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Araştırma Makalesi - Research Article

W-Yönlü Eğrilerden Elde Edilen Regle Yüzeyler

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ÖZ

Bu çalışmada, bir Frenet eğrisinin W-yönlü eğrisi dayanak eğrisi olarak kullanılıp elde edilen bazı özel regle yüzeyler tanımlanmıştır. Bu regle yüzeylerin açılabilirlik ve minimallik karakterizasyonlarını verdik. Esas eğri ve regle yüzeyin dayanak eğrisi arasındaki ilişkiyi, dayanak eğrisinin geodezik eğri, asimptotik eğri ve eğrilik çizgisi olması açısından inceledik. Ayrıca bazı örnekler de verildi.

Anahtar Kelimeler- Regle yüzey, W-yönlü eğri, Asimptotik eğri, Geodezik eğri, Eğrilik çizgisi

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Ruled Surfaces With W-direction Curves

ABSTRACT

In this study, some special ruled surfaces were determined which were constituted by using the base curve as the W-direction curves of a Frenet curve. We gave the characterizations of developability and minimality of these ruled surfaces. We investigated the relation between the primary curve and the base curve of ruled surfaces, we also inquired the occurence geodesic curve, asymptotic line and principal line of the W-direction curve on the ruled surface. Terminally, some examples were given.

Keywords- Ruled surface, W-direction curve, Asymptotic curve, Geodesic curve, Line of curvature

I. INTRODUCTION

The curve theory is a subsection of geometry which takes notice of curves in Euclidean space or different spaces by using calculus of differentiation and integration. Curve pairs such as Bertrand curves, involute-evolute curves, Mannheim partner curves and W-direction curves are associated curves which are researched in most subject matter in differential curve theory. Since these remarkable curves could be featured by the properties and behavior of the primary curves of them, operating with these associated curves is a pleasant outlook.

The most frequently utilised headings to qualify curves are cylindrical or general helix and slant helix. A *general helix* in E^3 is determined as; its tangent vector field performs a constant angle with a fixed direction. In the event that, the principal normal vector field of a curve performs a constant angle with a fixed direction, that curve is entitled as *slant helix*. Izumiya and Takeuchi acquired that a curve is a slant helix if and only if the function

$$\delta(s)=\left(\frac{\kappa^2}{\left(\kappa^2+\tau^2\right)^{3/2}}.\left(\frac{\tau}{\kappa}\right)'\right)(s) \tag{1}$$

is a constant function which is the geodesic curvature of the principal normal indicatrix curve [6].

Recently, Macit and Düldül described W-direction curve, W-rectifying curve and V-direction curve of a Frenet curve in E^3 and also principal direction curve, B_1-direction curve B_2-direction curve an B_2-rectifying curve in E^4. By integrating vector fields procured from the Frenet frame or Darboux frame among a curve, the associated curves mentioned above were imparted. The relationship of the curvature and the torsion among the associated curves and their primary curve and the Frenet vector fields is contributed [10].

The associated curves of a Frenet curve were examined in three dimensional compact Lie group G, by Kızıltuğ and Önder. The principal normal direction curve and principal normal donor curve were introduced, and certain attributions of those curves were gained in G [7].

Also associated curves were studied by Körpınar et al, in [8]. They defined these curves by using the Bishop frame and called the new curves as M_1-direction curve, M_2-direction curve, M_1-donor curve and M_2-donor curve. They gave some characterizations of these new curves.

In [12], Scofield gave a curve called constant precession which is defined by the property that the curve is traversed with unit speed, its centrode (Darboux vector field) revolves about a fixed axis with constant angle and constant speed. A curve of constant precession has a characterization with curvature and torsion which is $\kappa(s)=\omega\sin(\mu s),\quad \tau(s)=\omega\cos(\mu s)$ where $\omega\rangle 0$ and μ are constants.

The theory about the ruled surfaces can be seen in literature [3,4,5,6,13].

In this study, we investigated the normal surface and binormal surface which are ruled surfaces, by acquiring the sole curve as W-direction curve. We imparted beneficial results in the matter of being developable and minimal surface and being asymptotic curve, geodesic curve, line of curvature of the sole curves. We also illustrated these surfaces.

