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Local comparison of two sixth order solvers using only the first derivative

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Abstract

Two efficient sixth order solvers are compared involving Banach space valued operators. Earlier papers use hypotheses up to the seventh derivative that do not appear in the solver in the local convergence analysis. But we use hypotheses only on the first derivative. Hence, we expand the applicability of these solvers. We use examples to test the older as well as our results.

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1. Introduction

In this paper, we compare two solvers with sixth order of convergence for solving the nonlinear equation

$$F(x) = 0. \tag{1}$$

Here, $F : \Omega \subseteq \mathcal{X} \longrightarrow \mathcal{Y}$ is differentiable, $\Omega \subseteq \mathcal{X}$ is a convex subset in the Banach space $\mathcal{X}$, and $\mathcal{Y}$ is a Banach space. Solving equations like (1) is important in computational sciences, since problems usually reduce to such equations [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]. The convergence of the following two sixth order solvers are carried out using [8, 13] assumptions up to the seventh derivative of F, even though

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these derivatives do not appear in these solvers [13]

$$
\begin{aligned}
y_n &= x_n - \frac{2}{3}F'(x_n)^{-1}F(x_n) \\
z_n &= x_n - \frac{1}{2}[3F'(y_n) - F'(x_n)]^{-1} \\
&\quad \times [3F'(y_n) + F'(x_n)]F'(x_n)^{-1}F(x_n) \\
x_{n+1} &= z_n - \frac{1}{2}[3F'(y_n) - F'(x_n)]^{-1} \\
&\quad \times [3F'(y_n) + F'(x_n)]^2 F'(x_n)^{-1}F(z_n)
\end{aligned}
\tag{2}
$$

and

$$
\begin{aligned}
y_n &= x_n - \frac{2}{3}F'(x_n)^{-1}F(x_n) \\
z_n &= x_n - [\frac{-5}{8}I + \frac{3}{8}(F'(y_n)^{-1}F'(x_n))^2] \\
&\quad \times F'(x_n)^{-1}F(x_n) \\
x_{n+1} &= z_n - [\frac{-9}{4}I + \frac{15}{8}F'(y_n)^{-1}F'(x_n) \\
&\quad + \frac{11}{8}F'(x_n)^{-1}F'(y_n)]F'(y_n)^{-1}F(z_n)
\end{aligned}
\tag{3}
$$

The order of convergence six was shown in [13, 8], respectively using derivatives up to order seven and for $\mathcal{X} = \mathcal{Y} = \mathbb{R}^k$. But these derivatives do not appear in these solvers. Hence, the assumptions on the seventh order derivative limit the application of these solvers. For example: Let $\mathcal{X} = \mathcal{Y} = \mathbb{R}$, $\Omega = [-\frac{1}{2}, \frac{3}{2}]$. Define $\mathcal{F}$ on Ω by

$$
\mathcal{F}(x) = x^3 \log x^2 + x^5 - x^4
$$

Then, we have $p_* = 1$, and

$$
\mathcal{F}'(x) = 3x^2 \log x^2 + 5x^4 - 4x^3 + 2x^2,
$$
$$
\mathcal{F}''(x) = 6x \log x^2 + 20x^3 - 12x^2 + 10x,
$$
$$
\mathcal{F}'''(x) = 6 \log x^2 + 60x^2 = 24x + 22.
$$

Obviously $\mathcal{F}'''(x)$ is not bounded on Ω. So, the convergence of solvers (2) and (3) are not guaranteed by the analysis in [13, 8]. In this study we use only assumptions on the first derivative to prove our results.

The rest of the paper is structured as follows. Section 2 and Section 3 contain the local convergence analysis of method (2) and method (3), respectively. The numerical examples appear in the concluding Section 4.

2. Local convergence of solver (2)

Let $\varphi_0 : [0, \infty) \longrightarrow [0, \infty)$ be a continuous and increasing function with $\varphi_0(0) = 0$. Assume equation

$$
\varphi_0(t) = 1
\tag{1}
$$

has a minimal positive solution ρ_0. Set $I_0 = [0, \rho_0)$. Let functions $\varphi : I_0 \longrightarrow [0, \infty)$, $\varphi_1 : I_0 \longrightarrow [0, \infty)$ be continuous and increasing with $\varphi_0(0) = 0$. Define functions $\bar{\varphi}_1, \bar{\bar{\varphi}}_1$ on the interval I_0 by

$$
\bar{\varphi}_1(t) = \frac{\int_0^1 \varphi((1-\theta)t)d\theta + \frac{1}{3}\int_0^1 \varphi_1(\theta t)d\theta}{1 - \varphi_0(t)}
$$

and

$$
\bar{\bar{\varphi}}_1(t) = \bar{\varphi}_1(t) - 1.
$$

Assume

$$\frac{\bar{\varphi}_1(0)}{3} - 1 < 0. \tag{2}$$

We have by these definition $\bar{\bar{\varphi}}_1(0) = -1$ and $\bar{\bar{\varphi}}_1 \longrightarrow \infty$ as $t \longrightarrow \rho_0^-$. Then, the intermediate value theorem assures the existence of solutions for equation $\bar{\bar{\varphi}}_1(t) = 0$ in $(0, \rho_0)$. Denote by r_1 the minimal such solution.

