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INTERVAL OSCILLATION CRITERIA FOR IMPULSIVE CONFORMABLE FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we derive new interval oscillation criteria for impulsive conformable fractional differential equations having fixed moments of impulse actions. The results are extended to a more general class of nonlinear impulsive conformable fractional differential equations. Examples are also given to illustrate the relevance of the result.

1. Introduction

In recent years fractional differential equations are recognized as an excellent source of knowledge in modelling dynamical processes in self similar and porous structures, electrical networks, probability and statistics, visco elasticity, electro chemistry of corrosion, electro dynamics of complex medium, polymer rheology, industrial robotics, economics, biotechnology etc. For the theory and applications of fractional differential equations we refer the monographs [10, 18]. But the most commonly used definitions are based on the integration with singular kernel and which are nonlocal: Riemann-Liouville derivative and Caputo derivative. Moreover for this type of derivative useful product rule and chain rule are not applicable. But in 2014 Khalil et. al [9] introduced a new fractional derivative called the conformable derivative which is closely similar to classical derivative.

The oscillation of fractional differential equations as a new research field has received significant attention and some interesting results have already been obtained. We refer to [2, 3, 4, 5, 6, 14, 22, 24] and the references quoted therein.

The oscillation theory of impulsive differential equations, which provide a natural description of observed evolution processes, are regarded as important mathematical tools for the better understanding of several real world problems in applied

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sciences. For further details and applications one can refer the monographs [1, 12] and reference cited therein.

In [13], Q.L. Li and W.S. Cheng considered the following interval oscillation criteria for second order forced delay differential equation under impulses effects of the form

$$(p(t)x'(t))' + q(t)x(t-\tau) + \sum_{i=1}^{n} q_i(t)\Phi_{\alpha_i}(x(t-\tau)) = f(t), \quad t \neq t_k$$
$$x(t_k^+) = a_k x(t_k), \quad x'(t_k^+) = b_k x'(t_k), \quad k = 1, 2, \cdots$$

where $0 \le t_0 < t_1 < \cdots < t_k < \cdots$, $\lim_{k\to\infty} t_k = \infty$, $p, q, q_i, f \in PLC[t_0, \infty)$. By using the Riccati technique, some interesting oscillation results were obtained.

In the last decades, interval oscillation of impulsive differential equations was arousing the interest of many researchers, see [7, 8, 11, 15, 16, 19, 20, 21, 23, 25] and the references cited therein. Most of the existing literature concentrated on interval oscillation criteria for case of without delay and only very few papers appeared for case of with delay. As far as author knowledge, it seems that there has been no paper dealing with interval oscillation criteria for impulsive conformable fractional differential equations.

Motivated by this gap, we propose to initiate the following model of the form

$$\left.\begin{array}{l} T_\alpha\left(r(t)g\left(T_\alpha x(t)\right)\right) + q(t)x(t-\rho) + \sum\limits_{i=1}^{n} q_i(t)f_i(x(t-\rho)) = f(t), \quad t \neq t_k \\ x(t_k^+) = a_k x(t_k), \quad T_\alpha(x(t_k^+)) = b_k T_\alpha(x(t_k)), \quad k = 1, 2, \cdots, \end{array}\right\} \tag{1}$$

where T_α denotes the conformable fractional derivative of order $0 < \alpha \le 1$.

In the sequel, we assume that the following hypotheses (H) hold:

(H_1) $r(t) \in C^\alpha([t_0, \infty), (0, \infty))$, $q(t), q_i(t), f(t) \in PLC([t_0, \infty), \mathbb{R})$, $i = 1, 2, \cdots, n$, where PLC represents the class of functions which are piecewise continuous in t with discontinuities of first kind only at $t = t_k$, $k = 1, 2, \cdots$, and left continuous at $t = t_k, a_k, b_k$ are real-valued sequences satisfying $a_k > -1, a_k \le b_k$, $k = 1, 2, \cdots$, $t - \rho < t$, $\lim_{t\to\infty} t - \rho = \infty$, $0 < t_0 < t_1 < \cdots < t_k < \cdots$, $\lim_{k\to\infty} t_k = \infty$.

(H_2) $f_i, g \in C(\mathbb{R}, \mathbb{R})$ are convex in $[0, \infty)$ with $xf_i(x) > 0$ and $\frac{f_i(x)}{x} \ge \epsilon_i > 0$ for $x \neq 0$, $i = 1, 2, \cdots, n$, $xg(x) > 0$, $g(x) \le \eta x$ for $x \neq 0$, $g^{-1} \in C(\mathbb{R}, \mathbb{R})$ are continuous functions with $xg^{-1}(x) > 0$ for $x \neq 0$ and there exist positive constant ζ such that $g^{-1}(xy) \le \zeta g^{-1}(x)g^{-1}(y)$ for $xy \neq 0$ and

$$\int_{t_0}^{\infty} s^{\alpha-1} g^{-1}\left(\frac{1}{r(s)}\right) ds = \infty.$$

(H_3) For any $T \ge 0$ there exists intervals $[c_1, d_1]$ and $[c_2, d_2]$ contained in $[T, \infty)$ such that $c_1 < d_1 \le d_1 + \rho \le c_2 < d_2$, $c_j, d_j \notin \{t_k\}$, $j = 1, 2$, $k = 1, 2, \cdots$

and $r(t) > 0$, $q(t) \geq 0$, $q_i(t) \geq 0$, $i = 1, 2, \cdots, n$ for $t \in [c_1-\rho, d_1] \cup [c_2-\rho, d_2]$ and $f(t)$ has different signs in $[c_1 - \rho, d_1]$ and $[c_2 - \rho, d_2]$, for instance, let

$$f(t) \leq 0 \quad \text{for} \quad t \in [c_1 - \rho, d_1] \quad \text{and} \quad f(t) \geq 0 \quad \text{for} \quad t \in [c_2 - \rho, d_2].$$

Denote

$$J(s) := \max\{j : \ t_0 < t_j < s\}, \quad r_j := \max\{r(t) : \ t \in [c_j, d_j]\}, \quad j = 1, 2$$

$$J_p(c_j, d_j) = \{p \in C^\alpha[c_j, d_j], \ \ p(t) \neq 0, \ p(c_j) = p(d_j) = 0, \ j = 1, 2\}.$$

For two constants $c, d \notin \{t_k\}$ with $c < d$ and a function $\varphi \in C([c, d], \mathbb{R})$, we define an operator $\Phi : C([c, d], \mathbb{R}) \to \mathbb{R}$ by

$$\Phi_c^d[\varphi] = \begin{cases} 0, & J(c) = J(d) \\ \varphi(t_{J(c)+1})\tau(c) + \sum_{k=J(c)+2}^{J(d)} \varphi(t_i)\sigma(t_i), & J(c) < J(d), \end{cases}$$

where

$$\tau(c) = t_{I(c)+1}^{1-\alpha} \frac{a_{J(c)+1} - b_{J(c)+1}}{a_{J(c)+1}(t_{J(c_1)+1}^\alpha - c_1^\alpha)}$$

and

$$\sigma(t) = t_j^{1-\alpha} \frac{a_j - b_j}{a_j(t_j^\alpha - t_{j-1}^\alpha)}.$$

This paper is organized as follows: In Section 2, we present some definitions and results that will be needed in the sequel. The main results are given in Section 3. In Section 4, some examples is considered to illustrate the main results.

