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f-Asymptotically $\mathscr{I}_\sigma$-Equivalence of Real Sequences

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Abstract

In this study, we present the notions of strongly asymptotically $\mathscr{I}$-invariant equivalence, f-asymptotically $\mathscr{I}$-invariant equivalence, strongly f-asymptotically $\mathscr{I}$-invariant equivalence and asymptotically $\mathscr{I}$-invariant statistical equivalence for real sequences. Also, we investigate some relationships among them.

Keywords: *Ideal, Asymptotically equivalence, $\mathscr{I}$-convergence, Modulus function, Invariant convergence.*
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1. Introduction, Definitions and Notations

Throughout the paper $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{R}$ denotes the set of real numbers. The concept of convergence of a real sequence was extended to statistical convergence independently by Fast [2], Schoenberg [23] and then statistical convergence has been studied by many authors. The idea of $\mathscr{I}$-convergence was introduced by Kostyrko et al. [4] as a generalization of statistical convergence.
Several authors including Raimi [18], Schaefer [22], Mursaleen and Edely [9], Mursaleen [11], Savaş [19, 20], Nuray and Savaş [13], Pancaroğlu and Nuray [15] and some authors have studied on invariant convergent sequences. The concept of strongly σ-convergence was defined by Mursaleen [10]. Then Savaş and Nuray [21] introduced the concepts of σ-statistical convergence and lacunary σ-statistical convergence and gave some inclusion relations. After that Nuray et al. [14] defined the concepts of σ-uniform density of a subset A of $\mathbb{N}$, $\mathscr{I}_\sigma$-convergence and investigated relationships between $\mathscr{I}_\sigma$-convergence and invariant convergence also $\mathscr{I}_\sigma$-convergence and $[V_\sigma]_p$-convergence. Also, Pancaroḡlu and Nuray [15] studied statistical lacunary invariant summability.
Marouf [8] presented definitions for asymptotically equivalence and asymptotic regular matrices. Patterson [16] presented the concept of asymptotically statistical equivalent sequences for non-negative summability matrices. Ulusu [24] studied the concept of asymptotically ideal invariant equivalence.
The Modulus function was introduced by Nakano [12]. Maddox [7], Pehlivan [17] and many authors used a modulus function f to define some new concepts and give some inclusion theorems. Kumar and Sharma [5] studied on lacunary equivalent sequences by ideals and a modulus function.

Now, we recall the basic concepts and some definitions and notations (See [1, 3, 4, 6, 7, 8, 14, 16, 17]).

Let σ be a mapping such that $\sigma : \mathbb{N}^+ \to \mathbb{N}^+$ (the set of positive integers). A continuous linear functional ψ on ℓ_∞, the space of bounded sequences, is said to be an invariant mean or a σ-mean if it satisfies the following conditions:

1. $\psi(x_n) \geq 0$, when the sequence (x_n) has $x_n \geq 0$ for all n,
2. $\psi(e) = 1$, where $e = (1,1,1,\ldots)$ and
3. $\psi(x_{\sigma(n)}) = \psi(x_n)$ for all $(x_n) \in \ell_\infty$.

The mappings σ are assumed to be one-to-one and such that $\sigma^m(n) \neq n$ for all $m,n \in \mathbb{N}^+$, where $\sigma^m(n)$ denotes the m th iterate of the mapping σ at n. Thus ψ extends the limit functional on c, the space of convergent sequences, in the sense that $\psi(x_n) = \lim x_n$ for all $(x_n) \in c$.

A family of sets $\mathscr{I} \subseteq 2^{\mathbb{N}}$ is called an ideal if and only if
(i) $\emptyset \in \mathscr{I}$,
(ii) For each $A, B \in \mathscr{I}$ we have $A \cup B \in \mathscr{I}$,
(iii) For each $A \in \mathscr{I}$ and each $B \subseteq A$ we have $B \in \mathscr{I}$.