II. PRELIMINARIES

Let a curve be $\beta: I\to E^3$, $\{T,N,B\}$ state the Frenet frame and *s* is the arclength parameter of β. $T(s)=\beta'(s)$ is entitled the *unit tangent vector* of β at s. The *curvature* of β is designated as $\kappa(s)=\|\beta''(s)\|$ and the *torsion* is calculated as $\tau(s)=\|B'(s)\|$. N(s) is the *unit principal normal vector* and $\beta''(s)=\kappa(s)N(s)$. Also the unit vector $B(s)=T(s)\times N(s)$ is entitled the *unit binormal vector* of β at *s.* Then the renowned Frenet formula possess as;

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$$\begin{aligned} T'(s) &= \kappa(s)N(s), \\ N'(s) &= -\kappa(s)T(s) + \tau(s)B(s), \\ B'(s) &= -\tau(s)N(s). \end{aligned} \tag{2}$$

For *s* arc-length parametered curve β , the Frenet vectors are figured out by;

$$\begin{aligned} T(s) &= \beta'(s), \\ N(s) &= \frac{\beta''(s)}{\|\beta''(s)\|}, \\ B(s) &= T(s) \times N(s). \end{aligned} \tag{3}$$

For the curve $\beta : I \to E^3$ with *s* parameter of arc-length, the vector

$$W(s) = \tau(s)T(s) + \kappa(s)B(s) \tag{4}$$

is denominated the *Darboux vector* of β .This W(s) vector is the rotation vector of trihedral of β in the event that a point goes along the curve β by means of curvature $\kappa \neq 0$. *Frenet curve* β is unit speed curve if it possess the curvature which is not zero and $\beta''(s) \neq 0$.

Definition: Let a Frenet curve in E^3 be β and the Darboux vector field which is unit be W. *W-direction curve* of β is the integration curve of $W(s)$. Nominately, demonstrating $\overline{\beta}$ as the W-direction curve of β , $W(s) = \overline{\beta}'(s)$ is written. Herein $W(s) = \frac{1}{\sqrt{\kappa^2 + \tau^2}}(\tau T + \kappa B)$ [10].

For a Frenet curve β and its W-direction curve $\overline{\beta}$, the Frenet members are $\{T, N, B, \kappa, \tau\}$ and $\{\overline{T}, \overline{N}, \overline{B}, \overline{\kappa}, \overline{\tau}\}$ respectively. The relationship of Frenet members among the W-direction curve and the primary curve are contributed in [10] as;

$$\begin{aligned} \overline{T} &= \frac{\tau}{\sqrt{\kappa^2 + \tau^2}} T + \frac{\kappa}{\sqrt{\kappa^2 + \tau^2}} B, \\ \overline{N} &= -\frac{\kappa}{\sqrt{\kappa^2 + \tau^2}} T + \frac{\tau}{\sqrt{\kappa^2 + \tau^2}} B, \\ \overline{B} &= -N, \\ \overline{\kappa} &= \frac{|\tau\kappa' - \tau'\kappa|}{\kappa^2 + \tau^2}, \qquad \overline{\tau} = \sqrt{\kappa^2 + \tau^2}. \end{aligned} \tag{5}$$

Theorem: Let a curve be β which is not a general helix and $\overline{\beta}$ be the W-direction curve of it. Thereafter β is a slant helix necessary and sufficient condition $\overline{\beta}$ is a general helix [10].

Theorem: Let a curve be β which is not planar and $\overline{\beta}$ be the W-direction curve of it. Thereafter $\overline{\beta}$ is a straight line necessary and sufficient condition β is a general helix [10].

Theorem: For a curve, $\frac{\tau}{\kappa} = \text{constant}$ necessary and sufficient condition that curve is a general helix [11].

A surface in R^3 is entitled *ruled surface* which could be depicted as the points set scan the surface via a straight line scanning among the surface. Hence forth it holds a manner parametrization

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$$\Phi(s,v)=\alpha(s)+v\delta(s). \tag{6}$$

Here α is entitled *base curve* which is a curve lounging on the surface and *director curve* is δ. The straight lines of the surface are denominated as rulings. When we are utilizing ruled surface equation, we postulate that δ is on no account zero and α' is not likewise zero.