2. Preliminaries

In this section, we recall some definitions and results which will be used in our main results.

Definition 1. *A solution of equation* (1) *is called oscillatory if it has arbitrarily large zeros, otherwise it is called non-oscillatory. Equation* (1) *is called oscillatory if all its solutions are oscillatory.*

We use the following definition introduced by R.R. Khalil et al. [9].

Definition 2. *Given $f : [0, \infty) \to \mathbb{R}$. Then the conformable fractional derivative of f of order α is defined by*

$$T_\alpha(f)(t) = \lim_{\epsilon \to 0} \frac{f(t + \epsilon t^{1-\alpha}) - f(t)}{\epsilon}$$

for all $t > 0, \alpha \in (0, 1]$.

If f is α-differentiable in some $(0, a), a > 0$ and $\lim_{t \to 0^+} f^{(\alpha)}(t)$ exists, then define

$$f^{(\alpha)}(0) = \lim_{t \to 0^+} f^{(\alpha)}(t).$$

Definition 3. *$I_\alpha^a(f)(t) = I_1^a(t^{\alpha-1}f) = \int_a^t \dfrac{f(x)}{x^{1-\alpha}}dx$, where the integral is the usual Riemann improper integral, and $\alpha \in (0,1)$.*

Conformable fractional derivative has the following properties :

Theorem 4. *Let $\alpha \in (0,1]$ and f, g be α- differentiable at a point $t > 0$. Then*

(i) $T_\alpha(af + bg) = aT_\alpha(f) + bT_\alpha(g)$, *for all* $a, b \in \mathbb{R}$.
(ii) $T_\alpha(t^p) = pt^{p-\alpha}$, *for all* $p \in \mathbb{R}$.
(iii) $T_\alpha(\lambda) = 0$, *for all constant functions* $f(t) = \lambda$.
(iv) $T_\alpha(fg) = fT_\alpha(g) + gT_\alpha(f)$.
(v) $T_\alpha\left(\frac{f}{g}\right) = \dfrac{gT_\alpha(f) - fT_\alpha(g)}{g^2}$.
(vi) *If f is differentiable, then* $T_\alpha(f)(t) = t^{1-\alpha}\dfrac{df}{dt}(t)$.

3. Main Results

In this section, we established some new interval oscillation criteria for the equation (1) by using Riccati transformation.

Theorem 5. *Assume that conditions $(H_1)-(H_3)$ hold, furthermore for any $T \geq 0$ there exist c_j, d_j satisfying with $T \leq c_1 < d_1$, $T \leq c_2 < d_2$ and $p(t) \in J_p(c_1, d_1)$ such that*

$$\int_{c_j}^{d_j} \left[(p'(t))^2 t^{2-2\alpha}\eta r(t) + w(t)p^2(t)(1-\alpha)t^{-\alpha}\right] dt - \int_{c_j}^{t_{J(c_j)+1}} Q(t)p^2(t)M^j_{J(c_j)}(t)dt$$
$$- \sum_{k=J(c_j)+1}^{J(d_j)-1} \int_{t_k}^{t_{k+1}} Q(t)p^2(t)M^j_{J(c_j)}(t)dt - \int_{t_{J(d_j)}}^{d_j} Q(t)p^2(t)M^j_{J(d_j)}(t)dt \leq \Lambda(p, c_j, d_j) \tag{2}$$

where $Q(t) = q(t) + \sum\limits_{i=1}^{n} \epsilon_i q_i(t)$, $\Lambda(p, c_j, d_j) = 0$ for $J(c_j) = J(d_j)$ and

$$\Lambda(p, c_j, d_j) = r_j \left\{ p^2(t_{J(c_j)+1}) t^{1-\alpha}_{J(c_j)+1} \frac{a_{J(c_j)+1} - b_{J(c_j)+1}}{a_{J(c_j)+1}(t^\alpha_{J(c_j)+1} - c_j^\alpha)} + \sum_{k=J(c_j)+2}^{J(d_j)} p^2(t_k)t_k^{1-\alpha} \frac{a_k - b_k}{a_k(t_k^\alpha - t^\alpha_{k-1})} \right\}$$

for $J(c_j) < J(d_j)$, $j = 1, 2$

$$M_k^j(t) = \begin{cases} \dfrac{\rho\alpha}{\rho\alpha a_k + b_k(t^\alpha - t_k^\alpha)} \dfrac{(t-\rho)^\alpha - (t_k - \rho)^\alpha}{t_k^\alpha - (t_k - \rho)^\alpha}, & t \in (t_k, t_k + \rho) \\ \dfrac{(t-\rho)^\alpha - t_k^\alpha}{t^\alpha - t_k^\alpha}, & t \in [t_k + \rho, t_{k+1}), \end{cases}$$

then every solution of problem (1) is oscillatory.

Proof. Assume to the contrary that $x(t)$ is a non-oscillatory solution of (1). Without loss of generality we may assume that $x(t)$ is an eventually positive solution of (1). Then there exists $t_1 \geq t_0$ such that $x(t) > 0$ for $t \geq t_1$. Therefore it follows from (1) that

$$T_\alpha\left[r(t)g(T_\alpha(x(t)))\right] = f(t) - q(t)x(t-\rho) - \sum_{i=1}^{n} q_i(t)f_i(x(t-\rho)) \quad \text{for} \quad t \in [t_1, \infty).$$

Thus $T_\alpha\left[r(t)g(T_\alpha(x(t)))\right] \geq 0$ or $T_\alpha\left[r(t)g(T_\alpha(x(t)))\right] < 0$, $t \geq t_1$ for some $t_1 \geq t_0$. We now claim that

$$T_\alpha\left[r(t)g(T_\alpha(x(t)))\right] \geq 0 \quad \text{for} \quad t \geq t_1. \tag{3}$$

Suppose not, then $T_\alpha\left[r(t)g(T_\alpha(x(t)))\right] < 0$ and there exists $t_2 \in [t_1, \infty)$ such that $T_\alpha\left[r(t_2)g(T_\alpha(x(t_2)))\right] < 0$. Since $r(t)g(T_\alpha(x(t)))$ is strictly decreasing on $[t_1, \infty)$. It is clear that

$$r(t)g(T_\alpha(x(t))) < r(t_2)g(T_\alpha(x(t_2))) := -\mu$$

where $\mu > 0$ is a constant for $t \in [t_2, \infty)$, we have

$$\begin{aligned}
r(t)g(T_\alpha(x(t))) &< -\mu \\
T_\alpha(x(t)) &< g^{-1}\left(\frac{-\mu}{r(t)}\right) \\
T_\alpha(x(t)) &\leq -\zeta_1 g^{-1}\left(\frac{1}{r(t)}\right), \quad \text{where} \quad \zeta_1 = \zeta g^{-1}(\mu) \text{ for } t \in [t_2, \infty).
\end{aligned}$$