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An ideal $\mathscr{I} \subseteq 2^{\mathbb{N}}$ is called non-trivial if $\mathbb{N} \notin \mathscr{I}$ and a non-trivial ideal $\mathscr{I} \subseteq 2^{\mathbb{N}}$ is called admissible if $\{n\} \in \mathscr{I}$ for each $n \in \mathbb{N}$. Throughout the paper, we let $\mathscr{I}$ be an admissible ideal in $\mathbb{N}$.

Let $A \subseteq \mathbb{N}$ and

$$s_m = \min_n \left| A \cap \left\{ \sigma(n), \sigma^2(n), ..., \sigma^m(n) \right\} \right|$$

and

$$S_m = \max_n \left| A \cap \left\{ \sigma(n), \sigma^2(n), ..., \sigma^m(n) \right\} \right|.$$

If the limits

$$\underline{V}(A) = \lim_{m\to\infty} \frac{s_m}{m} \text{ and } \overline{V}(A) = \lim_{m\to\infty} \frac{S_m}{m}$$

exist, then they are called a lower σ-uniform density and an upper σ-uniform density of the set A, respectively. If $\underline{V}(A) = \overline{V}(A)$, then $V(A) = \underline{V}(A) = \overline{V}(A)$ is called the σ-uniform density of A.
Denote by $\mathscr{I}_\sigma$ the class of all $A \subseteq \mathbb{N}$ with $V(A) = 0$. Obviously $\mathscr{I}_\sigma$ is an admissible ideal in $\mathbb{N}$.

A sequence $x = (x_k)$ is said to be $\mathscr{I}_\sigma$-convergent to L if for every $\varepsilon > 0$, the set

$$A_\varepsilon = \{k : |x_k - L| \geq \varepsilon\}$$

belongs to $\mathscr{I}_\sigma$, i.e., $V(A_\varepsilon) = 0$. It is denoted by $\mathscr{I}_\sigma - \lim x_k = L$.

Two non-negative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically equivalent if

$$\lim_k \frac{x_k}{y_k} = 1$$

(denoted by $x \sim y$).

Two non-negative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically statistical equivalent of multiple L if for every $\varepsilon > 0$,

$$\lim_n \frac{1}{n} \left| \left\{ k \leq n : \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} \right| = 0$$

(denoted by $x \overset{S_L}{\sim} y$) and simply asymptotically statistical equivalent if $L = 1$.

Two non-negative sequences $x = (x_k)$ and $y = (y_k)$ are said to be strongly asymptotically equivalent of multiple L with respect to the ideal $\mathscr{I}$ if for every $\varepsilon > 0$,

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^{n} \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} \in \mathscr{I}$$

(denoted by $x \overset{\mathscr{I}(\omega)}{\sim} y$) and simply strongly asymptotically equivalent with respect to the ideal $\mathscr{I}$ if $L = 1$.
Two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically $\mathscr{I}_\sigma$-equivalent of multiple L if for every $\varepsilon > 0$,

$$\widetilde{A_\varepsilon} = \left\{ k \in \mathbb{N} : \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} \in \mathscr{I}_\sigma,$$

i.e., $V(A_\varepsilon) = 0$. It is denoted by $x \overset{\mathscr{I}_\sigma^L}{\sim} y$.

A function $f : [0,\infty) \to [0,\infty)$ is called a modulus if

1. $f(x) = 0$ if and if only if $x = 0$,
2. $f(x+y) \leq f(x) + f(y)$,
3. f is increasing,
4. f is continuous from the right at 0.

