Advances in the Theory of Nonlinear Analysis and its Applications **4** (2020) No. 2, 112–120.
https://doi.org/10.31197/atnaa.649269
Available online at www.atnaa.org
Research Article

Advances in the Theory of Nonlinear Analysis and its Applications

ISSN: 2587-2648 Peer-Reviewed Scientific Journal

Iterative approximation of common fixed points of generalized nonexpansive maps in convex metric spaces

Venkata Ravindranadh Babu Gutti[a], Satyanarayana Gedala[b]

[a] *Department of Mathematics, Andhra University, Visakhapatnam-530 003, India.*
[b] *Present address: Department of Mathematics, Andhra University, Visakhapatnam-530 003, India.*
Permanent address: Department of Mathematics, Dr. Lankapalli Bullayya college, Visakhapatnam-530 013, India.

Abstract

We define SP-iteration procedure associated with three selfmaps T_1, T_2, T_3 defined on a nonempty convex subset of a convex metric space X and prove Δ-convergence of this iteration procedure to a common fixed point of T_1, T_2, T_3 under the hypotheses that each T_i is either an α-nonexpansive map or a Suzuki nonexpansive map in the setting of uniformly convex metric spaces. Also, we prove the strong convergence of this iteration procedure to a common fixed point of T_1, T_2, T_3 under certain additional hypotheses namely either semi-compact or condition (D).

Keywords: SP-iteration procedure, α-nonexpansive map,
Suzuki nonexpansive map, common fixed point, Δ-convergence, strong convergence, uniformly convex metric space.
2010 MSC: 47H10, 54H25.

1. Introduction

In 1965, Browder [4] and Gőhde [13] proved that every nonexpansive selfmap of a nonempty closed convex and bounded subset of a uniformly convex Banach space has a fixed point. Browder and Petryshyn [5, 6],

Email addresses: `gvr_babu@hotmail.com` (Venkata Ravindranadh Babu Gutti), `gedalasatyam@gmail.com` (Satyanarayana Gedala)

Received Nov. 20, 2019, Accepted: April 24, 2020, Online: April 26, 2020.

Senter and Dotson [19] used iteration procedures to approximate fixed points of nonexpansive maps in the setting of Banach spaces.

In 1970, Takahashi [22] introduced the concept of convexity in metric spaces as follows.

Definition 1.1. *Let (X, d) be a metric space. A map $W : X \times X \times [0, 1] \to X$ is said to be a 'convex structure' on X if*

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y) \tag{1}$$

for $x, y, u \in X$ and $\lambda \in [0, 1]$.

By a convex metric space, we mean a metric space (X, d) together with a convex structure W and we denote it by (X, d, W).

Remark 1.2. *Every normed linear space $(X, ||.||)$ is a convex metric space. But there are convex metric spaces which are not normed linear spaces [3, 16, 22].*

A nonempty subset K of X is said to be *'convex'* if $W(x, y, \lambda) \in K$ for $x, y \in K$ and $\lambda \in [0, 1]$.

Das and Debata [7] studied the convergence of common fixed points of a pair of quasi nonexpansive maps T_1, T_2 by using the following iteration procedure in the setting of Banach spaces under certain hypotheses. Let X be a Banach space, K a nonempty convex subset of X, $T_1, T_2 : K \to K$ be selfmaps of K. For $x_1 \in K$,

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T_1(\beta_n T_2 x_n + (1 - \beta_n)x_n) \tag{2}$$

where $\alpha_n, \beta_n \in [0, 1]$ for $n \in \mathbb{N}$, where $\mathbb{N}$ denote the set of all natural numbers.

Shimizu and Takahashi [20] introduced the notion of uniform convexity in convex metric spaces as follows.

Definition 1.3. [20] *A convex metric space (X, d, W) is said to be uniformly convex if for any $\epsilon > 0$, there exists $\alpha = \alpha(\epsilon)$ such that for all $r > 0$ and $x, y, z \in X$ with $d(z, x) \leq r$, $d(z, y) \leq r$ and $d(x, y) \geq r\epsilon$,*

$$d(z, W(x, y, \frac{1}{2})) \leq r(1 - \alpha) < r.$$

Takahashi and Tamura [23] studied the weak convergence of the iteration procedure (2) when both T_1 and T_2 are nonexpansive maps in the setting of Banach spaces, provided $0 < a \leq \alpha_n, \beta_n \leq b < 1$ for $n \in \mathbb{N}$.

Let $T : K \to K$ be a map and K, a nonempty subset of a metric space (X, d). We denote $F(T) = \{x \in K : Tx = x\}$, the set of all fixed points of T.

A map $T : K \to K$ is said to be a quasi nonexpansive map if $F(T) \neq \emptyset$ and $d(Tx, p) \leq d(x, p)$ for all $x \in K$ and $p \in F(T)$.