Integrating the above inequality from t_2 to t, we have

$$x(t) \leq x(t_2) - \zeta_1 \int_{t_2}^{t} s^{\alpha-1} g^{-1}\left(\frac{1}{r(s)}\right) ds.$$

Letting $t \to \infty$, we get $\lim_{t\to+\infty} x(t) = -\infty$ which contradiction proves that (3) holds. Define the Riccati transformation

$$w(t) := \frac{r(t)g(T_\alpha(x(t)))}{x(t)}.$$

It follows from (1) that $w(t)$ satisfies

$$T_\alpha(w(t)) \leq \frac{f(t)}{x(t)} - \left[q(t) + \sum_{i=1}^{n} \epsilon_i q_i(t)\right] \frac{x(t-\rho)}{x(t)} - \frac{w^2(t)}{\eta r(t)}.$$

By the assumption, we can choose $c_1, d_1 \geq t_0$ such that $r(t) > 0$, $q(t) \geq 0$ and $q_i(t) \geq 0$ for $t \in [c_1 - \rho, d_1]$, $i = 1, 2, \cdots, n$ and $f(t) \leq 0$ for $t \in [c_1 - \rho, d_1]$ from (1) we can easily to see that

$$t^{1-\alpha} w'(t) \leq -\frac{w^2(t)}{\eta r(t)} - Q(t)\frac{x(t-\rho)}{x(t)}. \tag{4}$$

For $t = t_k,\ k = 1, 2, \cdots$, one has

$$w(t_k^+) = \frac{r(t_k^+)g(T_\alpha(x(t_k^+)))}{x(t_k^+)} \leq \frac{b_k}{a_k} w(t_k).$$

At first, we consider the case in which $J(c_1) < J(d_1)$. In this case, all the impulsive moments in $[c_1, d_1]$ are $t_{J(c_1)+1},\ t_{J(c_1)+2}, \cdots,\ t_{J(d_1)}$. Choose an $p(t) \in J_p(c_1, d_1)$ and multiplying by $p^2(t)$ on both sides on (4), integrating it from c_1 to d_1, we obtain

$$\begin{aligned}
&\int_{c_1}^{t_{J(c_1)+1}} p^2(t)t^{1-\alpha}w'(t)dt + \int_{t_{J(c_1)+1}}^{t_{J(c_1)+2}} p^2(t)t^{1-\alpha}w'(t)dt \\
&+ \cdots + \int_{t_{J(d_1)}}^{d_1} p^2(t)t^{1-\alpha}w'(t)dt \\
&\leq -\int_{c_1}^{t_{J(c_1)+1}} p^2(t)\frac{w^2(t)}{\eta r(t)}dt - \int_{t_{J(c_1)+1}}^{t_{J(c_1)+2}} p^2(t)\frac{w^2(t)}{\eta r(t)}dt - \cdots - \int_{t_{J(d_1)}}^{d_1} p^2(t)\frac{w^2(t)}{\eta r(t)}dt \\
&- \int_{c_1}^{t_{J(c_1)+1}} p^2(t)Q(t)\frac{x(t-\rho)}{x(t)}dt - \int_{t_{J(c_1)+1}}^{t_{J(c_1)+1}+\rho} p^2(t)Q(t)\frac{x(t-\rho)}{x(t)}dt \\
&- \int_{t_{J(c_1)+1}+\rho}^{t_{J(c_1)+2}} p^2(t)Q(t)\frac{x(t-\rho)}{x(t)}dt - \cdots - \int_{t_{J(c_1)-1}+\rho}^{t_{J(d_1)}} p^2(t)Q(t)\frac{x(t-\rho)}{x(t)}dt \\
&- \int_{t_{J(d_1)}}^{d_1} p^2(t)Q(t)\frac{x(t-\rho)}{x(t)}dt.
\end{aligned}$$

Using the integration by parts on the left-hand side, and noting that the condition $p(c_1) = p(d_1) = 0$, we get

$$\begin{aligned}
&\sum_{k=J(c_1)+1}^{J(d_1)} p^2(t_k)t_k^{1-\alpha}\left[w(t_k) - w(t_k^+)\right] \leq \int_{c_1}^{d_1}\left[p'(t)t^{1-\alpha}\sqrt{\eta r(t)} - \frac{p(t)w(t)}{\sqrt{\eta r(t)}}\right]^2 dt \\
&- \int_{c_1}^{t_{J(c_1)+1}} p^2(t)Q(t)\frac{x(t-\rho)}{x(t)}dt \\
&- \sum_{k=J(c_1)+1}^{J(d_1)-1}\left[\int_{t_k}^{t_k+\rho} p^2(t)Q(t)\frac{x(t-\rho)}{x(t)}dt + \int_{t_k+\rho}^{t_{k+1}} p^2(t)Q(t)\frac{x(t-\rho)}{x(t)}dt\right] \\
&- \int_{t_{J(d_1)}}^{d_1} p^2(t)Q(t)\frac{x(t-\rho)}{x(t)}dt + \int_{c_1}^{d_1} t^{2-2\alpha}\eta r(t)(p'(t))^2 dt \\
&+ \int_{c_1}^{d_1}(1-\alpha)t^{-\alpha}p^2(t)w(t)dt. \qquad (5)
\end{aligned}$$

There are several cases to consider to estimate $\frac{x(t-\rho)}{x(t)}$.

Case 1: For $t \in (t_k, t_{k+1}] \subset [c_1, d_1]$. If $t \in (t_k, t_{k+1}] \subset [c_1, d_1]$, since $t_{k+1} - t_k > \rho$,

we consider two sub cases:
Case 1.1: If $t \in [t_k + \rho, t_{k+1}]$, then $t - \rho \in [t_k, t_{k+1} - \rho]$ and there are no impulsive moments in $(t - \rho, t)$, then for any $t \in [t_k + \rho, t_{k+1}]$ one has

$$x(t) - x(t_k^+) = T_\alpha(x(\xi)) \left(\frac{t^\alpha - t_k^\alpha}{\alpha} \right), \quad \xi \in (t_k, t).$$

Since $r(t)g(T_\alpha(x(t)))$ is non-increasing

$$x(t) \geq T_\alpha(x(\xi)) \left(\frac{t^\alpha - t_k^\alpha}{\alpha} \right) > \frac{r(t)g(T_\alpha(x(t)))}{r(\xi)} \left(\frac{t^\alpha - t_k^\alpha}{\alpha} \right).$$

From the fact that $r(t)$ is nondecreasing, we get

$$\frac{r(t)g(T_\alpha(x(t)))}{x(t)} < \frac{\alpha r(\xi)}{t^\alpha - t_k^\alpha} < \frac{\alpha r(t)}{t^\alpha - t_k^\alpha}.$$

We obtain

$$\frac{T_\alpha(x(t))}{x(t)} < \frac{\alpha}{t^\alpha - t_k^\alpha}.$$

Integrating it from $t - \rho$ to t, we have

$$\frac{x(t-\rho)}{x(t)} > \frac{(t-\rho)^\alpha - t_k^\alpha}{t^\alpha - t_k^\alpha}.$$

