10. *Let Φ and Ψ be two integrable functions on $[a,b]$ satisfying the condition (1.4), let p be a positive function on $[a,b]$ and let h be a measurable, increasing, positive function on $(a,b]$ with $h\in C^1([a,b])$. Then, the following inequality holds*

$$\begin{aligned}
&\Big|{}_kI_{a,h}^{\alpha}[p(t)]\ {}_kI^{\beta}[p\Phi\Psi(t)]+{}_kI_{a,h}^{\beta}[p(t)]\ {}_kI^{\alpha}[p\Phi\Psi(t)] \qquad (2.31)\\
&-\ {}_kI_{a,h}^{\alpha}[p\Phi(t)]_kI_{a,h}^{\beta}[p\Psi(t)]-\ {}_kI_{a,h}^{\beta}[p\Phi(t)]_kI_{a,h}^{\alpha}[p\Psi(t)]\Big|^2\\
\leq\ &\Big\{\left[\left(M_kI_{a,h}^{\alpha}[p(t)]-\ {}_kI_{a,h}^{\alpha}[p\Phi(t)]\right)\left({}_kI_{a,h}^{\beta}[p\Phi(t)]-m_kI_{a,h}^{\beta}[p(t)]\right)\right]\\
&+\left[\left(M_kI_{a,h}^{\beta}[p(t)]-\ {}_kI_{a,h}^{\beta}[p\Phi(t)]\right)\left({}_kI_{a,h}^{\alpha}[p\Phi(t)]-m_kI_{a,h}^{\alpha}[p(t)]\right)\right]\Big\}\\
&\times\Big\{\left[\left(N_kI_{a,h}^{\alpha}[p(t)]-\ {}_kI_{a,h}^{\alpha}[p\Psi(t)]\right)\left({}_kI_{a,h}^{\beta}[p\Psi(t)]-n_kI_{a,h}^{\beta}[p(t)]\right)\right]\\
&+\left[\left(N_kI_{a,h}^{\beta}[p(t)]-\ {}_kI_{a,h}^{\beta}[p\Psi(t)]\right)\left({}_kI_{a,h}^{\alpha}[p\Psi(t)]-n_kI_{a,h}^{\alpha}[p(t)]\right)\right]\Big\},
\end{aligned}$$

where $\alpha,\beta>0, k>0$.

We need the two following lemmas.

Lemma 2.11. *Let Φ and Ψ be two integrable functions on $[a,b]$ satisfying the condition (1.4), let p be a positive function on $[a,b]$ and let h be a measurable, increasing, positive function on $(a,b]$ with $h\in C^1([a,b])$. Then for all $\alpha,\beta>0$, we have*

$$\begin{aligned} &\Big\{ {}_kI^{\alpha}_{a,h}[p(t)]\ {}_kI^{\beta}_{a,h}[p\Phi\Psi(t)] + {}_kI^{\beta}_{a,h}[p(t)]\ {}_kI^{\alpha}_{a,h}[p\Phi\Psi(t)] \\ &- {}_kI^{\alpha}_{a,h}[p\Phi(t)]\,{}_kI^{\beta}_{a,h}[p\Psi(t)] - {}_kI^{\beta}_{a,h}[p\Phi(t)]\,{}_kI^{\alpha}_{a,h}[p\Psi(t)]\Big\}^2 \\ \leq\ &\Big\{ {}_kI^{\alpha}_{a,h}[p(t)]\ {}_kI^{\beta}_{a,h}\left[p\Phi^2(t)\right] + {}_kI^{\beta}_{a,h}[p(t)]\ {}_kI^{\alpha}_{a,h}\left[p\Phi^2(t)\right] \\ &- 2\,{}_kI^{\alpha}_{a,h}[p\Phi(t)]\,{}_kI^{\beta}_{a,h}[p\Phi(t)]\Big\} \\ &\times\Big\{ {}_kI^{\alpha}_{a,h}[p(t)]\ {}_kI^{\beta}_{a,h}\left[p\Psi^2(t)\right] + {}_kI^{\beta}_{a,h}[p(t)]\ {}_kI^{\alpha}_{a,h}\left[p\Psi^2(t)\right] \\ &- 2\,{}_kI^{\alpha}_{a,h}[p\Psi(t)]\,{}_kI^{\beta}_{a,h}[p\Psi(t)]\Big\}. \end{aligned} \tag{2.32}$$