Suzuki [21] introduced a map with condition (C) in Banach spaces and under metric space setting it is as follows. Let K be a nonempty subset of a metric space (X, d). A map $T : K \to K$ is said to satisfy condition (C) if

$$\frac{1}{2}d(x, Tx) \leq d(x, y) \ \ implies \ \ d(Tx, Ty) \leq d(x, y) \ \ for \ \ all \ \ x, y \in K. \tag{3}$$

We call a map that satisfies condition (C), a Suzuki nonexpansive map. Aoyama and Kohsaka [1] introduced α-nonexpansive maps in Banach spaces and under metric space setting it is as follows. Let K be a nonempty subset of a metric space X. A map $T : K \to K$ is said to be an α-nonexpansive map for some $\alpha < 1$ if

$$d(Tx, Ty)^2 \leq \alpha d(Tx, y)^2 + \alpha d(x, Ty)^2 + (1 - 2\alpha)d(x, y)^2 \ \ for \ \ x, y \in K. \tag{4}$$

Aoyama and Kohsaka [1] observed the following facts :

(i) 0-nonexpansive map is called a nonexpansive map,
(ii) $\frac{1}{2}$-nonexpansive map is called a nonspreading map,
(iii) $\frac{1}{3}$-nonexpansive map is called a hybrid map.

Remark 1.4. *Either a Suzuki nonexpansive map or an α-nonexpansive map with a nonempty fixed point set is a quasi nonexpansive map.*

For any bounded sequence $\{x_n\}$ in a metric space (X, d), the asymptotic radius with respect to $K \subseteq X$ is defined by $r_K(\{x_n\}) = \inf_{x \in K}\{r(x, \{x_n\})\}$
where $r(x, \{x_n\})\} = \limsup_{n\to\infty} d(x, x_n)$ and the asymptotic center of $\{x_n\}$ with respect to K is defined by $A_K(\{x_n\}) = \{y \in K : r(y, \{x_n\}) = r_K(\{x_n\})\}$.

A sequence $\{x_n\}$ in a metric space (X, d) is said to $\Delta-$converges to a point x in X if x is the unique asymptotic center for every subsequence $\{x_{n_k}\}$ of the sequence $\{x_n\}$. In this case, we write $\Delta - \lim_{n\to\infty} x_n = x$.

A map $T : K \to K$ is said to be *semi-compact* if every bounded sequence $\{x_n\}$ in K with $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ has a convergent subsequence.

Dhompongsa, Inthakon and Takahashi [8] proved that the sequence $\{x_n\}$ generated by the iteration procedure (2) converges weakly to a common fixed point of T_1 and T_2, where T_1 is a nonspreading map and T_2 is a Suzuki nonexpansive map in the setting of Hilbert spaces, provided
$\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$ and $\liminf_{n\to\infty} \beta_n(1 - \beta_n) > 0$.

In 2011, Phuengrattana and Suantai [18] introduced a three step iteration procedure namely SP-iteration procedure to approximate fixed points of a continuous nondecreasing function defined on a closed interval on the real line and proved that this iteration procedure converges faster than Mann iteration procedure [15], Ishikawa iteration procedure [14] and Noor iteration procedure [17]. In the setting of normed linear spaces, SP-iteration procedure is defined as follows.

Let K be a nonempty convex subset of a normed linear space X,
$T : K \to K$ be a selfmap of K and for any $x_0 \in K$,

$$\begin{aligned} z_n &= (1 - \gamma_n)x_n + \gamma_n T x_n \\ y_n &= (1 - \beta_n)z_n + \beta_n T z_n \\ x_{n+1} &= (1 - \alpha_n)y_n + \alpha_n T y_n \end{aligned} \tag{5}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are sequences in $[0, 1]$.

In 2013, Wattanawitoon and Khamlae [25] considered the iteration procedure (2) to approximate common fixed point of T_1 and T_2, where T_1 is an $\alpha-$nonexpansive map and T_2 is a Suzuki nonexpansive map in the setting of Hilbert spaces, provided $0 < a \le \alpha_n, \beta_n \le b < 1$.

Uddin and Imdad [24] studied Δ-convergence and strong convergence of SP-iteration procedure to compute fixed points of Suzuki nonexpansive
mappings in Hadamard spaces.

Recently, Hafiz Fukhar-ud-din [11] considered one step iteration procedure and proved the following convergence theorem in the setting of convex metric spaces.