Case 1.2: If $t \in (t_k, t_k + \rho)$ then $t - \rho \in (t_k - \rho, t_k)$ and there is an impulsive moment t_k in $(t - \rho, t)$. Similar to Case 1.1, we obtain

$$x(t) - x(t_k - \rho) = T_\alpha(x(\xi_1)) \left(\frac{t^\alpha - (t_k - \rho)^\alpha}{\alpha} \right), \quad \xi_1 \in (t_k - \rho, t_k]$$

or

$$\frac{T_\alpha(x(t))}{x(t)} < \frac{\alpha}{t^\alpha - (t_k - \rho)^\alpha}.$$

Integrating it from $t - \rho$ to t, we get

$$\frac{x(t-\rho)}{x(t_k)} > \frac{(t-\rho)^\alpha - (t_k - \rho)^\alpha}{t_k^\alpha - (t_k - \rho)^\alpha} > 0, \quad t \in (t_k, t_k + \rho). \tag{6}$$

For any $t \in (t_k, t_k + \rho)$, we have

$$x(t) - x(t_k^+) < T_\alpha(x(t_k^+)) \left(\frac{t^\alpha - t_k^\alpha}{\alpha} \right), \quad \xi_2 \in (t_k, t).$$

Using the impulsive conditions in equation (1), we get

$$x(t) - a_k x(t_k) < b_k T_\alpha(x(t_k)) \left(\frac{t^\alpha - t_k^\alpha}{\alpha} \right)$$

$$\frac{x(t)}{x(t_k)} < b_k \frac{T_\alpha(x(t_k))}{x(t_k)} \left(\frac{t^\alpha - t_k^\alpha}{\alpha} \right) + a_k.$$

Using $\frac{T_\alpha(x(t_k))}{x(t_k)} < \frac{1}{\rho}$, we obtain

$$\frac{x(t)}{x(t_k)} < a_k + \frac{b_k}{\rho}\left(\frac{t^\alpha - t_k^\alpha}{\alpha}\right).$$

That is,

$$\frac{x(t_k)}{x(t)} > \frac{\rho\alpha}{\rho\alpha a_k + b_k(t^\alpha - t_k^\alpha)}. \tag{7}$$

From (6) and (7), we get

$$\frac{x(t-\rho)}{x(t)} > \frac{\rho\alpha}{\rho\alpha a_k + b_k(t^\alpha - t_k^\alpha)} \frac{(t-\rho)^\alpha - (t_k-\rho)^\alpha}{t_k^\alpha - (t_k-\rho)^\alpha} \geq 0.$$

Case 2: If $t \in [c_1, t_{J(c_1)+1}]$, we consider three sub cases:
Case 2.1: If $t_{J(c_1)} > c_1 - \rho$ and $t \in [t_{J(c_1)} + \rho, t_{J(c_1)+1}]$ then $t - \rho \in [t_{J(c_1)}, t_{J(c_1)+1} - \rho]$ and there are no impulsive moments in $(t-\rho, t)$. Making a similar analysis of the Case 1.1 and using Mean-value Theorem on $(t_{J(c_1)}, t_{J(c_1)+1}]$, we get

$$\frac{x(t-\rho)}{x(t)} > \frac{(t-\rho)^\alpha - t_{J(c_1)}^\alpha}{t^\alpha - t_{J(c_1)}^\alpha} > 0, \quad t \in [t_{J(c_1)} + \rho, t_{J(c_1)+1}].$$

Case 2.2: If $t_{J(c_1)} > c_1 - \rho$ and $t \in [c_1, t_{J(c_1)} + \rho)$, then $t - \rho \in [c_1 - \rho, t_{J(c_1)})$ and there is an impulsive moments $t_{J(c_1)}$ in $(t-\rho, t)$. Making a similar analysis of the Case 1.2, we have

$$\begin{aligned}\frac{x(t-\rho)}{x(t)} &> \frac{\rho\alpha}{\rho\alpha a_{J(c_1)} + b_{J(c_1)}(t^\alpha - t_{J(c_1)}^\alpha)} \frac{(t-\rho)^\alpha - (t_{J(c_1)}-\rho)^\alpha}{t_{J(c_1)}^\alpha - (t_{J(c_1)}-\rho)^\alpha} \\ &\geq 0, \quad t \in (c_1, t_{J(c_1)} + \rho).\end{aligned}$$

Case 2.3: If $t_{J(c_1)} < c_1 - \rho$, then for any $t \in [c_1, t_{J(c_1)+1}]$, $t - \rho \in [c_1 - \rho, t_{J(c_1)+1} - \rho]$ and there are no impulsive moments in $(t-\rho, t)$. Making a similar analysis of the Case 1.1, we obtain

$$\frac{x(t-\rho)}{x(t)} > \frac{(t-\rho)^\alpha - t_{J(c_1)}^\alpha}{t^\alpha - t_{J(c_1)}^\alpha} > 0, \quad t \in [c_1, t_{J(c_1)+1}].$$

Case 3: For $t \in (t_{J(d_1)}, d_1]$, there are three sub cases:
Case 3.1: If $t_{J(d_1)} + \rho < d_1$ and $t \in [t_{J(d_1)} + \rho, d_1]$ then $t - \rho \in [t_{J(d_1)}, d_1 - \rho]$ and there are no impulsive moments in $(t-\rho, t)$. Making a similar analysis of the Case 2.1, we have

$$\frac{x(t-\rho)}{x(t)} > \frac{(t-\rho)^\alpha - t_{J(d_1)}^\alpha}{t^\alpha - t_{J(d_1)}^\alpha} > 0, \quad t \in [t_{J(d_1)} + \rho, d_1].$$

Case 3.2: If $t_{J(d_1)} + \rho < d_1$ and $t \in [t_{J(d_1)}, t_{J(d_1)} + \rho)$, then $t - \rho \in [t_{J(d_1)} - \rho, t_{J(d_1)})$ and there is an impulsive moments $t_{J(d_1)}$ in $(t-\rho, t)$. Making a similar analysis of the Case 2.2, we obtain

$$\frac{x(t-\rho)}{x(t)} > \frac{\rho\alpha}{\rho\alpha a_{J(d_1)} + b_{J(d_1)}(t^\alpha - t^\alpha_{J(d_1)})} \frac{(t-\rho)^\alpha - (t_{J(d_1)} - \rho)^\alpha}{t^\alpha_{J(d_1)} - (t_{J(d_1)} - \rho)^\alpha} \geq 0.$$

Case 3.3: If $t_{J(d_1)} + \rho \geq d_1$, then for any $t \in (t_{J(d_1)}, d_1]$, we get $t - \rho \in (t_{J(d_1)} - \rho, d_1 - \rho]$ and there is an impulsive moments $t_{J(d_1)}$ in $(t-\rho, t)$. Making a similar analysis of the Case 3.2, we get

$$\frac{x(t-\rho)}{x(t)} > \frac{\rho\alpha}{\rho\alpha a_{J(d_1)} + b_{J(d_1)}(t^\alpha - t^\alpha_{J(d_1)})} \frac{(t-\rho)^\alpha - (t_{J(d_1)} - \rho)^\alpha}{t^\alpha_{J(d_1)} - (t_{J(d_1)} - \rho)^\alpha} \geq 0.$$