The *distribution parameter* of Φ given above is dedicated as;

$$\lambda=\frac{\det\left(\frac{d\alpha}{ds},\delta,\frac{d\delta}{ds}\right)}{\left\|\frac{d\delta}{ds}\right\|^2}. \tag{7}$$

The vector n which is standard unit normal vector field of Φ is identified by

$$n=\frac{\Phi_s\times\Phi_v}{\|\Phi_s\times\Phi_v\|} \tag{8}$$

where $\Phi_s=\frac{d\Phi}{ds}$ and $\Phi_v=\frac{d\Phi}{dv}$.

The Gaussian curvature and the mean curvature of a ruled surface Φ are calculated as;

$$K=\frac{eg-f^2}{EG-F^2} \tag{9}$$

and

$$H=\frac{Eg+Ge-2Ff}{2(EG-F^2)} \tag{10}$$

where $E=\langle\Phi_s,\Phi_s\rangle$, $F=\langle\Phi_s,\Phi_v\rangle$, $G=\langle\Phi_v,\Phi_v\rangle$, $e=\langle\Phi_{ss},n\rangle$, $f=\langle\Phi_{sv},n\rangle$ and $g=\langle\Phi_{vv},n\rangle$ [11].

For a ruled surface, asymptotic curves could be seen as rulings. Additionally, the ruled surface possesses negative Gaussian curvature every place . The distribution parameter vanishes necessary and sufficient condition the ruled surface is developable. Also its mean curvature vanishes necessary and sufficient condition it is minimal [4].

The general form of a ruled surface is imputed in equation (6). The ruled surfaces that are the *normal* and *binormal surface* are identified by

$$\begin{aligned}\Phi(s,v)&=\alpha(s)+vN(s),\\ \Phi(s,v)&=\alpha(s)+vB(s)\end{aligned} \tag{11}$$

where α is a curve and Frenet vector fields of α are $\{T,N,B\}$ [5,6].

Lets now give the concept of the Darboux frame of a curve on a surface. Let any curve be α with *s* arc-length parameter and its Frenet frame is $\{T,N,B\}$ among α. For a curve α lying on a surface, the frame $\{T,V,n\}$ among α curve is entitled the Darboux frame.

Hereby, T is the unit tangent vector of α, n is the unit normal vector of the surface and V is the unit vector imputed as $V=n\times T$. The relation among the derivatives and these frame vectors are

$$\begin{bmatrix} T' \\ V' \\ n' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_g & \kappa_n \\ -\kappa_g & 0 & \tau_g \\ -\kappa_n & -\tau_g & 0 \end{bmatrix} \begin{bmatrix} T \\ V \\ n \end{bmatrix} \tag{12}$$

where the geodesic curvature is κ_g, the normal curvature is κ_n and the geodesic torsion is τ_g [11]. These quatities are contributed with respect to the surface as

$$\begin{aligned} \kappa_g &= \langle n \times T, T' \rangle, \\ \kappa_n &= \langle \alpha'', n \rangle, \\ \tau_g &= \langle T, n \times n' \rangle. \end{aligned} \tag{13}$$

Following expressions are provided for a curve α which is lying on a surface [3]:

- The curve's geodesic curvature in regard to the surface vanishes if and only if α is a geodesic curve.
- The curve's normal curvature in regard to the surface vanishes if and only if α is an asymtotic curve.
- The curve's geodesic torsion in regard to the surface vanishes if and only if α is a line of curvature.

III. RULED SURFACES WITH W-DIRECTION CURVES

We will describe certain especial ruled surfaces in this part. These mentioned especial surfaces are constituted by utilizing the W-direction curve substituted for the base curve. Let a curve be β, *s* is its arc-length parameter and the W-direction curve of β is $\overline{\beta}$. The curve $\overline{\beta}$'s arc length parameter $\overline{s}$, can be acquired as $\overline{s} = s$ [1]. From the equation (11) the *normal* and *binormal surface* of $\overline{\beta}$ are given by

$$\Phi_1(s,v) = \overline{\beta}(s) + v\overline{N}(s), \tag{14}$$

$$\Phi_2(s,v) = \overline{\beta}(s) + v\overline{B}(s). \tag{15}$$

Remark: By the theorem given above; for a general helix curve β, the W-direction curve $\overline{\beta}$ of β, is a straight line. Since there is no Frenet 3-vectors of straight lines, it can not be created normal and binormal surfaces of straight lines. So we will take β curves which are not general helix, when we are creating the normal and binormal surfaces.

Theorem 1: For a Frenet curve β and its W-direction curve $\overline{\beta}$, the normal and binormal surface of $\overline{\beta}$ are not developable.