Theorem 1.5. *Let K be a nonempty, closed and convex subset of a complete and uniformly convex metric space X with continuous convex structure W. Let T be an α-nonexpansive selfmap on K, S a selfmap of K satisfying condition (C). For $x_1 \in K$, define*

$$x_{n+1} = W(Tx_n, W(Sx_n, x_n, \tfrac{\beta_n}{1-\alpha_n}), \alpha_n) \tag{6}$$

where $0 < a \le \alpha_n, \beta_n \le b < 1$ for $n \in \mathbb{N}$.
Let F be the set of all common fixed points of S and T. If $F \neq \emptyset$ then $\Delta - \lim_{n\to\infty} x_n = x \in F$. Moreover, if either S or T is semi-compact then $\{x_n\}$ converges strongly to a point of F.

For more literature on this topic, we refer to [10, 12] and related references there in.

The following lemmas are useful in developing this paper.

Lemma 1.6. [21] *Let T be a selfmap defined on a nonempty subset K of a metric space (X, d). If T satisfies condition (C) then $d(x, Ty) \le 3d(Tx, x) + d(x, y)$ for all $x, y \in K$.*

Lemma 1.7. [12] *Let K be a nonempty, closed and convex subset of a metric space X and T be an α-nonexpansive mapping on K. For any $x, y \in K$, the following two assertions hold:*

(i) *If* $0 \leq \alpha < 1$ *then* $d(x,Ty)^2 \leq \frac{1+\alpha}{1-\alpha} d(x,Tx)^2 + \frac{2}{1-\alpha}\{\alpha d(x,y) + d(Tx,Ty)\} d(x,Tx) + d(x,y)^2.$

(ii) *If* $\alpha < 0$ *then* $d(x,Ty)^2 \leq d(x,Tx)^2 + \frac{2}{1-\alpha}\{d(Tx,Ty) - \alpha d(Tx,y)\} d(x,Tx) + d(x,y)^2.$

A sequence $\{x_n\}_{n=0}^{\infty}$ in a metric space (X,d) is said to be a Fejér monotone sequence with respect to a subset C of X if $d(x_{n+1},p) \leq d(x_n,p)$ for all $p \in C$ and $n \in \mathbb{N} \cup \{0\}$.

For any subset A of a metric space (X,d) and $x \in X$, we denote
$dist(x,A) = \inf_{y \in A}\{d(x,y)\}$.

Lemma 1.8. [2] *Let K be a nonempty closed subset of a complete metric space (X,d) and $\{x_n\}$ a Fejér monotone sequence with respect to K. Then $\{x_n\}$ converges to some point $x \in K$ if and only if $\lim_{n\to\infty} dist(x_n,K) = 0$.*

Lemma 1.9. [9] *Let K be a nonempty, closed and convex subset of a uniformly convex complete metric space (X,d,W). Then every bounded sequence $\{x_n\}$ in X has a unique asymptotic center with respect to K.*

Lemma 1.10. [10] *Let X be a uniformly convex metric space with continuous convex structure W. Let $x \in X$ and $\{a_n\}$ be a sequence in $[b,c]$ for some $b,c \in (0,1)$. If $\{u_n\}$ and $\{v_n\}$ are sequences in X such that $\limsup_{n\to\infty} d(u_n,x) \leq r$, $\limsup_{n\to\infty} d(v_n,x) \leq r$ and $\lim_{n\to\infty} d(W(u_n,v_n,a_n),x) = r$ for some $r \geq 0$, then $\lim_{n\to\infty} d(u_n,v_n) = 0$.*

Motivated by the works of Das and Debata [7], Takahashi and Tamura [23], Dhompongsa, Inthakon and Takahashi [8], Wattanawitoon and Khamlae [25], Uddin and Imdad [24] and Hafiz Fukhar-ud-din [11], in this paper, we define SP-iteration procedure associated with three selfmaps T_1, T_2, T_3 in convex metric spaces and prove the Δ-convergence of this iteration procedure to a common fixed point of T_1, T_2, T_3 under the hypotheses that each T_i is either an α-nonexpansive map or a Suzuki nonexpansive map. Further, with an additional assumption that either any one of T_1, T_2, T_3 is semi-compact or T_1, T_2, T_3 satisfies condition (D), we prove the strong convergence of this iteration procedure to a common fixed point of T_1, T_2 and T_3.

2. Convergence of SP-iteration associated with three maps

Let (X,d,W) be a convex metric space and K, a nonempty convex subset of X. Let $T_1, T_2, T_3 : K \to K$ be three selfmaps of K. A point $x \in K$ is said to be a common fixed point of T_1, T_2, T_3 if $T_1x = T_2x = T_3x = x$. We denote the set of all common fixed points of T_1, T_2, and T_3 by $F = \bigcap_{i=1}^{3} F(T_i)$.