Assume equation

$$p(t) = 1, \tag{3}$$

has a minimal positive solution ρ_p where $p(t) = \frac{1}{3}(3\varphi_0(\bar{\varphi}_1(t)t) + \varphi_0(t))$. Set $\rho = \min\{\rho_0, \rho_p\}$ and $I = [0, \rho)$. Define functions $\bar{\varphi}_2$ and $\bar{\bar{\varphi}}_2$ on the interval I by

$$\bar{\varphi}_2(t) = \frac{\int_0^1 \varphi((1-\theta)t)d\theta}{1 - \varphi_0(t)}$$
$$+ \frac{3(\varphi_0(\bar{\varphi}_1(t)t) + \varphi_0(t))\int_0^1 \varphi_1(\theta t)d\theta}{4(1 - p(t))(1 - \varphi_0(t))}$$

and

$$\bar{\bar{\varphi}}_2(t) = \bar{\varphi}_2(t) - 1.$$

We get by these definitions $\bar{\bar{\varphi}}_2(0) = -1$ and $\bar{\bar{\varphi}}_2(t) \longrightarrow \infty$ as $t \longrightarrow \rho^-$. Denote by r_2 the minimal solution of equation $\bar{\bar{\varphi}}_2 = 0$ in $(0, \rho)$. Assume equation

$$\varphi_0(\bar{\varphi}_2(t)t) = 1 \tag{4}$$

has a minimal positive solution ρ_1. Set $\rho_2 = \min\{\rho, \rho_1\}$ and $I_1 = [0, \rho_2)$. Define functions $\bar{\varphi}_3$ and $\bar{\bar{\varphi}}_3$ on the interval I_1 by

$$\bar{\varphi}_3(t) = \left[\frac{\int_0^1 \varphi((1-\theta)\bar{\varphi}_2(t)t)d\theta}{1 - \varphi_0(\bar{\varphi}_2(t)t)} \right.$$
$$+ \frac{(\varphi_0(\bar{\varphi}_2(t)t) + \varphi_0(t))\int_0^1 \varphi_1(\theta\bar{\varphi}_2(t)t)d\theta}{(1 - \varphi_0(\bar{\varphi}_2(t)t))(1 - \varphi_0(t))}$$
$$\left. + \frac{a(t)b(t)\int_0^1 \varphi_1(\theta\bar{\varphi}_2(t)t)d\theta}{1 - \varphi_0(t)} \right] \bar{\varphi}_2(t)$$

and

$$\bar{\bar{\varphi}}_3(t) = \bar{\varphi}_3(t) - 1,$$

where

$$a(t) = \frac{3}{4} \frac{\varphi_0(\bar{\varphi}_1(t)t) + \varphi_0(t)}{1 - p(t)}$$

and

$$b(t) = \frac{8\varphi_1(\bar{\varphi}_1(t)t) + \varphi_0(t) + \varphi_0(\bar{\varphi}_1(t)t)}{4(1 - p(t))}.$$

We obtain by these definitions $\bar{\bar{\varphi}}_3(0) = -1$ and $\bar{\bar{\varphi}}_3(t) \longrightarrow \infty$ as $t \longrightarrow \rho_2^-$. Denote by r_3 the smallest solution of equation $\bar{\bar{\varphi}}_3(t) = 0$ in $(0, \rho_2)$. Define a radius of convergence r by

$$r = \min\{r_m\}, \, m = 1, 2, 3. \tag{5}$$

It then follows from (5) that

$$0 \le \varphi_0(t) < 1, \tag{6}$$

$$0 \le p(t) < 1, \tag{7}$$

$$0 \le \varphi_0(\bar{\varphi}_2(t)t) < 1, \tag{8}$$

$$0 \le a(t), \tag{9}$$

$$0 \le b(t) \tag{10}$$

and

$$0 \le \bar{\varphi}_m(t) < 1, \tag{11}$$

for each $t \in [0, r)$.

The conditions (A) shall be used:

(a1) $F : \Omega \subseteq \mathcal{X} \longrightarrow \mathcal{Y}$ is continuously differentiable, and there exists $x_* \in \Omega$ such that $F(x_*) = 0$ and $F'(x_*)^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$.

(a2) Function $\varphi_0 : [0, \infty) \longrightarrow [0, \infty)$ is continuous, increasing with $\varphi_0(0) = 0$ such that for each $x \in \Omega$

$$\|F'(x_*)^{-1}(F'(x) - F'(x_*))\| \le \varphi_0(\|x - x_*\|).$$

Set $\Omega_0 = \Omega \cap U(x_*, \rho_0)$, where ρ_0 is given in (1).