Combining all these cases, we have

$$\frac{x(t-\rho)}{x(t)} > \begin{cases} M^1_{J(c_1)}(t) & \text{for} \quad t \in [c_1, t_{J(c_1)+1}] \\ M^1_k(t) & \text{for} \quad t \in (t_k, t_{k+1}], \quad k = J(c_1)+1, \cdots, J(d_1)-1 \\ M^1_{J(d_1)}(t) & \text{for} \quad t \in (t_{J(d_1)+1}, d_1]. \end{cases}$$

Hence by (5), we have

$$\begin{aligned} &\sum_{k=J(c_1)+1}^{J(d_1)} p^2(t_k) t_k^{1-\alpha} \left[w(t_k) - w(t_k^+)\right] \\ &\qquad \leq \int_{c_1}^{t_{J(c_1)+1}} \left[(p'(t))^2 t^{2-2\alpha} \eta r(t) - p^2(t) Q(t) M^1_{J(c_1)}(t)\right] dt \\ &\qquad + \sum_{k=J(c_1)+1}^{J(d_1)-1} \int_{t_k}^{t_{k+1}} \left[(p'(t))^2 t^{2-2\alpha} \eta r(t) - p^2(t) Q(t) M^1_k(t)\right] dt \\ &\qquad + \int_{t_{J(d_1)}}^{d_1} \left[(p'(t))^2 t^{2-2\alpha} \eta r(t) - p^2(t) Q(t) M^1_{J(d_1)}(t)\right] dt \\ &\qquad + \int_{c_1}^{d_1} (1-\alpha) t^{-\alpha} p^2(t) w(t) dt \end{aligned} \tag{8}$$

Since $r(t)g(T_\alpha(x(t)))$ is non-increasing in $(c_1, t_{J(c_1)+1}]$. Thus

$$\begin{aligned} x(t) > x(t) - x(c_1) &= T_\alpha(x(\xi_3)) \left(\frac{t^\alpha - c_1^\alpha}{\alpha}\right) \\ &\geq \frac{r(t) g(T_\alpha(x(t)))}{r(\xi_3)} \left(\frac{t^\alpha - c_1^\alpha}{\alpha}\right), \quad \xi_3 \in (c_1, t) \end{aligned}$$

Letting $t \to t^-_{J(c_1)+1}$, it follows that

$$w(t_{J(c_1)+1}) < \frac{r_1}{t^\alpha_{J(c_1)+1} - c_1^\alpha}. \tag{9}$$

Similarly we can prove that on $(t_{k-1}, t_k]$, $k = J(c_1)+2, \cdots, J(d_1)$,

$$w(t_k) < \frac{r_1}{t_k^\alpha - t_{k-1}^\alpha}. \tag{10}$$

Hence (9) and (10), we have

$$\begin{aligned}
&\sum_{k=J(c_1)+1}^{J(d_1)} p^2(t_k) t_k^{1-\alpha} w(t_k) \left[\frac{a_k - b_k}{a_k}\right] \\
&\qquad \geq r_1 \left[p^2(t_{J(c_1)+1}) t_{J(c_1)+1}^{1-\alpha} \frac{a_{J(c_1)+1} - b_{J(c_1)+1}}{a_{J(c_1)+1}} \frac{1}{t_{J(c_1)+1}^\alpha - c_1^\alpha} \right. \\
&\qquad \left. + \sum_{k=J(c_1)+2}^{J(d_1)} p^2(t_k) t_k^{1-\alpha} \frac{a_k - b_k}{a_k} \frac{1}{t_k^\alpha - t_{k-1}^\alpha} \right] \\
&\qquad \geq \Lambda(p, c_1, d_1).
\end{aligned}$$

Thus we have

$$\sum_{k=J(c_1)+1}^{J(d_1)} p^2(t_k) t_k^{1-\alpha} w(t_k) \left[\frac{a_k - b_k}{a_k}\right] \geq \Lambda(p, c_1, d_1).$$

Therefore (8), we get

$$\begin{aligned}
&\int_{c_1}^{t_{J(c_1)+1}} \left[(p'(t))^2 t^{2-2\alpha} \eta r(t) - p^2(t) Q(t) M_{J(c_1)}^1(t) \right] dt \\
&+ \sum_{k=J(c_1)+1}^{J(d_1)-1} \int_{t_k}^{t_{k+1}} \left[(p'(t))^2 t^{2-2\alpha} \eta r(t) - p^2(t) Q(t) M_k^1(t) \right] dt \\
&+ \int_{t_{J(d_1)}}^{d_1} \left[(p'(t))^2 t^{2-2\alpha} \eta r(t) - p^2(t) Q(t) M_{J(d_1)}^1(t) \right] dt + \int_{c_1}^{d_1} (1-\alpha) t^{-\alpha} p^2(t) w(t) dt \\
&> \Lambda(p, c_1, d_1)
\end{aligned}$$

which contradicts (2).

If $J(c_1) = J(d_1)$ then $\Lambda(p, c_1, d_1) = 0$ and there are no impulsive moments in $[c_1, d_1]$. Similar to the proof of (8), we obtain

$$\int_{c_1}^{d_1} \left[(p'(t))^2 t^{2-2\alpha} \eta r(t) - p^2(t) Q(t) M_{J(c_1)}^1(t) + p^2(t)(1-\alpha) t^{-\alpha} w(t) \right] dt > 0$$

This again contradicts our assumption. Finally if $x(t)$ is eventually negative, we can consider $[c_2, d_2]$ and reach similar contradiction. The proof of theorem is complete. □

Following Kong [11] and Philos [17], we introduce a class of functions: Let $D = \{(t,s) : t_0 \le s \le t\}$, then a function $H_1, H_2 \in C(D, \mathbb{R})$ is said to belong to the class $\mathcal{H}$ if

(H_4) $H_1(t,t) = H_2(t,t) = 0$, $H_1(t,s) > 0, H_2(t,s) > 0$ for $t > s$ and

(H_5) H_1 and H_2 have partial derivatives $\frac{\partial H}{\partial t}$ and $\frac{\partial H}{\partial s}$ on D such that

$$\frac{\partial H_1}{\partial t} = h_1(t,s)H_1(t,s), \quad \frac{\partial H_2}{\partial s} = -h_2(t,s)H_2(t,s),$$

where $h_1, h_2 \in L_{loc}(D, \mathbb{R})$.

$$\begin{aligned}\Omega_{1,j} = &\int_{c_j}^{t_{J(c_j)+1}} H_1(t,c_j)Q(t)M^j_{J(c_j)}(t)dt + \sum_{k=J(c_i)+1}^{J(\lambda_j)-1} \int_{t_k}^{t_{k+1}} H_1(t,c_j)Q(t)M^j_k(t)dt \\ &+ \int_{t_{J(\lambda_j)}}^{\lambda_j} H_1(t,c_j)Q(t)M^j_{J(d_j)}(t)dt \\ &+ \int_{c_j}^{\lambda_j} H_1(t,c_j)\left[\frac{w^2(t)}{\eta r(t)} - w(t)t^{1-\alpha}h_1(t,c_j) - (1-\alpha)t^{-\alpha}w(t)\right]dt\end{aligned}$$

and

$$\begin{aligned}\Omega_{2,j} = &\int_{\lambda_j}^{t_{J(\lambda_j)+1}} H_2(d_j,t)Q(t)M^j_{J(\lambda_j)}(t)dt + \sum_{k=J(\lambda_j)+1}^{J(d_j)-1} \int_{t_k}^{t_{k+1}} H_2(d_j,t)Q(t)M^j_k(t)dt \\ &+ \int_{t_{J(d_j)}}^{d_j} H_2(d_j,t)Q(t)M^j_{J(d_j)}(t)dt \\ &+ \int_{\lambda_j}^{d_j} H_2(d_j,t)\left[\frac{w^2(t)}{\eta r(t)} + w(t)t^{1-\alpha}h_2(d_j,t) - (1-\alpha)t^{-\alpha}w(t)\right]dt.\end{aligned}$$