Proof: Bearing in mind that of the equation (7), contributed in (14) and (15) the normal and binormal surfaces' distribution parameters are

$$\lambda_{\Phi_1} = \frac{\det\left(\frac{d\overline{\beta}}{ds}, \overline{N}, \frac{d\overline{N}}{ds}\right)}{\left\|\frac{d\overline{N}}{ds}\right\|^2} \quad \text{and} \quad \lambda_{\Phi_2} = \frac{\det\left(\frac{d\overline{\beta}}{ds}, \overline{B}, \frac{d\overline{B}}{ds}\right)}{\left\|\frac{d\overline{B}}{ds}\right\|^2}. \tag{16}$$

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In deference to the equations conributed in (5) and

$$\frac{d\overline{N}}{ds} = -\left(\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\right)' T - \sqrt{\kappa^2+\tau^2}\, N + \left(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\right)' B, \qquad (17)$$
$$\frac{d\overline{B}}{ds} = \kappa T - \tau B$$

we have

$$\lambda_{\Phi_1} = \frac{(\kappa^2+\tau^2)^{5/2}}{(\tau\kappa' - \tau'\kappa)^2 + (\kappa^2+\tau^2)^3}, \qquad (18)$$
$$\lambda_{\Phi_2} = \frac{1}{\sqrt{\kappa^2+\tau^2}}.$$

The surfaces are not developable since the parameters above can not be zero.

Theorem 2: Let a Frenet curve be β, arc-length parameter of β be *s*, curvature and torsion of β be κ, τ and W-direction curve of β be $\overline{\beta}$. If the equations $\kappa^2+\tau^2 = c$ and $\frac{\tau}{\kappa} = a + b\int \frac{1}{\kappa^2}$; where a, b and c are constants, are provided then the normal surface of $\overline{\beta}$ is minimal and the binormal surface of $\overline{\beta}$ is not minimal.

Proof: We'll acquire the normal and binormal surfaces' mean curvatures in (9) and (10) for being minimal. For the normal surface imputed in (9), having regard to the fact the equations (1), the sequent equations are acquired

$$E_1 = A^2 + C^2 + D^2, \qquad (19)$$
$$F_1 = -AX + DY = 0,$$
$$G_1 = 1$$

and

$$e_1 = \frac{1}{Z_1}\left(-(A' + C\kappa)CY - (AY + DX)(A\kappa - C' - D\tau) - (D' - C\tau)CX\right), \qquad (20)$$
$$f_1 = \frac{1}{Z_1}\left(C(X'Y - Y'X) + \sqrt{\kappa^2+\tau^2}(AY + DX)\right),$$
$$g_1 = 0$$

where

$$X = \frac{\kappa}{\sqrt{\kappa^2+\tau^2}}, \qquad Y = \frac{\tau}{\sqrt{\kappa^2+\tau^2}} \qquad (21)$$
$$Z_1 = \left\|(\Phi_1)_s \times (\Phi_1)_v\right\|.$$

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$$\begin{aligned} n_1 &= \frac{1}{Z_1}\left((-YC)\vec{T} - (YA + XD)\vec{N} - (XC)\vec{B}\right), \\ A &= -X'v + Y, \\ C &= v\sqrt{\kappa^2 + \tau^2} \qquad \text{and} \qquad D = Y'v + X. \end{aligned} \tag{22}$$

For the binormal surface imparted in (15);

$$\begin{aligned} E_2 &= 1 + v^2(\kappa^2 + \tau^2), \\ F_2 &= 0, \\ G_2 &= 1, \end{aligned} \tag{23}$$

and 8

$$\begin{aligned} e_2 &= \frac{1}{Z_2}\left(v\frac{\kappa\kappa' + \tau\tau'}{\sqrt{\kappa^2 + \tau^2}} + \left(\frac{\tau}{\kappa}\right)' \frac{\kappa^2(1 + v^2(\kappa^2 + \tau^2))}{\kappa^2 + \tau^2} \right), \\ f_2 &= \frac{\sqrt{\kappa^2 + \tau^2}}{Z_2}, \\ g_2 &= 0 \end{aligned} \tag{24}$$

where $Z_2 = \sqrt{1 + v^2(\kappa^2 + \tau^2)}$ and $n_2 = \dfrac{1}{Z_2}\left((X - v\tau)\vec{T} - (Y + v\kappa)\vec{B}\right).$ (25)