(a3) Functions $\varphi : [0, \rho_0) \longrightarrow [0, \infty)$, $\varphi_1 : [0, \rho_0) \longrightarrow [0, \infty)$ are continuous, increasing with $\varphi(0) = 0$ such that for each $x, y \in \Omega_0$

$$\|F'(x_*)^{-1}(F'(y) - F'(x))\| \le \varphi(\|y - x\|)$$

and

$$\|F'(x_*)^{-1}F'(x)\| \le \varphi_1(\|x - x_*\|).$$

(a4) $\bar{U}(x_*, r) \subset \Omega, \rho_0, \rho_p, \rho_1$ given in (1), (3), (4), respectively exist and (2) holds.

(a5) There exists $\bar{r} \ge r$ such that

$$\int_0^1 \varphi_0(\theta\bar{r})d\theta < 1.$$

Set $\Omega_1 = \Omega \cap \bar{U}(x_*, \bar{r})$.

Next, the local convergence analysis of solver (2) is shown using conditions (A), and the preceding notation.

Theorem 2.1. *Under the hypotheses (A) further suppose $x_0 \in U(x_*, r) - \{x_*\}$. Then, the following assertions hold for solver (2)*

$$\{x_n\} \subset U(x_*, r), \tag{12}$$

$$\lim_{n \longrightarrow \infty} x_n = x_*, \tag{13}$$

$$\|y_n - x_*\| \le \bar{\varphi}_1(\|x_n - x_*\|)\|x_n - x_*\| \le \|x_n - x_*\| < r, \tag{14}$$

$$\|z_n - x_*\| \le \bar{\varphi}_2(\|x_n - x_*\|)\|x_n - x_*\| \le \|x_n - x_*\|, \tag{15}$$

$$\|x_{n+1} - x_*\| \le \bar{\varphi}_3(\|x_n - x_*\|)\|x_n - x_*\| \le \|x_n - x_*\|, \tag{16}$$

and x_* is the only solution of equation $F(x) = 0$ in the set Ω_1 given in (a5), where the function $\bar{\varphi}_m$ are defined previously.

Proof. Estimates (14)-(16) are shown using mathematical induction. Let $x \in U(x_*, r) - \{x_*\}$. By (a1), (6) and (a2) we have

$$\begin{aligned} \|F'(x_*)^{-1}(F'(x) - F'(x_*))\| &\leq \varphi_0(\|x - x_*\|) \\ &\leq \varphi_0(r) < 1, \end{aligned} \tag{17}$$

which together with a Banach lemma on invertible operators [2, 10, 11] imply $F'(x)^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$, and

$$\|F'(x)^{-1}F'(x_*)\| \leq \frac{1}{1 - \varphi_0(\|x - x_*\|)}. \tag{18}$$

It also follows that y_0 is well defined by the first substep of solver (2) for $n = 0$. We can write by (a1)

$$\begin{aligned} F'(x_*)^{-1}F(x) &= F'(x_*)^{-1}(F(x) - F(x_*)) \\ &= \int_0^1 F'(x_*)^{-1}F'(x_* + \theta(x - x_*))d\theta(x - x_*), \end{aligned} \tag{19}$$

so by the second condition in (a3)

$$\|F'(x_*)^{-1}F(x)\| \leq \varphi_1(\|x - x_*\|)\|x - x_*\|, \tag{20}$$

where we also used $\|x_* + \theta(x - x_*) - x_*\| = \theta\|x - x_*\| \leq \|x - x_*\| < r$, so $x_* + \theta(x - x_*) \in U(x_*, r)$ for each $\theta \in [0, 1]$. By (5), (11) (for $m = 1$), (18) (for $x = x_0$), (20) (for $x = x_0$), first substep of solver (2) (for $n = 0$), we get in turn that

$$\begin{aligned} \|y_0 - x_*\| &= \|(x_0 - x_* - F'(x_0)^{-1}F(x_0)) + \frac{1}{3}F'(x_0)^{-1}F(x_0)\| \\ &\leq \|F'(x_0)^{-1}F'(x_*)\|\|\int_0^1 F'(x_*)^{-1}(F'(x_* + \theta(x_0 - x_*)) - F'(x_0))d\theta(x_0 - x_*)\| \\ &\quad + \frac{1}{3}\|F'(x_0)^{-1}F'(x_*)\|\|F'(x_*)^{-1}F(x_0)\| \\ &\leq \frac{\int_0^1 \varphi((1 - \theta)\|x_0 - x_*\|)d\theta + \frac{1}{3}\varphi_1(\theta\|x_0 - x_*\|)d\theta}{1 - \varphi_0(\|x_0 - x_*\|)}\|x_0 - x_*\| \\ &= \bar{\varphi}_1(\|x_0 - x_*\|)\|x_0 - x_*\| \leq \|x_0 - x_*\| < r, \end{aligned} \tag{21}$$

