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Research Article

Products Of Element Orders In Finite Groups

Nil MANSUROĞLU[1*]

[1]Department of Mathematics, Kırşehir Ahi Evran University, Kırşehir, Turkey

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Abstract

The aim of this note is to investigate the products of element orders in finite groups and to give some properties of such products. Let $\psi'(G)$ denote the product of element orders of a finite group *G*. As an immediate consequence, by using a different approach, we proved that $\psi'(G) < \psi'(C)$ where *G* is a non-cyclic finite group and *C* is a cyclic group of the same order.

Keywords: Finite group, group element orders, product of element orders

Sonlu Grupların Eleman Mertebelerinin Çarpımı

Öz

Bu çalışmanın amacı sonlu grupların eleman mertebelerinin çarpımlarını araştırmak ve bu çarpımların bazı özelliklerini vermektir. Bir sonlu grup *G* nin eleman mertebelerinin çarpımı $\psi'(G)$ olsun. Farklı bir yaklaşım kullanarak aynı mertebeye sahip *G* devirli olmayan grup ve *C* devirli grup olmak üzere $\psi'(G) < \psi'(C)$ olduğu ispatlandı.

Anahtar Kelimeler: Sonlu grup, grup eleman mertebeleri, eleman mertebeleri çarpımı

1. Introduction

The goal of this paper is to discuss the product of element orders in a finite group. Essentially, we start from the ground up by giving some properties on the sum of element orders in a finite group. Given a finite group G and the sum of element orders of G is defined by

$$\psi(G) = \sum_{x \in G} o(x),$$

where $o(x)$ denotes the order of $x \in G$. The problem of properties on sum of element orders of a finite group has been considered by various authors. The starting point for this note is given by the papers (Amiri vd., 2009), (Herzog vd., 2018) and (Mansuroğlu, 2018) which investigated the sums of element orders of finite groups. Now we give some properties on such sums.

Lemma 1.1. (Lemma C, (Amiri vd., 2009)) Let p be the largest prime divisor of an integer $n > 1$. Then $\varphi(n) \geq \frac{n}{p}$,where φ is the Euler's function.

Corollary 1.2. (Corollary D, (Amiri vd., 2009)) Let $C > 1$ be a cyclic group, and let p be the largest prime divisor of $|C|$. Then

$$\psi(C) > \frac{|C|^2}{p}.$$

In this note we investigate some properties of the product of element orders of a finite group *G*. Such product is defined by

$$\psi'(G) = \prod_{x \in G} o(x),$$

where $o(x)$ denotes the order of $x \in G$. Some properties of the function ψ' have been studied in (Tarnauceanu, 2018) for finite abelian groups and in (Garonzi, 2017) for the problem of detecting structural properties of a finite group. Therefore, a working knowledge of group theory in (Isaacs, 2008) and (Robinson, 1996) should be all that is necessary to follow along in the paper.

2. The Main Results

Now, in this section we give our main results.

Lemma 2.1. Let *C* be a cyclic group. Then the product of element orders in *G* is calculated by

$$\psi'(C) = \prod_{d|n} d^{\varphi(d)},$$

where φ is the Euler's function.

Proof. The orders of the elements in *C* divide the order of *C*. The divisor *d* of *n* represents the order of at least one element among the elements in *C*. The function $\varphi(d)$ gives the number of the elements with order *d*. Thus, the product of element orders in *C* is $\prod_{d|n} d^{\varphi(d)}$.□

Corollary 2.2. Let *n* be an odd prime and *C* be a cyclic group. Then

$$\psi'(C) = n^{n-1}.$$

Proof. The divisors of *n* are only 1 and *n*. By applying Lemma 2.1, we obtain

$$\psi'(C) = 1^{\varphi(1)}.n^{\varphi(n)} = n^{n-1}.$$

□

Lemma 2.3. Let *G* be a non-cyclic group, $K \in Syl_p(G)$, $K \trianglelefteq G$ and *K* is cyclic. Let $a \in G$, suppose that the coset Ka has order *m* as an element of G/K. Then

$$\psi'(Ka) \leq m^{|K|}\psi'(K).$$

Proof. Each element in *Ka* has the form *ua* for some element *u* in *K*. The proof of Lemma A in (Amiri vd., 2009) shows that for $u \in K$, $o(ua) \leq mo(u)$. Then

$$\begin{aligned}\psi'(Ka) &= \prod_{u \in K} o(ua) \\ &\leq \prod_{u \in K} mo(u) \\ &= m^{|K|} \prod_{u \in K} o(u) \\ &= m^{|K|}\psi'(K),\end{aligned}$$