We define SP-iteration procedure associated with three selfmaps in the setting of convex metric spaces as follows. Let K be a nonempty convex subset of a convex metric space X, and $T_1, T_2, T_3 : K \to K$ be three selfmaps. For $x_0 \in K$,

$$\begin{aligned} z_n &= W(T_1x_n, x_n, \gamma_n) \\ y_n &= W(T_2z_n, z_n, \beta_n) \\ x_{n+1} &= W(T_3y_n, y_n, \alpha_n) \end{aligned} \tag{7}$$

where $\alpha_n, \beta_n, \gamma_n \in [0,1]$ for $n \in \mathbb{N} \cup \{0\}$.

Lemma 2.1. *Let K be a nonempty convex subset of a convex metric space (X,d,W). Let $T_1, T_2, T_3 : K \to K$ be selfmaps of K such that $F \neq \emptyset$. Assume that each T_i is either an α-nonexpansive or a Suzuki nonexpansive map. For any $x_0 \in K$, let $\{x_n\}_{n=0}^{\infty}$ be the sequence generated by SP-iteration procedure associated with three selfmaps* (7). *Then*

(i) $\{x_n\}$ *is a Fejér monotone sequence with respect to* F,

(ii) $\lim_{n\to\infty} d(x_n, p)$ *exists for each* $p \in F$, *and*

(iii) $\lim_{n\to\infty} dist(x_n, F)$ *exists.*

Proof. Let $p \in F$ and $n \in \mathbb{N} \cup \{0\}$. We consider

$$\begin{aligned} d(x_{n+1}, p) &= d(W(T_3 y_n, y_n, \alpha_n), p) \\ &\leq \alpha_n d(T_3 y_n, p) + (1-\alpha_n) d(y_n, p) \\ &\leq \alpha_n d(y_n, p) + (1-\alpha_n) d(y_n, p) \quad \text{(since } T_3 \text{ is quasi nonexpansive)} \\ &= d(y_n, p). \end{aligned} \tag{8}$$

We now consider

$$\begin{aligned} d(y_n, p) &= d(W(T_2 z_n, z_n, \beta_n), p) \\ &\leq \beta_n d(T_2 z_n, p) + (1-\beta_n) d(z_n, p) \\ &\leq \beta_n d(z_n, p) + (1-\beta_n) d(z_n, p) \quad \text{(since } T_2 \text{ is quasi nonexpansive)} \\ &= d(z_n, p). \end{aligned} \tag{9}$$

Now, we consider

$$\begin{aligned} d(z_n, p) &= d(W(T_1 x_n, x_n, \gamma_n), p) \\ &\leq \gamma_n d(T_1 x_n, p) + (1-\gamma_n) d(x_n, p) \\ &\leq \gamma_n d(x_n, p) + (1-\gamma_n) d(x_n, p) \quad \text{(since } T_1 \text{ is quasi nonexpansive)} \\ &= d(x_n, p) \ \textit{for } n \in \mathbb{N} \cup \{0\} \ . \end{aligned} \tag{10}$$

Therefore from the inequalities (8), (9) and (10), we have

$$d(x_{n+1}, p) \leq d(x_n, p) \tag{11}$$

for all $p \in F$ and $n \in \mathbb{N} \cup \{0\}$, that is, $\{x_n\}$ is a Fejér monotone sequence with respect to F.

We observe from the inequality (11) that the sequence $\{d(x_n, p)\}_{n=0}^{\infty}$ is a decreasing sequence of nonnegative real numbers so that $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in F$.

By using the inequality (11), it is easy to see that $dist(x_{n+1}, F) \leq d(x_n, p)$ for all $p \in F$ so that $dist(x_{n+1}, F) \leq dist(x_n, F)$ for $n \in \mathbb{N} \cup \{0\}$ and hence $\lim_{n\to\infty} dist(x_n, F)$ exists. □

Lemma 2.2. *Let K be a nonempty closed and convex subset of a uniformly convex metric space X with continuous convex structure W. Let $T_1, T_2, T_3 : K \to K$ be selfmaps and assume that each T_i is either an α-nonexpansive map or a Suzuki nonexpansive map. Assume that $F \neq \emptyset$. Let $\alpha_n, \beta_n, \gamma_n \in [a, b] \subseteq (0, 1)$ for $n \in \mathbb{N} \cup \{0\}$. For any $x_0 \in K$, let $\{x_n\}_{n=0}^{\infty}$ be the sequence generated by the SP-iteration procedure associated with three selfmaps (7). Then $\lim_{n\to\infty} d(x_n, T_i x_n) = 0$ for $i = 1, 2, 3$.*

Proof. It follows from (ii) of Lemma 2.1 that for each $p \in F$, there exists $c \geq 0$ such that

$$\lim_{n\to\infty} d(x_n, p) = c.$$

Therefore from the inequalities (8), (9) and (10), we have
$d(x_{n+1}, p) \leq d(y_n, p) \leq d(z_n, p) \leq d(x_n, p)$ and hence

$$\lim_{n\to\infty} d(y_n, p) = c, \text{ and } \lim_{n\to\infty} d(z_n, p) = c.$$