so $y_0 \in U(x_*, r)$ and (14) holds for $n = 0$. We must show $(3F'(y_0) - F'(x_0))^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$, so z_0 will be defined by the second substep of solver (2) for $n = 0$. Using (5), (7), (a2), and (21), we obtain in turn that

$$\begin{aligned} &\|(2F'(x_*))^{-1}[3(F'(y_0) - F'(x_*)) - (F'(x_0) - F'(x_*))]\| \\ &\leq \frac{1}{2}[3\|F'(x_*)^{-1}(F'(y_0) - F'(x_*))\| \\ &\quad + \|F'(x_*)^{-1}(F'(x_0) - F'(x_*))\|] \\ &\leq \frac{1}{3}(3\varphi_0(\|y_0 - x_*\|) + \varphi_0(\|x_0 - x_*\|)) \\ &\leq \frac{1}{3}(3\varphi_0(\bar{\varphi}_1(\|x_0 - x_*\|)\|x_0 - x_*\|) + \varphi_0(\|x_0 - x_*\|)) \\ &= p(\|x_0 - x_*\|) \leq p(r) < 1, \end{aligned} \tag{22}$$

so

$$\|(3F'(y_0) - F'(x_0))^{-1}F'(x_*)\| \leq \frac{1}{2(1 - p(\|x_0 - x_*\|))}. \tag{23}$$

Then, by (5), (11) (for $m = 2$), (18) (for $x = x_)$), (20) (for $x = x_0$), (21), (23), and the second substep of solver (2) for $n = 0$, we have in turn that

$$
\begin{aligned}
\|z_0 - x_*\| &= \|(x_0 - x_* - F'(x_0)^{-1}F(x_0)) \\
&\quad + [I - \frac{1}{2}(3F'(y_0) - F'(x_0))^{-1}(3F'(y_0) + F'(x_0))] \\
&\quad \times F'(x_0)^{-1}F(x_0)\| \\
&= \|(x_0 - x_* - F'(x_0)^{-1}F(x_0)) \\
&\quad + \frac{3}{2}(3F'(y_0) - F'(x_0))^{-1}((F'(y_0) - F'(x_*)) + (F'(x_*) - F'(x_0)) \\
&\quad \times F'(x_0)^{-1}F(x_0)\| \\
&\leq \|(x_0 - x_* - F'(x_0)^{-1}F(x_0))\| \\
&\quad + \frac{3}{2}\|(3F'(y_0) - F'(x_0))^{-1}F'(x_*)\|[\|F'(x_*)^{-1}((F'(y_0) - F'(x_*))\| \\
&\quad + \|F'(x_*)^{-1}(F'(x_*) - F'(x_0))\|] \\
&\quad \times \|F'(x_0)^{-1}F(x_*)\|\|F'(x_*)^{-1}F(x_0)\| \\
&\leq \left[\frac{\int_0^1 \varphi((1-\theta)\|x_0 - x_*\|)d\theta}{1 - \varphi_0(\|x_0 - x_*\|)} \right. \\
&\quad \left. + \frac{3}{4}\frac{(\varphi_0(\|y_0 - x_*\|) + \varphi_0(\|x_0 - x_*\|))\int_0^1 \varphi_1(\theta\|x_0 - x_*\|)d\theta}{(1 - \varphi_0(\|x_0 - x_*\|))(1 - p(\|x_0 - x_*\|))} \right] \\
&\quad \times \|x_0 - x_*\| \\
&\leq \bar{\varphi}_2(\|x_0 - x_*\|)\|x_0 - x_*\| \leq \|x_0 - x_*\| < r, \quad (24)
\end{aligned}
$$

so $z_0 \in U(x_*, r)$, and (15) holds for $n = 0$. Clearly, x_1 is well defined by the third substep of solver (2) for $n = 0$, and (23). Let $A_0 = I - \frac{1}{2}(3F'(y_0) - F'(x_0))^{-1}(3F'(y_0) + F'(x_0))$ and $B_0 = I + \frac{1}{2}(3F'(y_0) - F'(x_0))^{-1}(3F'(y_0) + F'(x_0))$. Then, we have in turn that

$$
\begin{aligned}
\|A_0\| &= \|(3F'(y_0) - F'(x_0))^{-1}[3F'(y_0) - F'(x_0) - \frac{1}{2}(3F'(y_0) + F'(x_0))]\| \\
&= \frac{3}{2}\|[(3F'(y_0) - F'(x_0))^{-1}F'(x_*)] \\
&\quad \times [F'(x_*)^{-1}(F'(y_0) - F'(x_*)) + F'(x_*)^{-1}(F'(x_*) - F'(x_0))]\| \\
&\leq \frac{3}{4}\frac{\varphi_0(\|y_0 - x_*\|) + \varphi_0(\|x_0 - x_*\|)}{1 - p(\|x_0 - x_*\|)} \\
&\leq a(\|x_0 - x_*\|). \quad (25)
\end{aligned}
$$