By using the equation (10), we acquire the mean curvatures as

$$\begin{aligned} H_1 &= \frac{e_1}{2E_1}, \\ H_2 &= \frac{e_2}{2E_2}. \end{aligned} \tag{26}$$

e_2 was obtained above and if we elaborate e_1, we find that

$$e_1 = \frac{1}{Z_1}\left(v^2.\frac{2(\tau'\kappa - \tau\kappa')(\kappa\kappa' + \tau\tau') - (\tau'\kappa - \tau\kappa')'(\kappa^2 + \tau^2)}{(\kappa^2 + \tau^2)^{3/2}} + v\left(1 + v\left(\frac{\tau}{\kappa}\right)' \frac{1}{1 + \left(\frac{\tau}{\kappa}\right)^2}\right).\frac{\kappa\kappa' + \tau\tau'}{\sqrt{\kappa^2 + \tau^2}} \right) \tag{27}$$

If $\kappa\kappa' + \tau\tau' = 0$ and $\tau'\kappa - \tau\kappa' = \text{constant}$, then $e_1=0$. The first equation yields $\kappa^2 + \tau^2 = c$ and solution of second equation which is a linear differential equation is $\frac{\tau}{\kappa} = a + b\int \frac{1}{\kappa^2}$. Since β cannot be taken as general helix, $\left(\frac{\tau}{\kappa}\right)' \neq 0$, $\kappa^2\left(1 + v^2\left(\kappa^2 + \tau^2\right)\right) \rangle 0$ and also if $\kappa\kappa' + \tau\tau' = 0$, then $e_2 \neq 0$.

Theorem 3: Let a Frenet curve be β, arc-length parameter of β be *s*, curvature and torsion of β be κ, τ and W-direction curve of β be $\bar{\beta}$. If the equation $\frac{\tau}{\kappa} = \tan(cs + d)$ is satisfied, then the base curve $\bar{\beta}$ of

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the normal surface of $\bar{\beta}$ is geodesic curve, where c and d are constants. Also the base curve $\bar{\beta}$ of the binormal surface of $\bar{\beta}$ is geodesic curve.

Proof: Now firstly we acquire the normal and binormal surfaces' geodesic curvatures. By using the equation (13), the geodesic curvatures are

$$\begin{aligned}\kappa_{g_1} &= \langle n_1 \times \bar{T}, \bar{T}' \rangle,\\ \kappa_{g_2} &= \langle n_2 \times \bar{T}, \bar{T}' \rangle.\end{aligned} \tag{28}$$

By utilizing the same procedure with the former proof and $\bar{T} = Y\vec{T} + X\vec{B}$ we possess

$$\begin{aligned}n_1 \times \bar{T} &= \frac{1}{Z_1}(YA + XD)\left(-X\vec{T} + Y\vec{B}\right),\\ n_2 \times \bar{T} &= -\frac{1}{Z_2}\vec{N}.\end{aligned} \tag{29}$$

In the face of the equation $\bar{T}' = Y'\vec{T} + X'\vec{B}$, we gain

$$\begin{aligned}\kappa_{g_1} &= -\frac{1}{Z_1}(YA + XD)(XY' - YX'),\\ \kappa_{g_2} &= 0.\end{aligned} \tag{30}$$

By using appropriate statements and $Y'X - YX' = \dfrac{\tau'\kappa - \tau\kappa'}{\kappa^2 + \tau^2}$, we obtain finally that

$$\kappa_{g_1} = -\frac{1}{Z_1}\left(1 + v\left(\frac{\tau}{\kappa}\right)'\frac{\kappa^2}{\kappa^2+\tau^2}\right)\left(\frac{\tau}{\kappa}\right)'\frac{\kappa^2}{\kappa^2+\tau^2} \tag{31}$$

Since β cannot be taken as general helix, namely $\left(\frac{\tau}{\kappa}\right)' \neq 0$ and also $\kappa \neq 0$, if $v\left(\frac{\tau}{\kappa}\right)'\frac{\kappa^2}{\kappa^2+\tau^2} = -1$,then $\kappa_{g_1} = 0$. If we elaborate the equation $v\left(\frac{\tau}{\kappa}\right)'\frac{\kappa^2}{\kappa^2+\tau^2} = -1$,we can see that $\left(\frac{\tau}{\kappa}\right)'\frac{\kappa^2}{\kappa^2+\tau^2} = -\frac{1}{v}$ which only can be satisfied in the condition that both sides of the equation should be constant. Because the right and left sides of equations depend on the parameters v and s. Hence we take v as constant and finally we have the equation $\left(\frac{\tau}{\kappa}\right)'\frac{\kappa^2}{\kappa^2+\tau^2} = c$, where c is constant. The solution of the last equation is $\frac{\tau}{\kappa} = \tan(cs + d)$.