A modulus may be unbounded (for example $f(x) = x^p$, $0 < p < 1$) or bounded (for example $f(x) = \frac{x}{x+1}$).
Let f be a modulus function. Two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be f-asymptotically equivalent of multiple L with respect to the ideal $\mathscr{I}$ provided that, for every $\varepsilon > 0$

$$\left\{ k \in \mathbb{N} : f\left(\left| \frac{x_k}{y_k} - L \right| \right) \geq \varepsilon \right\} \in \mathscr{I}$$

(denoted by $x \overset{\mathscr{I}(f)}{\sim} y$) and simply f-asymptotically $\mathscr{I}$-equivalent if $L = 1$.
Let f be a modulus function. Two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be strongly f-asymptotically equivalent of multiple L with respect to the ideal $\mathscr{I}$ provided that, for every $\varepsilon > 0$

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^{n} f\left(\left| \frac{x_k}{y_k} - L \right| \right) \geq \varepsilon \right\} \in \mathscr{I}$$

(denoted by $x \overset{\mathscr{I}(\omega_f)}{\sim} y$) and simply strongly f-asymptotically $\mathscr{I}$-equivalent if $L = 1$.

Lemma 1.1. *[17] Let f be a modulus and $0 < \delta < 1$. Then, for each $x \geq \delta$ we have $f(x) \leq 2f(1)\delta^{-1}x$.*

2. Main Results

Definition 2.1. *Two non-negative sequences $x=(x_k)$ and $y=(y_k)$ are said to be strongly asymptotically $\mathscr{I}$-invariant equivalent of multiple L if for every $\varepsilon>0$,*

$$\left\{n\in\mathbb{N}:\frac{1}{n}\sum_{k=1}^{n}\left|\frac{x_k}{y_k}-L\right|\geq\varepsilon\right\}\in\mathscr{I}_\sigma$$

(denoted by $x\overset{[\mathscr{I}_\sigma^L]}{\sim}y$) and simply strongly asymptotically $\mathscr{I}$-invariant equivalent if $L=1$.

Definition 2.2. *Let f be a modulus function. Two non-negative sequences $x=(x_k)$ and $y=(y_k)$ are said to be f-asymptotically $\mathscr{I}$-invariant equivalent of multiple L if for every $\varepsilon>0$,*

$$\left\{k\in\mathbb{N}:f\left(\left|\frac{x_k}{y_k}-L\right|\right)\geq\varepsilon\right\}\in\mathscr{I}_\sigma$$

(denoted by $x\overset{\mathscr{I}_\sigma^L(f)}{\sim}y$) and simply f-asymptotically $\mathscr{I}$-invariant equivalent if $L=1$.

Definition 2.3. *Let f be a modulus function. Two non-negative sequences $x=(x_k)$ and $y=(y_k)$ are said to be strongly f-asymptotically $\mathscr{I}$-invariant equivalent of multiple L if for every $\varepsilon>0$,*

$$\left\{n\in\mathbb{N}:\frac{1}{n}\sum_{k=1}^{n}f\left(\left|\frac{x_k}{y_k}-L\right|\right)\geq\varepsilon\right\}\in\mathscr{I}_\sigma$$

(denoted by $x\overset{[\mathscr{I}_\sigma^L(f)]}{\sim}y$) and simply strongly f-asymptotically $\mathscr{I}$-invariant equivalent if $L=1$.

Theorem 2.4. *Let f be a modulus function. Then, for two non-negative sequences $x=(x_k)$ and $y=(y_k)$, we have $x\overset{[\mathscr{I}_\sigma^L]}{\sim}y\Rightarrow x\overset{[\mathscr{I}_\sigma^L(f)]}{\sim}y$.*

Proof. Suppose that $x\overset{[\mathscr{I}_\sigma^L]}{\sim}y$ and $\varepsilon>0$ be given. Also, select $0<\delta<1$ such that $f(t)<\varepsilon$, for $0\leq t\leq\delta$. Then, we have

$$\frac{1}{n}\sum_{k=1}^{n}f\left(\left|\frac{x_k}{y_k}-L\right|\right)=\frac{1}{n}\sum_{\substack{k=1\\ \left|\frac{x_k}{y_k}-L\right|\leq\delta}}^{n}f\left(\left|\frac{x_k}{y_k}-L\right|\right)+\frac{1}{n}\sum_{\substack{k=1\\ \left|\frac{x_k}{y_k}-L\right|>\delta}}^{n}f\left(\left|\frac{x_k}{y_k}-L\right|\right)$$

and so by Lemma 1.1

$$\frac{1}{n}\sum_{k=1}^{n}f\left(\left|\frac{x_k}{y_k}-L\right|\right)<\varepsilon+\left(\frac{2f(1)}{\delta}\right)\frac{1}{n}\sum_{k=1}^{n}\left|\frac{x_k}{y_k}-L\right|.$$