Similarly, we get

$$
\begin{aligned}
\|B_0\| &= \|(3F'(y_0) - F'(x_0))^{-1}[3F'(y_0) - F'(x_0) + \frac{1}{2}(3F'(y_0) + F'(x_0))] \\
&= \frac{1}{2}\|[(3F'(y_0) - F'(x_0))^{-1}F'(x_*)] \\
&\quad \times [8F'(x_*)^{-1}F(y_0) + F'(x_*)^{-1}(F'(y_0) - F'(x_*)) \\
&\quad + F'(x_*)^{-1}(F'(x_*) - F'(x_0))]\| \\
&\leq \frac{8\varphi_1(\|y_0 - x_*\|) + \varphi_0(\|x_0 - x_*\|) + \varphi_0(\|y_0 - x_*\|)}{4(1 - p(\|x_0 - x_*\|))} \\
&\leq b(\|x_0 - x_*\|). \quad (26)
\end{aligned}
$$

Moreover, by (5), (11) (for $m = 3$), (18), (23)-(25), and the third substep of solver (2) for $n = 0$, we get in

turn that

$$
\begin{aligned}
\|x_1 - x_*\| &= \|(z_0 - x_* - F'(z_0)^{-1}F(z_0)) + F'(z_0)^{-1}F(z_0) \\
&\quad + \Big\{ -I + [I - (\tfrac{1}{2}(3F'(y_0) - F'(x_0))^{-1}(3F'(y_0) + F'(x_0)))^2 \Big\} \\
&\quad \times F'(x_0)^{-1}F(z_0)\| \\
&= \|(z_0 - x_* - F'(z_0)^{-1}F(z_0)) + F'(z_0)^{-1}[(F'(x_0) - F'(x_*)) \\
&\quad + (F'(x_*) - F''(z_0))]F(z_0) + A_0 B_0 F'(x_0)^{-1}F(z_0)\| \\
&\leq \Bigg\{ \frac{\int_0^1 \varphi((1-\theta)\|z_0 - x_*\|)d\theta}{1 - \varphi_0(\|z_0 - x_*\|)} \\
&\quad + \frac{(\varphi_0(\|z_0 - x_*\|) + \varphi_0(\|x_0 - x_*\|)) \int_0^1 \varphi_1(\theta\|z_0 - x_*\|)d\theta}{(1 - \varphi_0(\|z_0 - x_*\|))(1 - \varphi_0(\|x_0 - x_*\|))} \\
&\quad + \frac{a(\|x_0 - x_*\|)b(\|x_0 - x_*\|) \int_0^1 \varphi_1(\theta\|z_0 - x_*\|)d\theta}{1 - \varphi_0(\|z_0 - x_*\|)} \Bigg\} \|z_0 - x_*\| \\
&\leq \bar{\varphi}_2(\|x_0 - x_*\|)\|x_0 - x_*\| \leq \|x_0 - x_*\|,
\end{aligned}
\tag{27}
$$

so $x_1 \in U(x_*, r)$, and (16) holds for $n = 0$. The induction for (14)-(16) is completed by simply substituting x_0, y_0, z_0, x_1 by x_k, y_k, z_k, x_{k+1}, respectively in the preceding estimations. Furthermore, from the estimation

$$
\|x_{k+1} - x_k\| \leq c\|x_k - x_*\| < r, \; c = \bar{\varphi}_3(\|x_0 - x_*\|) \in [0,1),
\tag{28}
$$

we deduce that $\lim_{k\to\infty} x_k = x_*$, and $x_{k+1} \in U(x_*, r)$. Set $T = \int_0^1 F'(y_* + \theta(x_* - y_*))d\theta$ for $y_* \in \Omega_1$ with $F(y_*) = 0$. Finally, for (a2) and (a5), we obtain that

$$
\begin{aligned}
\|F'(x_*)^{-1}(T - F'(x_*))\| &\leq \int_0^1 \varphi_0(\theta\|x_* - y_*\|)d\theta \\
&\leq \int_0^1 \varphi_0(\theta\bar{r})d\theta < 1,
\end{aligned}
$$

so $x^* = y^*$, since $T^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$, and $0 = F(y_*) - F(x_*) = T(y_* - x_*)$. $\qquad\square$

Remark 2.2.   *1. By (a2), and the estimate*

$$
\begin{aligned}
\|F'(x^*)^{-1}F'(x)\| &= \|F'(x^*)^{-1}(F'(x) - F'(x^*)) + I\| \\
&\leq 1 + \|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \leq 1 + \varphi_0(\|x - x^*\|)
\end{aligned}
$$

second condition in (a3) can be dropped, and φ_1 be defined as

$$
\varphi_(t) = 1 + \varphi_0(t).
$$

2. *The results obtained here can be used for operators F satisfying autonomous differential equations [2, 3, 4] of the form*

$$
F'(x) = T(F(x))
$$

where T is a continuous operator. Then, since $F'(x^*) = T(F(x^*)) = T(0)$, we can apply the results without actually knowing x^*. For example, let $F(x) = e^x - 1$. Then, we can choose: $T(x) = x + 1$.