as required. □

Corollary 2.4. Let *G* be a non-cyclic group, $K \in Syl_p(G)$, $K \trianglelefteq G$ and *K* is cyclic. Then

$$\psi'(G) \leq \psi'(K)^{|G/K|}\, \psi'(G/K)^{|K|}.$$

Proof. By $o(Ka)$, we denote the order of a coset *Ka* viewed as an element of G/K. By applying Lemma 2.3 to each coset of *K* in *G*, we obtain

$$\psi'(G) = \prod_{Ka \in \frac{G}{K}} \psi'(Ka)$$

$$\leq \prod_{Ka \in \frac{G}{K}} o(Ka)^{|K|}\, \psi'(K)$$

$$= \psi'(K)^{|G/K|} \prod_{Ka \in G/K} o(Ka)^{|K|}.$$

Therefore, $\psi'(G) \leq \psi'(K)^{|G/K|}\psi'(G/K)^{|K|}$.□

Corollary 2.5. Let $C > 1$ be a cyclic group, let p be the largest prime divisor of $|C|$. Then

$$\psi'(C) > |C|^{|C|/p}.$$

Proof. Suppose that $|C| = n$. There are $\varphi(n)$ elements of order n in $|C|$. Hence $\psi'(C) > n^{\varphi(n)} \geq n^{n/p}$. □

Now we present a significant approach which is differed from (Garonzi, 2017) for proving the following theorem.

Theorem 2.6. Let G be a non-cyclic finite group and C be a cyclic finite group of the same order. Then

$$\psi'(G) < \psi'(C).$$

Proof. Let C be cyclic of order n and suppose that G has order n and $\psi'(G) \geq \psi'(C)$. We need to show that G is cyclic. For $n = 1$, this is trivial, that is, G is cyclic. Now we assume that $n > 1$ and we use the induction method on n. By Corollary 2.5,

$$\psi'(G) \geq \psi'(C) > n^{\frac{n}{p}} > \frac{n^{\frac{n}{p}}}{p} > \frac{n}{p}.$$

There exists an element a in G such that $o(a) > \frac{n}{p}$. Therefore, $|G:\langle a\rangle| < p$ and $\langle a\rangle$ has a Sylow p-subgroup K of G and K is cyclic. Moreover, $\langle a\rangle \subseteq N_G(K)$ and hence $|G:N_G(K)| < p$, this means that $K \unlhd G$. By Corollary 2.4, $\psi'(G) \leq \psi'(K)^{|G/K|}\,\psi'(G/K)^{|K|}$. Let L be the Sylow p-subgroup of C, K and L are cyclic groups with equal orders, so $K \cong L$ and $\psi'(K) = \psi'(L)$. Therefore, $\psi'(K)^{|G/K|}\psi'(G/K)^n \geq \psi'(G) \geq \psi'(C) = \psi'(L)^{|C/L|}\psi'(C/L)^n$. Since $\psi'(K) = \psi'(L)$, by cancelling we obtain that $\psi'(G/K) \geq \psi'(C/L)$. It follows that K is central in G. Since K is central and G/K is cyclic, G is abelian. Moreover, $G = K \times T$, where $T \cong G/K$ is cyclic. This shows that G is a product of cyclic groups of coprime orders, so that G is cyclic. This completes the proof of theorem. □

Lemma 2.7. If C is a cyclic finite p-group of order p^r for some prime p, then

$$\psi'(C) = \prod_{i=0}^{r} p^{i\varphi(p^i)} = p^{\sum_{i=1}^{r} i\varphi(p^i)}.$$

Proof. The function $\varphi(d)$ represents the number of the elements with order d. By Lemma 2.1,

$$\begin{aligned}\psi'(C) &= \prod_{d|n} d^{\varphi(d)} \\ &= 1.p^{\varphi(p)}.p^{2\varphi(p^2)} \ldots p^{r\varphi(p^r)} \\ &= \prod_{i=0}^{r} p^{i\varphi(p^i)} \\ &= p^{\sum_{i=1}^{r} i\varphi(p^i)}.\end{aligned}$$

□

Example 2.8. Let C be a cyclic finite 2-group of order 24. Then

$$\begin{aligned}\psi'(C) &= \prod_{d|16} d^{\varphi(d)} = 1.2^1.4^2.8^4.16^8 \\ &= \prod_{i=0}^{4} 2^{i\varphi(2^i)}.\end{aligned}$$

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