Since $\lim_{n\to\infty} d(z_n, p) = c$, we have $\lim_{n\to\infty} d(W(T_1 x_n, x_n, \gamma_n), p) = c$.
Since T_1 is a quasi nonexpansive map, we have

$\limsup_{n\to\infty} d(T_1x_n, p) \le \limsup_{n\to\infty} d(x_n, p) = c.$
Now by applying Lemma 1.10, we have

$$\lim_{n\to\infty} d(x_n, T_1x_n) = 0. \tag{12}$$

We consider
$d(z_n, x_n) = d(W(T_1x_n, x_n, \gamma_n), x_n) \le \gamma_n\ d(T_1x_n, x_n) \le b\ d(T_1x_n, x_n).$
On letting $n \to \infty$, we have

$$\lim_{n\to\infty} d(z_n, x_n) = 0. \tag{13}$$

Since $\lim_{n\to\infty} d(y_n, p) = c$, we have $\lim_{n\to\infty} d(W(T_2z_n, z_n, \beta_n), p) = c$.
Since T_2 is a quasi nonexpansive map, we have
$\limsup_{n\to\infty} d(T_2z_n, p) \le \limsup_{n\to\infty} d(z_n, p) = c.$
Therefore by Lemma 1.10, we have

$$\lim_{n\to\infty} d(z_n, T_2z_n) = 0. \tag{14}$$

We consider

$$\begin{aligned} d(y_n, x_n) &= d(W(T_2z_n, z_n, \beta_n), x_n) \\ &\le \beta_n d(T_2z_n, x_n) + (1-\beta_n) d(z_n, x_n) \\ &\le \beta_n d(T_2z_n, z_n) + d(z_n, x_n) \text{ for } n \in \mathbb{N}\cup\{0\}. \end{aligned}$$

Now, on letting $n \to \infty$ it follows from (13) and (14) that

$$\lim_{n\to\infty} d(y_n, x_n) = 0. \tag{15}$$

Since $\lim_{n\to\infty} d(x_{n+1}, p) = c$, we have $\lim_{n\to\infty} d(W(T_3y_n, y_n, \alpha_n), p) = c$.
Since T_3 is a quasi nonexpansive map, we have
$\limsup_{n\to\infty} d(T_3y_n, p) \le \limsup_{n\to\infty} d(y_n, p) = c.$
Again by Lemma 1.10, we have

$$\lim_{n\to\infty} d(y_n, T_3y_n) = 0. \tag{16}$$

Now, we prove $\lim_{n\to\infty} d(x_n, T_ix_n) = 0$ for $i = 2, 3$ by considering the following cases.
<u>*Case (i)*</u> : T_2 is a Suzuki nonexpansive map.
By using triangle inequality and Lemma 1.6, it is easy to see that

$$\begin{aligned} d(x_n, T_2x_n) &\le d(x_n, z_n) + d(z_n, T_2x_n) \\ &= 2d(z_n, x_n) + 3d(z_n, T_2z_n) \text{ for } n \in \mathbb{N}\cup\{0\}. \end{aligned}$$

Therefore it follows from (13) and (14) that $\lim_{n\to\infty} d(x_n, T_2x_n) = 0$.
<u>*Case (ii)*</u> : T_2 is an α-nonexpansive map for $0 \le \alpha < 1$.
By (i) of Lemma 1.7, we have
$d(z_n, T_2x_n)^2 \le \frac{1+\alpha}{1-\alpha} d(z_n, T_2z_n)^2 + \frac{2}{1-\alpha}\{\alpha d(z_n, x_n) + A\} d(z_n, T_2z_n) + d(z_n, x_n)^2$
where $A = \sup\{d(T_2z_n, T_2x_n) : n \in \mathbb{N}\cup\{0\}\}$.
On letting $n \to \infty$, it follows from (13) and (14) that $\lim_{n\to\infty} d(z_n, T_2x_n) = 0$.
Now by using the triangle inequality, it follows that $\lim_{n\to\infty} d(x_n, T_2x_n) = 0$.
<u>*Case (iii)*</u> : T_2 is an α-nonexpansive map for $\alpha < 0$.
By (ii) of Lemma 1.7, we have

$$\begin{aligned} d(z_n, T_2x_n)^2 &\le d(z_n, T_2z_n)^2 + \tfrac{2}{1-\alpha}\{d(T_2z_n, T_2x_n) - \alpha d(T_2z_n, x_n)\} d(z_n, T_2z_n) \\ &\qquad + d(z_n, x_n)^2 \\ &\le d(z_n, T_2z_n)^2 + \tfrac{2}{1-\alpha}\{A - \alpha d(T_2z_n, z_n) - \alpha d(z_n, x_n)\} d(z_n, T_2z_n) \\ &\qquad + d(z_n, x_n)^2 \end{aligned}$$

where A is defined as in case (ii).