3. *The local results obtained here can be used for projection solvers such as the Arnoldi's solver, the generalized minimum residual solver (GMRES), the generalized conjugate solver(GCR) for combined Newton/finite projection solvers and in connection to the mesh independence principle can be used to develop the cheapest and most efficient mesh refinement strategies [2, 3, 4].*

4. Let $\varphi_0(t) = L_0 t$, and $\varphi(t) = Lt$. The parameter $r_A = \frac{2}{2L_0+L}$ was shown by us to be the convergence radius of Newton's solver [2]

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n) \text{ for each } n = 0, 1, 2, \cdots \tag{29}$$

under the conditions (a1)-(a3) (φ_1 is not used). It follows that the convergence radius r of solver (2) cannot be larger than the convergence radius r_A of the second order Newton's solver (29). As already noted in [2, 3, 4] r_A is at least as large as the convergence ball given by Rheinboldt [11]

$$r_{TR} = \frac{2}{3L_1},$$

where L_1 is the Lipschitz constant on Ω, $L_0 \leq L_1$ and $L \leq L_1$. In particular, for $L_0 < L_1$ or $L < L_1$, we have that

$$r_{TR} < r_A$$

and

$$\frac{r_{TR}}{r_A} \to \frac{1}{3} \ as \ \frac{L_0}{L_1} \to 0.$$

That is our convergence ball r_A is at most three times larger than Rheinboldt's. The same value for r_{TR} was given by Traub [14].

5. It is worth noticing that solver (2) is not changing, when we use the conditions (A) of Theorem 2.1 instead of the stronger conditions used in [13]. Moreover, we can compute the computational order of convergence (COC) defined by

$$\xi = \ln\left(\frac{\|x_{n+1} - x^*\|}{\|x_n - x^*\|}\right) / \ln\left(\frac{\|x_n - x^*\|}{\|x_{n-1} - x^*\|}\right)$$

or the approximate computational order of convergence

$$\xi_1 = \ln\left(\frac{\|x_{n+1} - x_n\|}{\|x_n - x_{n-1}\|}\right) / \ln\left(\frac{\|x_n - x_{n-1}\|}{\|x_{n-1} - x_{n-2}\|}\right).$$

This way we obtain in practice the order of convergence in a way that avoids the bounds given in [13] involving estimates up to the seventh Fréchet derivative of operator F.

3. Local convergence of solver (3)

The local convergence analysis of solver (3) is analogous to the corresponding one in Theorem 2.1 but some of the majorizing functions $\bar{\varphi}_m$ change to $\bar{\psi}_m$ defined below. Let $\varphi_0, \rho_0, \varphi, \varphi_1, r_1$ be as in Theorem 2.1, $\bar{\psi}_1 = \bar{\varphi}_1$ and $\bar{\bar{\psi}}_1 = \bar{\bar{\varphi}}_1$. Assume equation

$$\varphi_0(\bar{\psi}_1(t)t) = 1 \tag{1}$$

has a minimal positive solution ρ_1. Set $\rho = \min\{\rho_0, \rho_1\}$, and $S_0 = [0, \rho)$. Define functions $\bar{\psi}_2$ and $\bar{\bar{\psi}}_2$ on the interval by

$$\bar{\psi}_2(t) = \left\{ \frac{\int_0^1 \varphi((1-\theta)t)d\theta}{1 - \varphi_0(t)} \right.$$

$$\left. + \frac{3}{8}\frac{(\varphi_0(\bar{\psi}_1(t)t) + \varphi_0(t))(\varphi_1(\bar{\psi}_1(t)t) + \varphi_1(t))\int_0^1 \varphi_1(\theta t)d\theta}{(1 - \varphi_0(\psi_1(t)t))^2(1 - \varphi_0(t))} \right\}$$

and

$$\bar{\bar{\psi}}_2(t) = \bar{\psi}_2(t) - 1.$$

We have by these definition that $\bar{\bar{\psi}}_2(0) = -1$, and $\bar{\bar{\psi}}_2(t) \longrightarrow \infty$ as $t \longrightarrow \rho^-$. Denote by R_2 the minimal solution of equation $\bar{\bar{\psi}}_2(t) = 0$ in $(0, \rho)$.