Theorem 6. *Assume that conditions $(H_1)-(H_3)$ hold, furthermore for any $T \ge 0$ there exist c_j, d_j satisfying $(H_4), (H_5)$ with $c_1 < \lambda_1 < d_1 \le c_2 < \lambda_2 < d_2$. If there exists $H_1, H_2 \in \mathcal{H}$ such that*

$$\frac{1}{H_1(\lambda_1,c_1)}\Omega_{1,1} + \frac{1}{H_2(d_1,\lambda_1)}\Omega_{2,1} > \Lambda(H_1,H_2;c_j,d_j), \tag{11}$$

where

$$\Lambda(H_1,H_2;c_j,d_j) = -\left\{\frac{r_j}{H_1(\lambda_j,c_j)}\Phi^{\lambda_j}_{c_j}[H_1(.,c_j)] + \frac{r_j}{H_2(d_j,\lambda_j)}\Phi^{d_j}_{\lambda_j}[H_2(d_j,.)]\right\},$$

then every solution of problem (1) *is oscillatory.*

Proof. Suppose to the contrary that there is a non-oscillatory solution $x(t)$ of problem (1). Notice whether or not there are impulsive moments in $[c_1, \lambda_1]$ and $[\lambda_1, d_1]$, we should consider the following cases $J(c_1) < J(\lambda_1) < J(d_1)$, $J(c_1) = J(\lambda_1) <$

$J(d_1)$, $J(c_1) < J(\lambda_1) = J(d_1)$ and $J(c_1) = J(\lambda_1) = J(d_1)$. Moreover, the impulsive moments of $x(t-\rho)$ having following two cases, $t_{J(\lambda_s)} + \rho > \lambda_s$ and $t_{J(\lambda_s)} + \rho \le \lambda_s$. Consider the case $J(c_1) < J(\lambda_1) < J(d_1)$, with $t_{J(\lambda_s)} + \rho > \lambda_s$. For this case, the impulsive moments are $t_{J(\lambda_1)+1},\ t_{J(\lambda_1)+2}, \cdots,\ t_{J(d_1)}$ in $[\lambda_1, d_1]$.

Multiplying by $H_1(t, c_1)$ on both sides on (4), integrating it from c_1 to λ_1, we obtain

$$\int_{c_1}^{\lambda_1} H_1(t,c_1) t^{1-\alpha} w'(t)dt \le -\int_{c_1}^{\lambda_1} H_1(t,c_1)Q(t)\frac{x(t-\rho)}{x(t)}dt - \int_{c_1}^{\lambda_1} H_1(t,c_1)\frac{w^2(t)}{\eta r(t)}dt.$$

Applying integration by parts formula on the L.H.S, we get,

$$\begin{aligned}
&\sum_{k=J(c_1)+1}^{J(\lambda_1)} H_1(t_k,c_1)t_k^{1-\alpha}\left[w(t_k) - w(t_k^+)\right] - H_1(\lambda_1,c_1)\lambda_1^{1-\alpha}w(\lambda_1)\\
&\quad - \int_{c_1}^{\lambda_1} w(t)\left[h_1(t,c_1)H_1(t,c_1)t^{1-\alpha} + H_1(t,c_1)(1-\alpha)t^{-\alpha}\right]dt\\
&\le -\int_{c_1}^{\lambda_1} H_1(t,c_1)Q(t)\frac{x(t-\rho)}{x(t)}dt - \int_{c_1}^{\lambda_1}\frac{w^2(t)}{\eta r(t)}H_1(t,c_1)dt \qquad (12)
\end{aligned}$$

By Theorem 5, we divide the interval $[c_1, \lambda_1]$ into several and calculating the function $\dfrac{x(t-\rho)}{x(t)}$, we obtain

$$\begin{aligned}
\int_{c_1}^{\lambda_1} H_1(t,c_1)Q(t)\frac{x(t-\rho)}{x(t)}dt \ge &\int_{c_1}^{t_{J(c_1)+1}} H_1(t,c_1)Q(t)M^1_{J(c_1)}(t)dt\\
&+\sum_{k=J(c_1)+1}^{J(\lambda_1)-1}\int_{t_k}^{t_{k+1}} H_1(t,c_1)Q(t)M^1_{k(t)}dt\\
&+\int_{t_{J(\lambda_1)}}^{\lambda_1} H_1(t,c_1)Q(t)M^1_{J(\lambda_1)}(t)dt. \qquad (13)
\end{aligned}$$

From (12) and (13), we obtain

$$\begin{aligned}
&\int_{c_1}^{t_{J(c_1)+1}} H_1(t,c_1)Q(t)M^1_{J(c_1)}(t)dt + \sum_{k=J(c_1)+1}^{J(\lambda_1)-1}\int_{t_k}^{t_{k+1}} H_1(t,c_1)Q(t)M^1_{k(t)}dt\\
&+\int_{t_{J(\lambda_1)}}^{\lambda_1} H_1(t,c_1)Q(t)M^1_{J(\lambda_1)}(t)dt\\
&+\int_{c_1}^{\lambda_1} H_1(t,c_1)w(t)\left[\frac{w(t)}{\eta r(t)} - t^{1-\alpha}h_1(t,c_1) - (1-\alpha)t^{-\alpha}\right]dt\\
&\le -\sum_{k=J(c_1)+1}^{J(\lambda_1)} H_1(t_k,c_1)t_k^{1-\alpha}\left[\frac{a_k-b_k}{a_k}\right]w(t_k) - H_1(\lambda_1,c_1)\lambda_1^{1-\alpha}w(\lambda_1). \qquad (14)
\end{aligned}$$

On the other hand multiplying both sides of (4) by $H_2(d_1,t)$ and integrating from λ_1 to d_1 and using the similar of above, we get

$$\int_{\lambda_1}^{t_{J(\lambda_1)+1}} H_2(d_1,t)Q(t)M^1_{J(\lambda_1)}(t)dt + \sum_{k=J(\lambda_1)+1}^{J(d_1)-1}\int_{t_k}^{t_{k+1}} H_2(d_1,t_k)Q(t)M^1_{k(t)}dt$$
$$+ \int_{t_{J(d_1)}}^{d_1} H_2(d_1,t)Q(t)M^1_{J(d_1)}(t)dt$$
$$+ \int_{\lambda_1}^{d_1} H_2(d_1,t)w(t)\left[\frac{w(t)}{\eta r(t)} + t^{1-\alpha}h_2(d_1,t) - (1-\alpha)t^{-\alpha}\right]dt$$
$$\le -\sum_{k=J(\lambda_1)+1}^{J(d_1)} H_2(d_1,t_k)t_k^{1-\alpha}\left[\frac{a_k-b_k}{a_k}\right]w(t_k) + H_2(d_1,\lambda_1)\lambda_1^{1-\alpha}w(\lambda_1). \quad (15)$$