Proof. Using (2.13), we get

$$\begin{aligned} &\int_a^t\int_a^t {}^s_kH_{\alpha,h}(t,\tau)\ {}^s_kH_{\beta,h}(t,\rho)\,(\Phi(t)-\Phi(\tau))\,(\Psi(t)-\Psi(\rho))\,d\tau ds \\ =\ & {}^s_kJ^{\alpha}_{a,h}p(t)\ {}^s_kJ^{\beta}_{a,h}(p\Phi\Psi)(t) + {}^s_kJ^{\beta}_{a,h}p(t)\ {}^s_kJ^{\alpha}_{a,h}(p\Phi\Psi)(t) \\ &- {}^s_kJ^{\alpha}_{a,h}(p\Phi)(t)\ {}^s_kJ^{\beta}_{a,h}(p\Psi)(t) - {}^s_kJ^{\beta}_{a,h}(p\Phi)(t)\ {}^s_kJ^{\alpha}_{a,h}(p\Psi)(t). \end{aligned} \tag{2.33}$$

Then, by Cauchy Schwarz inequality, we get

$$\begin{aligned} &\Big[\left({}^s_kJ^{\alpha}_{a,h}[p(t)]\right)\left({}^s_kJ^{\beta}_{a,h}[p\Phi\Psi(t)]\right) + \left({}^s_kJ^{\beta}_{a,h}[p(t)]\right)\left({}^s_kJ^{\alpha}_{a,h}[p\Phi\Psi(t)]\right) \\ &- \left({}^s_kJ^{\alpha}_{a,h}[p\Phi(t)]\right)\left({}^s_kJ^{\beta}_{a,h}[p\Psi(t)]\right) - \left({}^s_kJ^{\beta}_{a,h}[p\Phi(t)]\right)\left({}^s_kJ^{\alpha}_{a,h}[p\Psi(t)]\right)\Big]^2 \\ \leq\ &\Big\{ {}^s_kJ^{\alpha}_{a,h}[p(t)]\ {}^s_kJ^{\beta}_{a,h}\left[p\Phi^2(t)\right] + {}^s_kJ^{\beta}_{a,h}[p(t)]\ {}^s_kJ^{\alpha}_{a,h}\left[p\Phi^2(t)\right] \\ &- 2\,{}^s_kJ^{\alpha}_{a,h}[p\Phi(t)]\,{}^s_kJ^{\beta}_{a,h}[p\Phi(t)]\Big\} \\ &\times\Big\{ {}^s_kJ^{\alpha}_{a,h}[p(t)]\ {}^s_kJ^{\beta}_{a,h}\left[p\Psi^2(t)\right] + {}^s_kJ^{\beta}_{a,h}[p(t)]\ {}^s_kJ^{\alpha}_{a,h}\left[p\Psi^2(t)\right] \\ &- 2\,{}^s_kJ^{\alpha}_{a,h}[p\Psi(t)]\,{}^s_kJ^{\beta}_{a,h}[p\Psi(t)]\Big\}. \end{aligned} \tag{2.34}$$

Now, applying limit to both sides of (2.34) for $s\longrightarrow -1^+$, we get (2.32). □

Lemma 2.12. *Let φ be an integrable function on $[a,b]$ satisfying the condition (1.4) on $[a,b]$, let p be a positive function on $[a,b]$ and let h be a measurable, increasing, positive function on $(a,b]$ with $h\in C^1([a,b])$. Then for all $\alpha,\beta>0$, we have*

$$\begin{aligned} &{}_kI^{\alpha}_{a,h}[p(t)]\ {}_kI^{\beta}_{a,h}\left[p\varphi^2(t)\right] + {}_kI^{\beta}_{a,h}[p(t)]\ {}_kI^{\alpha}_{a,h}\left[p\varphi^2(t)\right] \\ &-2\left({}_kI^{\alpha}_{a,h}[p\varphi(t)]\right)\left({}_kI^{\beta}_{a,h}[p\varphi(t)]\right) \\ =\ &\left(M\left({}_kI^{\beta}_{a,h}[p(t)]\right) - {}_kI^{\beta}_{a,h}[p\varphi(t)]\right)\left({}_kI^{\alpha}_{a,h}[p\varphi(t)] - m\left({}_kI^{\alpha}_{a,h}[p(t)]\right)\right) \\ &+\left(M\left({}_kI^{\alpha}_{a,h}[p(t)]\right) - {}_kI^{\alpha}_{a,h}[p\varphi(t)]\right)\left({}_kI^{\beta}_{a,h}[p\varphi(t)] - m\left({}_kI^{\beta}_{a,h}[p(t)]\right)\right) \\ &-\left({}_kI^{\alpha}_{a,h}[p(t)]\right)\left({}_kI^{\beta}_{a,h}[(M-\varphi(t))(\varphi(t)-m)\,p(t)]\right) \\ &-\left({}_kI^{\beta}_{a,h}[p(t)]\right)\left({}_kI^{\alpha}_{a,h}[(M-\varphi(t))(\varphi(t)-m)\,p(t)]\right). \end{aligned} \tag{2.35}$$