Theorem 4: Let a Frenet curve be β , arc-length parameter of β be *s* and W-direction curve of β be $\bar{\beta}$. The base curve $\bar{\beta}$ of the normal surface of $\bar{\beta}$ is asymptotic curve and the base curve $\bar{\beta}$ of the binormal surface of $\bar{\beta}$ is not asmptotic curve.

Proof: The normal and binormal surfaces' normal curvatures of $\bar{\beta}$ are computed by the equation (13) as

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$$\kappa_{n_1} = \langle \overline{\beta}'', n_1 \rangle, \qquad (32)$$
$$\kappa_{n_2} = \langle \overline{\beta}'', n_2 \rangle.$$

With an eye to $\overline{\beta}'' = \overline{T}' = Y'\overline{T} + X'\overline{B}$ and doing appropriate calculations we get

$$\kappa_{n_1} = -\frac{C}{Z_1}(XX' + YY'), \qquad (33)$$
$$\kappa_{n_2} = \frac{1}{Z_2}(XY' - YX').$$

By virtue of $XX' + YY' = 0$ and $XY' - YX' = \dfrac{\kappa\tau' - \tau\kappa'}{\kappa^2 + \tau^2}$, we facilely find out that

$$\kappa_{n_1} = 0, \qquad (34)$$
$$\kappa_{n_2} = \frac{1}{Z_2}\left(\frac{\tau}{\kappa}\right)' \frac{\kappa^2}{\kappa^2 + \tau^2}.$$

So the base curve $\overline{\beta}$ of the normal surface is asymptotic curve. Since $\left(\dfrac{\tau}{\kappa}\right)' \neq 0$ and $\kappa \neq 0$, then $\kappa_{n_2} \neq 0$.

Theorem 5: Let a Frenet curve be β, arc-length parameter of β be *s*, curvature and torsion of β be κ, τ and W-direction curve of β be $\overline{\beta}$. If the equation $\dfrac{\tau}{\kappa} = \tan(cs + d)$ is satisfied , then the base curve $\overline{\beta}$ of the normal surface of $\overline{\beta}$ is line of curvature. Also the base curve $\overline{\beta}$ of the binormal surface of $\overline{\beta}$ is line of curvature if and only if the equation $\dfrac{(\tau - \kappa)\sqrt{\kappa^2 + \tau^2}}{\kappa + \tau} = c$ satisfies, where c is constant.

Proof: By using the equation (13), the geodesic torsions of the normal and binormal surfaces of $\overline{\beta}$ are

$$\tau_{g_1} = \langle \overline{T}, n_1 \times n_1' \rangle, \qquad (35)$$
$$\tau_{g_2} = \langle \overline{T}, n_2 \times n_2' \rangle.$$

Subsequent to some calculations and using the annotations $r = -\dfrac{YC}{Z_1}$, $q = -\dfrac{YA + XD}{Z_1}$, $t = -\dfrac{XC}{Z_1}$ and $K = \left(\dfrac{X - v\tau}{Z_2}\right)'$, $L = \dfrac{X + Y + v(\kappa - \tau)}{Z_2}$, $M = \left(\dfrac{Y + v\kappa}{Z_2}\right)'$, we find

$$\begin{aligned} n_1 \times n_1' = & \left(\frac{XC}{Z_1}(r\kappa + q' - t\tau) - \frac{YA + XD}{Z_1}(t' + q\tau)\right)\overline{T} \\ & + \left(\frac{YC}{Z_1}(t' + q\tau) - \frac{XC}{Z_1}(r' - q\kappa)\right)\overline{N} \\ & + \left(\frac{YA + XD}{Z_1}(r' - q\kappa) - \frac{YC}{Z_1}(r\kappa + q' - t\tau)\right)\overline{B} \end{aligned} \qquad (37)$$

and

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$$n_2 \times n_2' = \left(\frac{Y+v\kappa}{Z_2}\right)L.\vec{T} + \left(\left(\frac{X-v\tau}{Z_2}\right)M - \left(\frac{Y+v\kappa}{Z_2}\right)K\right)\vec{N} + \left(\frac{X-v\tau}{Z_2}\right)L.\vec{B} \tag{38}$$