On letting $n \to \infty$, it is easy to see from (13) and (14) that
$\lim_{n\to\infty} d(z_n, T_2x_n) = 0$ and hence $\lim_{n\to\infty} d(x_n, T_2x_n) = 0$.
Case (iv) : T_3 is either a Suzuki nonexpansive map or an α-nonexpansive map for $\alpha < 1$.
By proceeding as in the above cases, it follows from (15) and (16) that
$\lim_{n\to\infty} d(x_n, T_3x_n) = 0$. □

Theorem 2.3. *Let K be a nonempty, closed and convex subset of a complete and uniformly convex metric space X with continuous convex structure W. Let $T_1, T_2, T_3 : K \to K$ be three selfmaps of K such that each T_i is either an α-nonexpansive map or a Suzuki nonexpansive map. Assume that $F \neq \emptyset$ and let $\alpha_n, \beta_n, \gamma_n \in [a, b] \subseteq (0, 1)$ for $n \in \mathbb{N} \cup \{0\}$. For any $x_0 \in K$, let $\{x_n\}$ be the sequence generated by SP-iteration procedure associated with three selfmaps* (7)*. Then there exists $x \in F$ such that $\Delta - \lim_n x_n = x$.*

Proof. By Lemma 2.1, we have $\{x_n\}$ is bounded. Therefore by Lemma 1.9, the sequence $\{x_n\}$ has a unique asymptotic center with respect to K, i. e., $A_K(\{x_n\}) = \{x\}$ for some $x \in K$. Similarly, if $\{x_{n_k}\}$ is a subsequence of the sequence $\{x_n\}$ then there exists $u \in K$ such that $A_K(\{x_{n_k}\}) = \{u\}$.

We substitute $x = x_n$ and $y = x$ in Lemma 1.6 if T_i a Suzuki nonexpansive map, and in Lemma 1.7 if T_i is an α-nonexpansive map through which it follows that $\limsup_{n\to\infty} d(x_n, T_ix) \leq \limsup_{n\to\infty} d(x_n, x)$ so that $r_K(\{x_n\}) = \limsup_{n\to\infty} d(x_n, T_ix)$ for $i = 1, 2, 3$. Therefore $T_ix \in A_K(\{x_n\}) = \{x\}$ for $i = 1, 2, 3$ so that $x \in F$. Similarly, we have $u \in F$.

Now, we prove that $x = u$. On the contrary, let $x \neq u$.
Since $u \in F$, it follows from Lemma 2.1 that $\lim_{n\to\infty} d(x_n, u)$ exists.
Now, we consider

$$\begin{aligned}\limsup_{k\to\infty} d(x_{n_k}, u) &< \limsup_{k\to\infty} d(x_{n_k}, x) \text{ (since } A_K(\{x_{n_k}\}) = \{u\})\\ &\leq \limsup_{n\to\infty} d(x_n, x)\\ &< \limsup_{n\to\infty} d(x_n, u) \text{ (since } A_K(\{x_n\}) = \{x\})\\ &= \lim_{n\to\infty} d(x_n, u) = \lim_{k\to\infty} d(x_{n_k}, u) = \limsup_{k\to\infty} d(x_{n_k}, u),\end{aligned}$$

a contradiction.
Therefore $A_K(\{x_{n_k}\}) = \{x\}$ for every subsequence $\{x_{n_k}\}$ of the sequence $\{x_n\}$, that is, $\Delta - \lim_{n\to\infty} x_n = x$. □

Theorem 2.4. *Under the hypotheses of Theorem 2.3, if any one of T_1, T_2, or T_3 is semi-compact then for $x_0 \in K$, the sequence $\{x_n\}$ generated by SP-iteration procedure* (7) *associated with three selfmaps converges strongly to a common fixed point of T_1, T_2, and T_3.*

Proof. Let T_i be semi-compact for some $i = 1, 2, 3$.
Now by Lemma 2.1 and Lemma 2.2, we have the sequence $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(x_n, T_ix_n) = 0$. Since T_i is semi-compact, the sequence $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ such that $\lim_{k\to\infty} d(x_{n_k}, x) = 0$ for some $x \in K$.