Proof. Let us multiplying (2.20) by ${}^s_kH_{\alpha,h}(t,\tau)\times{}^s_kH_{\beta,h}(t,\rho),(\tau,\rho)\in(a,t)^2, s\in\mathbb{R}-\{-1\}$ and integrating with respect to τ and ρ over

$(a,t)^2$, we get

$$\frac{1}{2k^2\Gamma_k^2(\alpha)}\left\{\int_a^t\int_a^t \lim_{s\longrightarrow -1^+}\left[\left(\frac{h^{s+1}(t)-h^{s+1}(\tau)}{s+1}\right)^{\frac{\alpha}{k}-1} h^s(\tau)h'(\tau)(p(\tau)\varphi(\tau)-mp(\tau))\right.\right.$$
$$\left.\times\left(\frac{h^{s+1}(t)-h^{s+1}(\rho)}{s+1}\right)^{\frac{\beta}{k}-1} h^s(\rho)h'(\rho)(Mp(\rho)-p(\rho)\varphi(\rho))\right]d\tau d\rho$$
$$+\int_a^t\int_a^t \lim_{s\longrightarrow -1^+}\left[\left(\frac{h^{s+1}(t)-h^{s+1}(\tau)}{s+1}\right)^{\frac{\alpha}{k}-1} h^s(\tau)h'(\tau)(Mp(\tau)-p(\tau)\varphi(\tau))\right.$$
$$\left.\times\left(\frac{h^{s+1}(t)-h^{s+1}(\rho)}{s+1}\right)^{\frac{\beta}{k}-1} h^s(\rho)h'(\rho)(p(\rho)\varphi(\rho)-mp(\rho))\right]d\tau d\rho$$
$$-\int_a^t\int_a^t \lim_{s\longrightarrow -1^+}\left[\left(\frac{h^{s+1}(t)-h^{s+1}(\tau)}{s+1}\right)^{\frac{\alpha}{k}-1} h^s(\tau)h'(\tau)(M-\varphi(\tau))(\varphi(\tau)-m)\right.$$
$$\left.\times\left(\frac{h^{s+1}(t)-h^{s+1}(\rho)}{s+1}\right)^{\frac{\beta}{k}-1} h^s(\rho)h'(\rho)p(\rho)p(\tau)\right]d\tau d\rho$$
$$-\int_a^t\int_a^t \lim_{s\longrightarrow -1^+}\left[\left(\frac{h^{s+1}(t)-h^{s+1}(\tau)}{s+1}\right)^{\frac{\alpha}{k}-1} h^s(\tau)h'(\tau)p(\tau)\right.$$
$$\left.\left.\times\left(\frac{h^{s+1}(t)-h^{s+1}(\rho)}{s+1}\right)^{\frac{\beta}{k}-1} h^s(\rho)h'(\rho)p(\rho)(M-\varphi(\rho))(\varphi(\rho)-m)\right]d\tau d\rho\right\}$$