Thus, for any $\gamma>0$

$$\left\{n\in\mathbb{N}:\frac{1}{n}\sum_{k=1}^{n}f\left(\left|\frac{x_k}{y_k}-L\right|\right)\geq\gamma\right\}\subseteq\left\{n\in\mathbb{N}:\frac{1}{n}\sum_{k=1}^{n}\left|\frac{x_k}{y_k}-L\right|\geq\frac{(\gamma-\varepsilon)\delta}{2f(1)}\right\}.$$

Since $x\overset{[\mathscr{I}_\sigma^L]}{\sim}y$, it follows the second set and so, the first set in above expression belongs to $\mathscr{I}_\sigma$. This proves that $x\overset{[\mathscr{I}_\sigma^L(f)]}{\sim}y$. □

Theorem 2.5. *If $\lim_{t\to\infty}\frac{f(t)}{t}=\alpha>0$, then for two non-negative sequences $x=(x_k)$ and $y=(y_k)$, we have $x\overset{[\mathscr{I}_\sigma^L(f)]}{\sim}y\Leftrightarrow x\overset{[\mathscr{I}_\sigma^L]}{\sim}y$.*

Proof. In Theorem 2.4, we showed that $x\overset{[\mathscr{I}_\sigma^L]}{\sim}y\Rightarrow x\overset{[\mathscr{I}_\sigma^L(f)]}{\sim}y$. Now, we must show that

$$x\overset{[\mathscr{I}_\sigma^L(f)]}{\sim}y\Rightarrow x\overset{[\mathscr{I}_\sigma^L]}{\sim}y.$$

Let $\lim_{t\to\infty}\frac{f(t)}{t}=\alpha>0$. Then, we have $f(t)\geq\alpha t$, for all $t\geq0$. Assume that $x\overset{[\mathscr{I}_\sigma^L(f)]}{\sim}y$. Since

$$\frac{1}{n}\sum_{k=1}^{n}f\left(\left|\frac{x_k}{y_k}-L\right|\right)\geq\frac{1}{n}\sum_{k=1}^{n}\alpha\left(\left|\frac{x_k}{y_k}-L\right|\right)=\alpha\left(\frac{1}{n}\sum_{k=1}^{n}\left|\frac{x_k}{y_k}-L\right|\right),$$

it follows that for each $\varepsilon>0$, we have

$$\left\{n\in\mathbb{N}:\frac{1}{n}\sum_{k=1}^{n}\left|\frac{x_k}{y_k}-L\right|\geq\varepsilon\right\}\subseteq\left\{n\in\mathbb{N}:\frac{1}{n}\sum_{k=1}^{n}f\left(\left|\frac{x_k}{y_k}-L\right|\right)\geq\alpha\varepsilon\right\}.$$

Since $x_k\overset{[\mathscr{I}_\sigma^L(f)]}{\sim}y_k$, it follows the second set and so, the first set in above expression belongs to $\mathscr{I}_\sigma$. This proves that

$$x_k\overset{[\mathscr{I}_\sigma^L(f)]}{\sim}y_k\Leftrightarrow x_k\overset{[\mathscr{I}_\sigma^L]}{\sim}y_k.$$

□

Definition 2.6. *Two non-negative sequences (x_k) and (y_k) are said to be asymptotically $\mathscr{I}$-invariant statistical equivalent of multiple L if for every $\varepsilon > 0$ and each $\gamma > 0$,*

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : \left|\frac{x_k}{y_k} - L\right| \geq \varepsilon\right\}\right| \geq \gamma\right\} \in \mathscr{I}_\sigma$$

(denoted by $x \overset{\mathscr{I}(S_\sigma^L)}{\sim} y$) and simply asymptotically $\mathscr{I}$-invariant statistical equivalent if $L = 1$.