Assume equation

$$\psi_0(\bar{\varphi}_2(t)t) = 1 \tag{2}$$

has a minimal solution ρ_2 in $(0, \rho)$. Set $S = [0, \rho_3)$, $\rho_3 = \min\{\rho, \rho_2\}$. Define functions $\bar{\psi}_3$ and $\bar{\bar{\psi}}_3$ on the interval S by

$$\bar{\psi}_3(t) \;=\; \left\{ \frac{\int_0^1 \varphi((1-\theta)\bar{\psi}_2(t)t)d\theta}{1 - \varphi_0(\bar{\psi}_2(t)t)} \right.$$
$$+ \frac{(\varphi_0(\bar{\psi}_2(t)t) + \varphi_0(\bar{\psi}_1(t)t))\int_0^1 \varphi_1(\theta\bar{\psi}_2(t)t)d\theta \int_0^1 \varphi_1(\theta\bar{\psi}_1(t)t)d\theta}{(1 - \varphi_0(\bar{\psi}_2(t)t))(1 - \varphi_0(\bar{\psi}_1(t)t))}$$
$$+ \left[\frac{15}{8} \frac{(\varphi_0(\bar{\psi}_1(t)t) + \varphi_0(t))}{1 - \varphi_0(\bar{\psi}_2(t)t)} + \frac{11}{8} \frac{(\varphi_0(\bar{\psi}_1(t)t) + \varphi_0(t))}{1 - \varphi_0(t)} \right.$$
$$\left. \left. \times \frac{\int_0^1 \varphi_1(\theta\bar{\psi}_2(t)t)d\theta}{1 - \varphi_0(\bar{\psi}_2(t)t)} \right] \right\} \bar{\psi}_2(t)$$

and

$$\bar{\bar{\psi}}_3(t) = \bar{\psi}_3(t) - 1.$$

We have again by these definitions $\bar{\bar{\psi}}_3(0) = -1$, and $\bar{\bar{\psi}}_3(t) \longrightarrow \infty$ as $t \longrightarrow \rho_3^-$. Denote by R_3 the minimal solution of equation $\bar{\bar{\psi}}_3(t) = 0$ in $(0, \rho_3)$. Define a radius of convergence R by

$$R = \min\{r_1, R_2, R_3\}. \tag{3}$$

The local convergence analysis uses conditions (H)

(h1) = (a1), (h2) = (a2), (h3) = (a3)
(h4) $\bar{U}(x_*, R) \subseteq \Omega$, ρ_0, ρ_1, ρ_2 exist and are given in (1), (1), (2), respectively and (2) holds.
(h5) There exists $\bar{R} \geq R$ such that $\int_0^1 \varphi_0(\theta\bar{R})d\theta < 1$.
    Set $\Omega_2 = \Omega \cap \bar{U}(x_*, \bar{R})$.

Theorem 3.1. *Under the hypotheses (H) further suppose $x_0 \in U(x_*, R) - \{x_*\}$. Then, the following assertions hold for solver (3)*

$$\{x_n\} \subset U(x_*, R),$$

$$\lim_{n \longrightarrow \infty} x_n = x_*,$$

$$\|y_n - x_*\| \leq \bar{\psi}_1(\|x_n - x_*\|)\|x_n - x_*\| \leq \|x_n - x_*\| < R, \tag{4}$$

$$\|z_n - x_*\| \leq \bar{\psi}_2(\|x_n - x_*\|)\|x_n - x_*\| \leq \|x_n - x_*\|, \tag{5}$$

$$\|x_{n+1} - x_*\| \leq \bar{\psi}_3(\|x_n - x_*\|)\|x_n - x_*\| \leq \|x_n - x_*\|, \tag{6}$$

and x_* is the only solution of equation $F(x) = 0$ in the set Ω_2 given in (h5), where the function $\bar{\psi}_m$ are defined previously.

Proof. As in Theorem 2.1 but using instead the identities

$$z_n - x_* \;=\; [(x_n - x_* - F'(x_n)^{-1}F(x_n)) + \frac{3}{8}[I - (F'(y_n)^{-1}F(x_n))^2]F'(x_n)^{-1}F(x_n)$$
$$=\; (x_n - x_* - F'(x_n)^{-1}F(x_n)) + \frac{3}{8}F'(y_n)^{-1}(F'(y_n) - F'(x_n))$$
$$\times F'(y_n)^{-1}(F'(y_n) + F'(x_n))F'(x_n)^{-1}F(x_n),$$

and

$$
\begin{aligned}
x_{n+1} - x_* &= (z_n - x_* - F'(z_n)^{-1}F(z_n)) + F'(z_n)^{-1}(F'(y_n) - F'(z_n))F'(y_n)^{-1}F(z_n) \\
&\quad + [\frac{26}{8}I - \frac{15}{8}F'(y_n)^{-1}F'(x_n) - \frac{11}{8}F'(x_n)^{-1}F'(y_n)]F'(y_n)^{-1}F(z_n) \\
&= (z_n - x_* - F'(z_n)^{-1}F(z_n)) + F'(z_n)^{-1}(F'(y_n) - F'(x_n))F'(y_n)^{-1}F(z_n) \\
&\quad + \frac{15}{8}F'(y_n)^{-1}(F'(y_n) - F'(x_n)) + \frac{11}{8}F'(x_n)^{-1}(F'(x_n) - F'(y_n))]F'(y_n)^{-1}F(z_n).
\end{aligned}
$$