By taking into account that $\overline{T} = YT + XB$,we obtain lastly

$$\tau_{g_1} = \frac{1}{Z_1^2}\left(1+v\left(\frac{\tau}{\kappa}\right)'\frac{\kappa^2}{\kappa^2+\tau^2}\right)\left(\sqrt{\kappa^2+\tau^2}+\left(\frac{\tau}{\kappa}\right)'\kappa^2\left(\frac{v}{\sqrt{\kappa^2+\tau^2}}-\frac{C}{\kappa^2+\tau^2}\right)\right) \tag{39}$$

and

$$\tau_{g_1} = \frac{\kappa+\tau+v(\kappa-\tau)\sqrt{\kappa^2+\tau^2}}{Z_2^2\sqrt{\kappa^2+\tau^2}} \tag{40}$$

For τ_{g_1}, if $v\left(\frac{\tau}{\kappa}\right)'\frac{\kappa^2}{\kappa^2+\tau^2} = -1$ which is equal to $\frac{\tau}{\kappa} = \tan(cs+d)$, the result is apparent.

Also if $\kappa+\tau+v(\kappa-\tau)\sqrt{\kappa^2+\tau^2} = 0$, then $\tau_{g_2} = 0$ .This equation yields $\frac{(\tau-\kappa)\sqrt{\kappa^2+\tau^2}}{\kappa+\tau} = \frac{1}{v}$. Since the right and left sides depend on s and v, respectively, both sides should be constant to satisfy the equation. Therefore if we take v as constant, we have the result.

Corollary 1: Let a Frenet curve be β, arc-length parameter of β be *s*, curvature and torsion of β be κ, τ and W-direction curve of β be $\overline{\beta}$. If the curve β is constant precession curve, then the base curve $\overline{\beta}$ of the normal surface of $\overline{\beta}$ is geodesic curve and line of curvature.

Proof: Let the curve β be constant precession curve. Then β has the curvature and torsion as $\kappa(s) = \omega\sin(cs)$, $\tau(s) = \omega\cos(cs)$, where $\omega \rangle 0$ and c are constants. Thus the ratio

$$\frac{\tau}{\kappa} = \tan(cs) \tag{41}$$

is obtained simply, which gives the condition in Theorem 3 and 5 in the case $d = 0$.

Example: Let the slant helix be

$$\beta(s) = \left(-\frac{3}{2}\cos\left(\frac{s}{2}\right)-\frac{1}{6}\cos\left(\frac{3s}{2}\right),\ -\frac{3}{2}\sin\left(\frac{s}{2}\right)-\frac{1}{6}\sin\left(\frac{3s}{2}\right),\ \sqrt{3}\cos\left(\frac{s}{2}\right)\right). \tag{42}$$

The W-direction curve of β is found as

$$\overline{\beta}(s) = \left(-\frac{9s}{8}-6\sin\left(\frac{s}{2}\right)-\frac{3}{4}\sin(s)-\frac{1}{16}\sin(2s),\ -\frac{1}{2}\cos(s),\ \frac{\sqrt{3}}{2}s\right)+(c_1, c_2, c_3) \tag{43}$$

where c_1, c_2, c_3 are constants [11].

We calculated principal normal and binormal vectors of W-direction curve $\overline{\beta}$;

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$$\overline{N} = \left(-\sin(s), \quad \cos(s), \quad 0\right),$$
$$\overline{B} = \left(\frac{\sqrt{3}}{2}\cos(s), \quad \frac{\sqrt{3}}{2}\sin(s), \quad -\frac{1}{2}\right). \tag{44}$$