Theorem 2.7. *Let f be a modulus function. Then, for two non-negative sequences $x = (x_k)$ and $y = (y_k)$, we have $x \overset{[\mathscr{I}_\sigma^L(f)]}{\sim} y \Rightarrow x \overset{\mathscr{I}(S_\sigma^L)}{\sim} y$.*

Proof. Assume that $x \overset{[\mathscr{I}_\sigma^L(f)]}{\sim} y$ and $\varepsilon > 0$ be given. Since

$$\frac{1}{n}\sum_{k=1}^{n} f\left(\left|\frac{x_k}{y_k} - L\right|\right) \geq \frac{1}{n}\sum_{\substack{k=1 \\ \left|\frac{x_k}{y_k} - L\right| \geq \varepsilon}}^{n} f\left(\left|\frac{x_k}{y_k} - L\right|\right) \geq f(\varepsilon)\frac{1}{n}\left|\left\{k \leq n : \left|\frac{x_k}{y_k} - L\right| \geq \varepsilon\right\}\right|,$$

it follows that for any $\gamma > 0$, we have

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : \left|\frac{x_k}{y_k} - L\right| \geq \varepsilon\right\}\right| \geq \frac{\gamma}{f(\varepsilon)}\right\} \subseteq \left\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n} f\left(\left|\frac{x_k}{y_k} - L\right|\right) \geq \gamma\right\}.$$

Since $x \overset{[\mathscr{I}_\sigma^L(f)]}{\sim} y$, so the second set belongs to $\mathscr{I}_\sigma$. Then, the first set belongs to $\mathscr{I}_\sigma$ and therefore $x \overset{\mathscr{I}(S_\sigma^L)}{\sim} y$. □

Theorem 2.8. *Let f be a modulus function. If f is bounded, then for two non-negative sequences $x = (x_k)$ and $y = (y_k)$, we have $x \overset{\mathscr{I}(S_\sigma^L)}{\sim} y \Leftrightarrow x \overset{[\mathscr{I}_\sigma^L(f)]}{\sim} y$.*

Proof. In Theorem 2.7, we showed that $x \overset{[\mathscr{I}_\sigma^L(f)]}{\sim} y \Rightarrow x \overset{\mathscr{I}(S_\sigma^L)}{\sim} y$. Now, we must show that

$$x \overset{\mathscr{I}(S_\sigma^L)}{\sim} y \Rightarrow x \overset{[\mathscr{I}_\sigma^L(f)]}{\sim} y.$$

Assume that f is bounded and $x \overset{\mathscr{I}(S_\sigma^L)}{\sim} y$. Since f is bounded, there exists an positive real number M such that $|f(x)| \leq M$, for all $x \geq 0$. Further using this fact, we have

$$\frac{1}{n}\sum_{k=1}^{n} f\left(\left|\frac{x_k}{y_k} - L\right|\right) = \frac{1}{n}\sum_{\substack{k=1 \\ \left|\frac{x_k}{y_k} - L\right| \geq \varepsilon}}^{n} f\left(\left|\frac{x_k}{y_k} - L\right|\right) + \frac{1}{n}\sum_{\substack{k=1 \\ \left|\frac{x_k}{y_k} - L\right| < \varepsilon}}^{n} f\left(\left|\frac{x_k}{y_k} - L\right|\right)$$

$$\leq \frac{M}{n}\left|\left\{k \leq n : \left|\frac{x_k}{y_k} - L\right| \geq \varepsilon\right\}\right| + f(\varepsilon).$$

This proves that $x_k \overset{[\mathscr{I}_\sigma^L(f)]}{\sim} y_k$. □

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