Dividing (14) and (15) by $H_1(\lambda_1,c_1)$ and $H_2(d_1,\lambda_1)$ respectively and adding them, we get

$$\frac{1}{H_1(\lambda_1,c_1)}\Omega_{1,1} + \frac{1}{H_2(d_1,\lambda_1)}\Omega_{2,1}$$
$$\le -\left[\frac{1}{H_1(\lambda_1,c_1)}\sum_{k=J(c_1)+1}^{J(\lambda_1)} H_1(t_k,c_1)t_k^{1-\alpha}\left[\frac{a_k-b_k}{a_k}\right]w(t_k)\right.$$
$$\left. + \frac{1}{H_2(d_1,\lambda_1)}\sum_{k=J(\lambda_1)+1}^{J(d_1)} H_2(d_1,t_k)t_k^{1-\alpha}\left[\frac{a_k-b_k}{a_k}\right]w(t_k)\right]. \quad (16)$$

Using the method as in (9) , we obtain

$$\left.\begin{aligned} -\sum_{k=J(c_1)+1}^{J(\lambda_1)} H_1(t_k,c_1)t_k^{1-\alpha}\left[\frac{a_k-b_k}{a_k}\right]w(t_k) &\le -r_1\Phi_{c_1}^{\lambda_1}[H_1(\cdot,c_1)] \\ -\sum_{k=J(\lambda_1)+1}^{J(d_1)} H_2(d_1,t_k)t_k^{1-\alpha}\left[\frac{a_k-b_k}{a_k}\right]w(t_k) &\le -r_1\Phi_{\lambda_1}^{d_1}[H_2(d_1,\cdot)]. \end{aligned}\right\} \quad (17)$$

From (16) and (17), we obtain

$$\frac{1}{H_1(\lambda_1,c_1)}\Omega_{1,1} + \frac{1}{H_2(d_1,\lambda_1)}\Omega_{2,1}$$
$$\le -\left\{\frac{r_1}{H_1(\lambda_1,c_1)}\Phi_{c_1}^{\lambda_1}[H_1(\cdot,c_1)] + \frac{r_1}{H_2(d_1,\lambda_1)}\Phi_{\lambda_1}^{d_1}[H_2(d_1,\cdot)]\right\}$$
$$\le \Lambda(H_1,H_2;c_1,d_1) \quad (18)$$

which is contradiction to the condition (11). Suppose $x(t) < 0$, we take interval $[c_2,d_2]$ for equation (1). The proof is similar and hence omitted. The proof is complete. □

4. Examples

In this section, we present some examples to illustrate our results established in Section 3.

Example 7. *Consider the following impulsive conformable fractional differential equations*

$$\left.\begin{array}{l} T_{\frac{1}{2}}\left(2\left(T_{\frac{1}{2}}(x(t))\right)\right)+mx(t-\frac{\pi}{8})+2mx(t-\frac{\pi}{8})=f(t), \quad t\neq 2k\pi\pm\frac{\pi}{4}, \\ x(t_k^+)=\frac{1}{3}x(t_k), \quad T_{\frac{1}{2}}(x(t_k^+))=\frac{2}{3}T_{\frac{1}{2}}(x(t_k)), \qquad k=1,2,\cdots. \end{array}\right\} \tag{19}$$

Here $\alpha=\frac{1}{2}$, $a_k=\frac{1}{3}$, $b_k=\frac{2}{3}$, $r(t)=2$, $q(t)=m$, $q_1(t)=2m$, $g(x)=x$, $f_1(x)=x$, $\eta=1$, $f(t)=\cos t-2t\sin t+3m\sin(t-\frac{\pi}{8})$ *and* m *is a positive constant. Also* $\rho=\frac{\pi}{8}$, $t_{k+1}-t_k=\pi/2>\pi/8$. *For any* $T>0$, *we choose* k *large enough such that* $T<c_1=4k\pi-\frac{\pi}{2}<d_1=4k\pi$ *and* $c_2=4k\pi+\frac{\pi}{8}<d_2=4k\pi+\frac{\pi}{2}$, $k=1,2,\cdots$. *Then there is an impulsive movement* $t_k=4k\pi-\frac{\pi}{4}$ *in* $[c_1,d_1]$ *and an impulsive moment* $t_{k+1}=4k\pi+\frac{\pi}{4}$ *in* $[c_2,d_2]$. *For* $\epsilon_1=1$, *we have* $Q(t)=3m$, *and we take* $p(t)=\sin 8t\in J_p(c_j,d_j)$, $j=1,2$, $t_{J(c_1)}=4k\pi-\frac{7\pi}{4}$, $t_{J(d_1)}=4k\pi-\frac{\pi}{4}$, *then by using simple calculation, the left side of equation* (2) *is the following :*

$$\begin{aligned}
&\int_{c_j}^{d_j}\left[(p'(t))^2t^{2-2\alpha}\eta r(t)+w(t)p^2(t)(1-\alpha)t^{-\alpha}\right]dt-\int_{c_j}^{t_{J(c_j)+1}}Q(t)p^2(t)M^j_{J(c_j)}(t)dt\\
&-\sum_{k=J(c_j)+1}^{J(d_j)-1}\int_{t_k}^{t_{k+1}}Q(t)p^2(t)M^j_{J(c_j)}(t)dt-\int_{t_{J(d_j)}}^{d_j}Q(t)p^2(t)M^j_{J(d_j)}(t)dt\\
&\leq\int_{4k\pi-\frac{\pi}{2}}^{4k\pi}\left(2t(8\cos 8t)^2+\frac{t^{\frac{1}{2}}\cos t}{\sin t}t^{-\frac{1}{2}}\sin^2(8t)\right)dt\\
&-3m\Bigg\{\int_{4k\pi-\frac{\pi}{2}}^{4k\pi-\frac{\pi}{4}}\sin^2(8t)\left(\frac{\left(t-\frac{\pi}{8}\right)^{\frac{1}{2}}-\left(4k\pi-\frac{7\pi}{4}\right)^{\frac{1}{2}}}{t^{\frac{1}{2}}-\left(4k\pi-\frac{7\pi}{4}\right)^{\frac{1}{2}}}\right)dt\\
&\quad+\int_{4k\pi-\frac{\pi}{4}}^{4k\pi-\frac{\pi}{8}}\sin^2(8t)\left(\frac{\frac{\pi}{16}}{\frac{\pi}{48}+\frac{2}{3}\left(t^{\frac{1}{2}}-\left(4k\pi-\frac{\pi}{4}\right)^{\frac{1}{2}}\right)}\right)\\
&\qquad\times\left(\frac{\left(t-\frac{\pi}{8}\right)^{\frac{1}{2}}-\left(4k\pi-\frac{\pi}{4}-\frac{\pi}{8}\right)^{\frac{1}{2}}}{\left(4k\pi-\frac{\pi}{4}\right)^{\frac{1}{2}}-\left(4k\pi-\frac{\pi}{4}-\frac{\pi}{8}\right)^{\frac{1}{2}}}\right)\\
&\quad+\int_{4k\pi-\frac{\pi}{8}}^{4k\pi}\sin^2(8t)\left(\frac{\left(t-\frac{\pi}{8}\right)^{\frac{1}{2}}-\left(4k\pi-\frac{\pi}{4}\right)^{\frac{1}{2}}}{t^{\frac{1}{2}}-\left(4k\pi-\frac{\pi}{4}\right)^{\frac{1}{2}}}\right)dt\Bigg\}\\
&\simeq 1182.67634-m(1.94487).
\end{aligned}$$