Now we prove that $x \in F$.
Case (i) : T_j is a Suzuki nonexpansive map for $j = 1, 2, 3$.
By using Lemma 1.6, we have $d(x_{n_k}, T_jx) \leq 3d(x_{n_k}, T_jx_{n_k}) + d(x_{n_k}, x)$ for all k so that $\lim_{k\to\infty} d(x_{n_k}, T_jx) = 0$ and hence $x \in F(T_j)$.
Case (ii) : T_j is an α-nonexpansive map for $0 \leq \alpha < 1$ and $j = 1, 2, 3$.
By Lemma 1.7, we have

$$\begin{aligned}d(x_{n_k}, T_jx)^2 &\leq \tfrac{1+\alpha}{1-\alpha} d(x_{n_k}, T_jx_{n_k})^2 + \tfrac{2}{1-\alpha}\{\alpha d(x_{n_k}, x) + d(T_jx_{n_k}, T_jx)\}\\ &\qquad d(x_{n_k}, T_jx_{n_k}) + d(x_{n_k}, x)^2\\ &\leq \tfrac{1+\alpha}{1-\alpha} d(x_{n_k}, T_jx_{n_k})^2 + \tfrac{2}{1-\alpha}\{\alpha d(x_{n_k}, x) + d(T_jx_{n_k}, x_{n_k})\\ &\qquad + d(x_{n_k}, T_jx)\} d(x_{n_k}, T_jx_{n_k}) + d(x_{n_k}, x)^2.\end{aligned}$$

On letting $k \to \infty$, we have $\lim_{k\to\infty} d(x_{n_k}, T_jx) = 0$ and hence $x \in F(T_j)$.

Case (iii) : T_j is an α-nonexpansive map for $\alpha < 0$ and $j = 1, 2, 3$.
By proceeding as in case (ii), it follows from Lemma 1.7 that $x \in F(T_j)$.
Hence by considering all the above cases, we have $x \in F$.

Therefore by Lemma 2.1, we have $\lim_{n\to\infty} d(x_n, x)$ exists and hence the sequence $\{x_n\}$ converges strongly to a point $x \in F$. □

We say that three selfmaps $T_1, T_2, T_3 : K \to K$ are said to satisfy *condition (D)* with respect to a subset C of K if there exists a nondecreasing fuction $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(t) > 0$ for all $t > 0$ such that

$$f(dist(x, C)) \leq \sum_{i=1}^{3} d(x, T_i x) \text{ for all } x \in K. \tag{17}$$

Theorem 2.5. *Under the hypotheses of Theorem 2.3, if T_1, T_2, and T_3 satisfy condition (D) with respect to F then the sequence $\{x_n\}_{n=0}^{\infty}$ generated by SP-iteration procedure* (7) *associated with three selfmaps converges strongly to a common fixed point of T_1, T_2, and T_3.*

Proof. By Lemma 2.2, we have $\lim_{n\to\infty} d(x_n, T_i x_n) = 0$ for $i = 1, 2, 3$ so that
$\lim_{n\to\infty} \sum_{i=1}^{3} d(x_n, T_i x_n) = 0$. Therefore from the inequality (17), we have
$\lim_{n\to\infty} f(dist(x_n, F)) = 0$.

We now prove that $\lim_{n\to\infty} dist(x_n, F) = 0$. On the contrary,
if $\lim_{n\to\infty} dist(x_n, F) \neq 0$ then there exist an $\epsilon > 0$ and a subsequence $\{x_{n_k}\}$ of the sequence $\{x_n\}$ such that $dist(x_{n_k}, F) \geq \epsilon$ for all k.
Therefore $f(dist(x_{n_k}, F)) \geq f(\epsilon) > 0$ for all k so that
$\lim_{n\to\infty} f(dist(x_n, F)) \neq 0$,
a contradiction.

Therefore $\lim_{n\to\infty} dist(x_n, F) = 0$.

By Lemma 2.1, we have the sequence $\{x_n\}$ is a Fejér monotone sequence with respect to F. Since T_1, T_2 and T_3 are quasi nonexpansive maps, we have F is closed. Therefore by using Lemma 1.8, we have the sequence $\{x_n\}$ converges strongly to a point of F. □

By choosing $\alpha = \frac{1}{2}$ or $\alpha = \frac{1}{3}$ in Theorem 2.3, Theorem 2.4 and Theorem 2.5, we have the following corollary.

Corollary 2.6. *Let K be a nonempty, closed and convex subset of a complete and uniformly convex metric space X with continuous convex structure W. Let $T_1, T_2, T_3 : K \to K$ be three selfmaps of K such that each T_i is a nonspreading map, a hybrid map, or a Suzuki nonexpansive map. Assume that $F \neq \emptyset$. Let $\alpha_n, \beta_n, \gamma_n \in [a, b] \subseteq (0, 1)$ for $n \in \mathbb{N} \cup \{0\}$. For any $x_0 \in K$, let $\{x_n\}$ be the sequence generated by SP-iteration procedure* (7) *associated with three selfmaps. Then*

(a) *there exists $x \in F$ such that $\Delta - \lim_n x_n = x$,*
(b) *if any one of T_1, T_2 and T_3 is semi-compact then $\{x_n\}$ converges strongly to a common fixed point of T_1, T_2, T_3, and*
(c) *if T_1, T_2, and T_3 satisfy condition (D) with respect to F then the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to a common fixed point of T_1, T_2, and T_3.*

Acknowledgements
The authors are thankful to the referee for his/her careful reading and valuable suggestions which improved the presentation of the paper.