The normal and binormal surfaces of W-direction curve $\overline{\beta}$ are imputed respectively by

$$\Phi_1(s,v) = \begin{pmatrix} -\frac{9s}{8} - 6\sin\left(\frac{s}{2}\right) - \frac{3}{4}\sin(s) - \frac{1}{16}\sin(2s) - v\sin(s) + c_1, \\ -\frac{1}{2}\cos(s) + v\cos(s) + c_2, \quad \frac{\sqrt{3}}{2}s + c_3 \end{pmatrix} \tag{45}$$

and

$$\Phi_2(s,v) = \begin{pmatrix} -\frac{9s}{8} - 6\sin\left(\frac{s}{2}\right) - \frac{3}{4}\sin(s) - \frac{1}{16}\sin(2s) + v\frac{\sqrt{3}}{2}\cos(s) + c_1, \\ -\frac{1}{2}\cos(s) + v\frac{\sqrt{3}}{2}\sin(s) + c_2, \quad \frac{\sqrt{3}}{2}s - v\frac{1}{2} + c_3 \end{pmatrix}. \tag{46}$$

The following figures shows the surfaces $\Phi_1(s,v)$ and $\Phi_2(s,v)$.

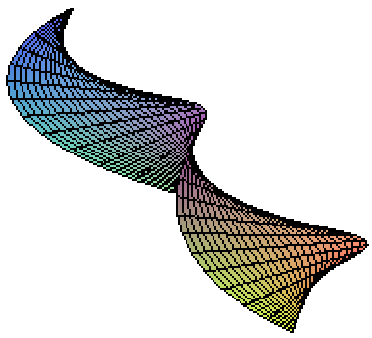

Figure 1. Normal surface of $\overline{\beta}$

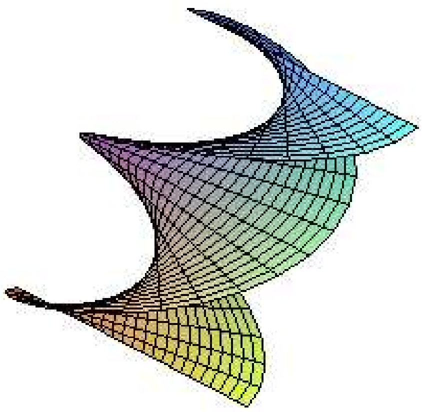

Figure 2. Binormal surface of $\overline{\beta}$

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REFERENCES

[1] Choi, J. H., & Kim, Y. H. (2012). Associated curves of a Frenet curve and their applications. *Applied Mathematics and Computation.*, 218, 9116-9124.

[2] Choi, J. H., Kim, Y. H., & Ali, A. T. (2012). Some associated curves of Frenet nonlightlike curves in E_1^3. *Journal of Mathematical Analysis and Applications*, 394, 712-723.

[3] Do Carmo, M. P. (1976). *Differential Geometry of Curves and Surfaces*. Prentice-Hall Inc., New Jersey.

[4] Gray, A. (1993). *Modern Differential Geometry of Curves and Surfaces with Mathematica 2nd ed.* Boca raton, Fl: Crc Press.

[5] Izumiya, S., & Takeuchi, N. (2003). Special curves and ruled surfaces. *Beitrage zur Algebra Und Geometrie, Contributions to Algebra and Geometry*, 44(1), 203-212.

[6] Izumiya, S., & Takeuchi, N. (2004). New special curves and developable surfaces. *Turkish Journal of Mathematics*, 28, 153-163.

[7] Kızıltuğ, S., & Önder, M. (2015). Associated curves of Frenet curves in three dimensional compact Lie group. *Miskolc Mathematical Notes*, 16(2), 953-964.

[8] Körpınar, T., Sarıaydın, M. T., & Turhan, E. (2013). Associated curves according to Bishop frame in Euclidean 3-space. *Advanced Modeling and Optimization*, 15(3), 713-717.

[9] Kula, L., & Yaylı, Y. (2005). On slant helix and spherical indicatrix. *Applied Mathematics and Computation*, 169, 600-607.

[10] Macit, N., & Düldül, M. (2014). Some new associated curves of a Frenet curve in E^3 and E^4. *Turkish Journal of Mathematics*, 38, 1023-1037.

[11] O'Neill, B. (2006). *Elementary Differential Geometry.* Academic Press.

[12] Scofield, P. D., (1995). Curves of constant precession. *American Mathematical Monthly*, 102(6), 531-537.

[13] Yu, Y., Liu, H., & Jung, S. D. (2014). Structure and characterization of ruled surfaces in Euclidean 3-space. *Applied Mathematics and Computation*, 233, 252-259.