for m large enough. On the other hand, note that $J(c_1) = k-1$, $J(d_1) = k$, $r_1 = 2$, we have $\Lambda(p, c_i, d_i) = 0$. Therefore the condition (2) is satisfied in $[c_1, d_1]$. Similarly, we can prove that for $t \in [c_2, d_2]$. Hence by Theorem 5, every solution of (19) is oscillatory. In fact $x(t) = \sin t$ is one such solution of problem (19).

Example 8. *Consider the following impulsive conformable fractional differential equations*

$$\left.\begin{array}{ll} T_{\frac{1}{3}}\left(3\left(T_{\frac{1}{3}}(x(t))\right)\right) + \frac{m}{2}x(t-\frac{\pi}{8}) + mx(t-\frac{\pi}{8}) = f(t), & t \neq 2k\pi \pm \frac{\pi}{4}, \\ x(t_k^+) = 4x(t_k), \quad T_{\frac{1}{3}}(x(t_k^+)) = 5T_{\frac{1}{3}}(x(t_k)), & k = 1, 2, \cdots. \end{array}\right\} \tag{20}$$

Here $\alpha = \frac{1}{3}$, $a_k = 4$, $b_k = 5$, $r(t) = 3$, $q(t) = \frac{m}{2}$, $q_1(t) = m$, $g(x) = 2x$, $f_1(x) = x$, $\eta = 2$, $f(t) = -4t^{\frac{1}{3}}\sin t - 6t^{\frac{4}{3}}\cos t + \frac{3m}{2}\cos(t-\frac{\pi}{8})$ and m is a positive constant. Also $\rho = \frac{\pi}{8}$, $t_{k+1} - t_k = \pi/2 > \pi/8$. For any $T > 0$, we choose k large enough such that $T < c_1 = 4k\pi - \frac{\pi}{2} < d_1 = 4k\pi$ and $c_2 = 4k\pi + \frac{\pi}{8} < d_2 = 4k\pi + \frac{\pi}{2}$, $k = 1, 2, \cdots$. Then there is an impulsive movement $t_k = 4k\pi - \frac{\pi}{4}$ in $[c_1, d_1]$ and an impulsive moment $t_{k+1} = 4k\pi + \frac{\pi}{4}$ in $[c_2, d_2]$. For $\epsilon_1 = 1$, we have $Q(t) = \frac{3m}{2}$, and we take $p(t) = \sin 16t \in J_p(c_j, d_j)$, $j = 1, 2$, $t_{J(c_1)} = 4k\pi - \frac{7\pi}{4}$, $t_{J(d_1)} = 4k\pi - \frac{\pi}{4}$, then by using simple calculation, the left side of Equation (2) is the following :

$$\begin{aligned}
&\int_{c_j}^{d_j} \left[(p'(t))^2 t^{2-2\alpha}\eta r(t) + w(t)p^2(t)(1-\alpha)t^{-\alpha}\right] dt - \int_{c_j}^{t_{J(c_j)+1}} Q(t)p^2(t)M^j_{J(c_j)}(t)dt \\
&- \sum_{k=J(c_j)+1}^{J(d_j)-1} \int_{t_k}^{t_{k+1}} Q(t)p^2(t)M^j_{J(c_j)}(t)dt - \int_{t_{J(d_j)}}^{d_j} Q(t)p^2(t)M^j_{J(d_j)}(t)dt \\
&\leq \int_{4k\pi-\frac{\pi}{2}}^{4k\pi} \left(6t^{\frac{4}{3}}(16\cos 16t)^2 - \frac{4t^{\frac{1}{3}}\sin t}{\cos t}\sin^2(16t)\right) dt \\
&- \frac{3m}{2}\Bigg\{\int_{4k\pi-\frac{\pi}{2}}^{4k\pi-\frac{\pi}{4}} \sin^2(16t)\left(\frac{\left(t-\frac{\pi}{8}\right)^{\frac{1}{3}} - \left(4k\pi - \frac{7\pi}{4}\right)^{\frac{1}{3}}}{t^{\frac{1}{3}} - \left(4k\pi - \frac{7\pi}{4}\right)^{\frac{1}{3}}}\right) dt \\
&\quad + \int_{4k\pi-\frac{\pi}{4}}^{4k\pi-\frac{\pi}{8}} \sin^2(16t)\left(\frac{\frac{\pi}{24}}{\frac{2\pi}{6} + 5\left(t^{\frac{1}{3}} - \left(4k\pi - \frac{\pi}{4}\right)^{\frac{1}{3}}\right)}\right)\left(\frac{\left(t-\frac{\pi}{8}\right)^{\frac{1}{3}} - \left(4k\pi - \frac{\pi}{4} - \frac{\pi}{8}\right)^{\frac{1}{3}}}{\left(4k\pi - \frac{\pi}{4}\right)^{\frac{1}{3}} - \left(4k\pi - \frac{\pi}{4} - \frac{\pi}{8}\right)^{\frac{1}{3}}}\right) \\
&\quad + \int_{4k\pi-\frac{\pi}{8}}^{4k\pi} \sin^2(16t)\left(\frac{\left(t-\frac{\pi}{8}\right)^{\frac{1}{3}} - \left(4k\pi - \frac{\pi}{4}\right)^{\frac{1}{3}}}{t^{\frac{1}{3}} - \left(4k\pi - \frac{\pi}{4}\right)^{\frac{1}{3}}}\right) dt\Bigg\} \\
&\qquad \simeq 32367.58257 - m(0.66712).
\end{aligned}$$

for m large enough. On the other hand, note that $J(c_1) = k-1$, $J(d_1) = k$, $r_1 = 3$, we have $\Lambda(p, c_i, d_i) = 0$. Therefore the condition (2) is satisfied in $[c_1, d_1]$. Similarly, we can prove that for $t \in [c_2, d_2]$. Hence by Theorem 5, every solution of (20) is oscillatory. In fact $x(t) = \cos t$ is one such solution of problems(20).

Remark 9. *In this paper, some new oscillation results are obtained, generalizing the results of* [13] *to impulsive conformable fractional differential equations. The improvement factors impulses, delay and forcing term that affect the interval qualitative properties of solution in the sequence of subintervals in* $[0, \infty)$*, were taken into account together. Our newly obtained results in this paper have improved and extended some of the results already prevailing in the existing literature.*

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