Notice that (4) is (14). Then, we have respectively for (5) and (6) that

$$
\begin{aligned}
\|z_n - x_*\| &\leq \Bigg\{ \frac{\int_0^1 \varphi((1-\theta)\|x_n - x_*\|)d\theta}{1 - \varphi_0(\|x_n - x_*\|)} \\
&\quad + \frac{3}{8}\frac{(\varphi_0(\|y_n - x_*\|) + \varphi_0(\|x_n - x_*\|))(\varphi_1(\|y_n - x_*\|) + \varphi_1(\|x_n - x_*\|))}{(1 - \varphi_0(\|y_n - x_*\|))^2(1 - \varphi_0(\|x_n - x_*\|))}\Bigg\} \\
&\quad \times \int_0^1 \varphi_1(\theta\|x_n - x_*\|)d\theta\|x_n - x_*\| \\
&\leq \bar\psi_2(\|x_n - x_*\|)\|x_n - x_*\| \leq \|x_n - x_*\|
\end{aligned}
$$

and

$$
\begin{aligned}
\|x_{n+1} - x_*\| &\leq \Bigg\{ \frac{\int_0^1 \varphi((1-\theta)\|z_n - x_*\|)d\theta}{1 - \varphi_0(\|z_n - x_*\|)} \\
&\quad + \frac{(\varphi_0(\|z_n - x_*\|) + \varphi_0(\|y_n - x_*\|))\int_0^1 \varphi_1(\theta\|z_n - x_*\|)d\theta}{(1 - \varphi_0(\|z_n - x_*\|))(1 - \varphi_0(\|y_n - x_*\|))} \\
&\quad + \Bigg[\frac{15(\varphi_0(\|y_n - x_*\|) + \varphi_0(\|x_n - x_*\|)}{8(1 - \varphi_0(\|y_n - x_*\|))} \\
&\quad + \frac{11}{8}\frac{(\varphi_0(\|y_n - x_*\|) + \varphi_0(\|x_n - x_*\|))}{1 - \varphi_0(\|z_n - x_*\|)}\Bigg] \\
&\quad \times \frac{\int_0^1 \varphi_1(\theta\|z_n - x_*\|)d\theta}{1 - \varphi_0(\|z_n - x_*\|)}\Bigg\} \|z_n - x_*\| \\
&\leq \bar\psi_3(\|x_0 - x_*\|)\|x_0 - x_*\| \leq \|x_0 - x_*\|.
\end{aligned}
$$

$\square$

4. Numerical Examples

Example 4.1. *Let $\mathcal{X} = \mathcal{Y} = C[0,1]$, the space of continuous functions defined on $[0,1]$ with the max norm. Let $\Omega = \overline{U}(0,1)$. Define function F on Ω by*

$$
F(\varphi)(x) = \varphi(x) - 5\int_0^1 x\theta\varphi(\theta)^3 d\theta. \tag{1}
$$

We have that

$$
F'(\varphi(\xi))(x) = \xi(x) - 15\int_0^1 x\theta\varphi(\theta)^2\xi(\theta)d\theta, \text{ for each } \xi \in \Omega.
$$

Then, we get that $x_* = 0$, $\varphi_0(t) = \frac{15}{2}t, \varphi(t) = \varphi_1(t) = 15t$. This way, we have that

$$
r_1 = 0.0571429, r_2 = 0.04803, r_3 = 0.0460721, R_2 = 0.0497658, R_3 = 0.0420623.
$$

Example 4.2. *Let $\mathcal{X} = \mathcal{Y} = \mathbb{R}^3$, $\Omega = U(0,1)$, $x_* = (0,0,0)^T$ and define F on Ω by*

$$F(x) = F(u_1, u_2, u_3) = (e^{u_1} - 1, \frac{e-1}{2}u_2{}^2 + u_2, u_3)^T. \tag{2}$$

For the points $u = (u_1, u_2, u_3)^T$, the Fréchet derivative is given by

$$F'(u) = \begin{pmatrix} e^{u_1} & 0 & 0 \\ 0 & (e-1)u_2 + 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Using the norm of the maximum of the rows and since $F'(x_*) = diag(1,1,1)$, we get by conditions (A) $\varphi_0(t) = (e-1)t$, $\varphi(t) = e^{\frac{1}{e-1}}t$, and $\varphi_1(t) = e^{\frac{1}{e-1}}t$.

$$r_1 = 0.343486, \ r_2 = 0.278341, r_3 = 0.271076, R_2 = 299594, R_3 = 0.251117.$$

Example 4.3. *Returning back to the motivational example at the introduction of this study, we have $\varphi_0(t) = \varphi(t) = a(t) = 96.662907t$, and $\varphi_1(t) = 1.0631$. Then, we have $r_1 = 0.0068884, \ r_2 = 0.00675474, \ r_3 = 0.00673327, R_1 = 0.00689513, R_2 = 0.00514578$.*

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