which implies that

$$\frac{1}{2k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\left\{\int_a^t\int_a^t\left[\left(\log\frac{h(t)}{h(\tau)}\right)^{\frac{\alpha}{k}-1}\left(\log\frac{h(t)}{h(\rho)}\right)^{\frac{\beta}{k}-1}\frac{h'(\tau)}{h(\tau)}\right.\right.$$
$$\left.\times\frac{h'(\rho)}{h(\rho)}(Mp(\rho)-p(\rho)\varphi(\rho))(p(\tau)\varphi(\tau)-mp(\tau))\right]d\tau d\rho$$
$$+\int_a^t\int_a^t\left[\left(\log\frac{h(t)}{h(\tau)}\right)^{\frac{\alpha}{k}-1}\left(\log\frac{h(t)}{h(\rho)}\right)^{\frac{\beta}{k}-1}\frac{h'(\rho)}{h(\rho)}\right.$$
$$\left.\times\frac{h'(\tau)}{h(\tau)}(Mp(\tau)-p(\tau)\varphi(\tau))(p(\rho)\varphi(\rho)-mp(\rho))\right]d\tau d\rho$$
$$-\int_a^t\int_a^t\left[\left(\log\frac{h(t)}{h(\tau)}\right)^{\frac{\alpha}{k}-1}\left(\log\frac{h(t)}{h(\rho)}\right)^{\frac{\beta}{k}-1}\frac{h'(\rho)}{h(\rho)}p(\rho)\right.$$
$$\left.\times\frac{h'(\tau)}{h(\tau)}p(\tau)(M-\varphi(\tau))(\varphi(\tau)-m)\right]d\tau d\rho$$
$$-\int_a^t\int_a^t\left[\left(\log\frac{h(t)}{h(\tau)}\right)^{\frac{\alpha}{k}-1}\left(\log\frac{h(t)}{h(\rho)}\right)^{\frac{\beta}{k}-1}\frac{h'(\tau)}{h(\tau)}p(\tau)\right.$$
$$\left.\left.\times\frac{h'(\rho)}{h(\rho)}p(\rho)(M-\varphi(\rho))(\varphi(\rho)-m)\right]d\tau d\rho\right\}.$$

Lemma 12 is thus proved. □

Now, we give the proof of the Theorem 10.

Proof. We have

$$(M-\Phi(t))(\Phi(t)-m)\geq 0 \text{ and } (N-\Psi(t))(\Psi(t)-m)\geq 0, \tag{2.36}$$

then, we observe that

$$\begin{aligned}&-\left({}_k^sJ_{a,h}^{\alpha}[p(t)]\right)\left({}_k^sJ_{a,h}^{\beta}[(M-\varphi(t))(\varphi(t)-m)p(t)]\right)\\&-\left({}_k^sJ_{a,h}^{\beta}[p(t)]\right)\left({}_k^sJ_{a,h}^{\alpha}[(M-\varphi(t))(\varphi(t)-m)p(t)]\right)\leq 0.\end{aligned} \tag{2.37}$$

And from Lemma 12, using the identity with the (s,k,h) −Riemann-Liouville fractional integral and applying Cauchy Schwarz integral inequality for double integrals with two parameters $\alpha,\beta>0$, the Lemma 11, (2.36), (2.37) and $\lim_{s\longrightarrow -1^+}\left({}_k^sJ_{a,h}^{\alpha},{}_k^sJ_{a,h}^{\beta}\right)$ we get (2.31). □

Theorem 2.13. *Let Φ be integrable functions on $[a,b]$ satisfying the condition (1.4), let p be a positive function on $[a,b]$ and let h be a measurable, increasing, positive function on (a,b) with $h \in C^1([a,b])$. Then for all $t > 0$, the following inequality holds*

$$\begin{aligned} & \Big| {}_kI^{\alpha}_{a,h}[p(t)]\ {}_kI^{\beta}\left[p\Phi^2(t)\right] + {}_kI^{\beta}_{a,h}[p(t)]\ {}_kI^{\alpha}\left[p\Phi^2(t)\right] \\ & - 2\ {}_kI^{\alpha}_{a,h}[p\Phi(t)]\,{}_kI^{\beta}_{a,h}[p\Phi(t)]\Big|^2 \\ \leq\ & 2\Big\{\Big[\Big(M_k I^{\alpha}_{a,h}[p(t)] - {}_kI^{\alpha}_{a,h}[p\Phi(t)]\Big)\Big({}_kI^{\beta}_{a,h}[p\Phi(t)] - m_k I^{\beta}_{a,h}[p(t)]\Big)\Big] \\ & + \Big[\Big(M_k I^{\beta}_{a,h}[p(t)] - {}_kI^{\beta}_{a,h}[p\Phi(t)]\Big)\Big({}_kI^{\alpha}_{a,h}[p\Phi(t)] - m_k I^{\alpha}_{a,h}[p(t)]\Big)\Big]\Big\}, \end{aligned} \tag{2.38}$$

where $\alpha, \beta > 0, k > 0.$

Proof. Applying Theorem 10 for $\Phi(x) = \Psi(x)$, we obtain (2.38). □

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