References

[1] K. Aoyama and F. Kohsaka, Fixed point theorem for α-nonexpansive mappings in Banach spaces, Nonlinear Anal., 74(13)(2011), 4387-4391.

[2] H. H. Bauschke and P. L. Combettes, Convex Analysis and Monotone Operator Theory in Hilbert spaces, Springer-Verlag, New York, 2011.

[3] M. Bridson and A. Haefliger, Metric spaces of non-positive curvature, Springer-Verlag, Berlin, Heidelberg, New York, 1999.

[4] F. E. Browder, Nonexpansive nonlinear operators in a Banach space, Proc. Natl. Acad. Sci. USA, 54(4)(1965), 1041-1044.

[5] F.E. Browder and W.V. Petryshyn, The solution by iteration of nonlinear functional equations in Banach spaces, Bull. Amer. Math. Soc., 72(3)(1966), 571-575.

[6] F.E. Browder and W.V. Petryshyn, Construction of fixed points of nonlinear mappings in Hilbert space, J. Math. Anal. Appl., 20(2)(1967), 197-228.

[7] G. Das and J.P. Debata, Fixed points of quasinonexpansive mappings, Indian J. Pure Appl. Math., 17(11)(1986), 1263-1269.

[8] S. Dhompongsa, W. Inthakon, and W. Takahashi, A weak convergence theorem for common fixed points of some generalized nonexpansive mappings and nonspreading mappings in a Hilbert space, Optimization, 60(6)(2011), 769-779.

[9] H. Fukhar-ud-din, Existence and approximation of fixed points in convex metric spaces, Carpathian J. Math., 30(2)(2014), 175-185.

[10] H. Fukhar-ud-din, One step iterative scheme for a pair of nonexpansive mappimgs in a convex metric space, Hacet. J. Math. Stat.,44(5)(2015), 1023-1031.

[11] H. Fukhar-ud-din, Iterative Process for an $\alpha-$Nonexpansive Mapping and a Mapping Satisfying Condition (C) in a Convex Metric space, Iranian Journal of Mathematical Sciences and Informatics, 14(1) (2019), 167-179.

[12] H. Fukhar-ud-din and A.R. Khan, Approximation of common fixed point of two quasi nonexpansive mappings in convex metric spaces, Mediterr. J. Math., (2018)15:77.

[13] D. Göhde, Zum Prinzip der kontraktiven Abbildung, Math. Nachr., 30(1965), 251-258.

[14] S. Ishikawa, Fixed points by a new iteration method, Proc. Amer. Math. Soc., 44 (1974), 147-150.

[15] W.R. Mann, Mean value methods in iteration, Proc. Amer. Math. Soc., 4(3) (1953), 506-510.

[16] M. Moosaei, Fixed point theorems in convex metric spaces, Fixed Point Theory Appl., 2012, Article 164, (2012), 6 pages.

[17] M.A. Noor, New approximation schemes for general variational inequalities, J. Math. Anal. Appl., 251(1)(2000), 217-229.

[18] W. Phuengrattana and S. Suantai, "On the rate of convergence of Mann, Ishikawa, Noor and SP-iterations for continuous functions on an arbitrary interval", J. Comput. Appl. Math., 235(9) (2011), 3006-3014.

[19] H.F. Senter and W. G. Dotson, Jr., Approximating fixed points of nonexpansive mappings, Proc. Amer. Math. Soc.,44(2) (1974), 375-380.

[20] T. Shimizu and W. Takahashi, Fixed points of multivalued mappings in certain convex metric spaces, Topol. Methods Nonlinear Anal., 8 (1996), 197-203.

[21] T. Suzuki, Fixed point theorems and convergence theorems for some generalized nonexpansive mappings, J. Math. Anal. Appl., 340(2)(2008), 1088-1095.

[22] W. Takahashi, A convexity in metric space and nonexpansive mappings, I, Kodai Math. Sem. Rep., 22(2)(1970), 142-149.

[23] W. Takahashi and T. Tamura, Convergence theorems for a pair of nonexpansive mappings, J. Convex Anal., 5(1)(1998), 45-56.

[24] I. Uddin and M. Imdad, Convergence of SP-iteration for generalized nonexpansive mapping in Hadamard spaces, Hacet. J. Math. Stat., 47(6) (2018), 1595-1604.

[25] K. Wattanawitoon and Y. Khamlae, Weak and Strong Convergence Theorems for an α-nonexpansive mapping and a generalized Nonexpansive Mapping in Hilbert spaes, Thai J. Math., 11(3) (2013),633-643.