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Blow Up and Growth of Solutions for A Parabolic Type Kirchhoff Equation with Multiple Nonlinearities

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Abstract

In this paper, we investigate a class of doublynonlinear parabolic Krichhoff-type equations. We give appropriate conditions in order to have nonexistence of global solutions or exponential growth incase of global existence.

Keywords: *Blow up, Exponential growth, Kirchhoff-type, Multiple nonlinearities, Parabolic equation.*
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1. Introduction

Our main interest lies in the following reaction-diffusion equations with multiple nonlinearities

$$\begin{cases} u_t - \Delta u_t - M(\|\nabla u\|^2)\Delta u + |u|^{q-2} u_t = |u|^{p-2} u, & \text{in } \Omega \times (0,T) \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0,T) \\ u(x,0) = u_0(x), & \text{in } \Omega \times (0,T) \end{cases} \tag{1.1}$$

where p, $q > 2$ are real numbers and Ω is a bounded domain in R^n $(n = 1,2,3)$ with smooth boundary $\partial\Omega$ and $0 < T < \infty$.
In the absence of the terms $\triangle u_t$ and $|u|^{q-2} u_t$, equation (1.1) become to the following equation

$$u_t - M(\|\nabla u\|^2)\Delta u = |u|^{p-2} u, \tag{1.2}$$

when $M(s) = a + bs$. Han and Li [1] used potential well method and variational method to the investigation of the long time behaviours of solution for problem (1.2). They obtain global exitence and blow up of solutions when initial energy is supercritical, critical or subcritical. Also Han et al. [2] investigated the upper and lower bounds for the blow-up time and gave a new blow-up criterion for problem (1.2) when the initial energy is positive.
Tuan et al. [3] considered

$$u_t - M(\|\nabla u\|^2)\Delta u = F(x,t,u(x,t)). \tag{1.3}$$

They gave the condition for the existence of the first time backward problem (1.3) and showed that problem (1.3) is ill-posed in the sense of Hadamard. In [4, 5] the authors studied problem (1.3) with initial-boundary conditions and F (nonlinear source) is limited.
Kundu et al. [6] studied the following problem with the initial and boundary conditions of Dirichlet type

$$u_t - (1 + \|\nabla u\|)\Delta u = F(x,t). \tag{1.4}$$

In [7, 8] the authors considered the following nonlinear parabolic equations

$$u_t - M(\|\nabla u\|)\Delta u = F(x,t), \tag{1.5}$$

where M has a nonlinear nonlocal form in u. Elliptic and hiperbolic Kirchhoff equation with initial value problems are investigated in a lot of works, see [9, 10, 11, 12, 13, 14]
This paper is organized as follows: In Section 2, we present some notations and stament of assumptions. In Section 3, the blow up of the solution is given and Section is devoted to show the exponential growth of solution.

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2. Preliminaries

In this section, we shall give some assumptions for the proof of our results. Let $\|.\|$, $\|.\|_p$ and $(u,v)=\int_\Omega u(x)v(x)dx$ denote the usual $L^2(\Omega)$ norm, $L^p(\Omega)$ norm and inner product of $L^2(\Omega)$, respectively.
$M(s)$ is a nonnegative C^1 function for $s\geq 0$ satisfying

$$M(s)=1+s^\gamma,\ \gamma>0.$$

For the numbers p and q, we suppose that

$$\begin{cases} 2<q<p\leq \frac{2(n-1)}{n-2} & \text{if } n\geq 3,\\ 2<q<p<+\infty & \text{if } n=1,2. \end{cases} \tag{2.1}$$

Similar to [15], we call $u(x,t)$ a solution of problem (1.1) on $\Omega\times[0,T)$ if

$$\begin{cases} u\in L^\infty(0,T;H_0^1(\Omega)),\\ u_t\in L^2(0,T;L^2(\Omega)),\\ |u|^{q-2}u_t\in L^2(\Omega\times[0,T)) \end{cases} \tag{2.2}$$

satisfying the initial condition $u(x,0)=u_0(x)$ and

$$(u_t,v)+(\nabla u_t,\nabla v)+\left(\left(1+\left(\int_\Omega|\nabla u|^2dx\right)^\gamma\right)\nabla u,\nabla v\right)+\left(|u|^{q-2}u_t,v\right)=\left(|u|^{p-2}u,v\right), \tag{2.3}$$

for all $v\in C(0,T;H_0^1(\Omega))$.
In this paper, we assume that the problem (1.1) has a unique regular local solution (see [16]).
The energy functional associated with problem (1.1) is

$$E(t)=-\frac{1}{p}\|u\|_p^p+\frac{1}{2}\|\nabla u\|^2+\frac{1}{2(\gamma+1)}\|\nabla u\|^{2(\gamma+1)}, \tag{2.4}$$

where $u\in H_0^1(\Omega)$.
Multiplying the first equation in (1.1) by u_t, integrating over Ω, we have

$$E'(t)=-\|u_t\|^2-\|\nabla u_t\|^2-\int_\Omega|u|^{q-2}u_t^2dx<0, \tag{2.5}$$

and then

$$E(t)\leq E(0). \tag{2.6}$$

3. Blow up of solutions

In this section, we state and prove the blow up result for the problem (1.1).

Theorem 3.1. *Suppose that (2.1) hold and $p>2(\gamma+1)$, $u_0\in H_0^1(\Omega)$ and u is a local solution of the system (1.1), and $E(0)<0$. Then the solution of the system (1.1) blows up in finite time.*

Proof. Let

$$H(t)=-E(t). \tag{3.1}$$

By the definition of $H(t)$ and (2.5)

$$H'(t)=-E'(t)\geq 0. \tag{3.2}$$

Consequently, by $E(0)<0$, we have

$$H(0)=-E(0)>0. \tag{3.3}$$

By integrating (3.2) on $[0,t]$, we obtain

$$0<H(0)\leq H(t). \tag{3.4}$$

From (2.4) and (3.1)

$$H(t)-\frac{1}{p}\|u\|_p^p=-\frac{1}{2}\|\nabla u\|^2-\frac{1}{2(\gamma+1)}\|\nabla u\|^{2(\gamma+1)}<0, \tag{3.5}$$

then, we have

$$0<H(0)\leq H(t)\leq\frac{1}{p}\|u\|_p^p. \tag{3.6}$$

We define

$$\Psi(t)=H^{1-\sigma}(t)+\frac{\varepsilon}{2}\|u\|^2+\frac{\varepsilon}{2}\|\nabla u\|^2, \tag{3.7}$$

where $\varepsilon>0$ is a small parameter to be chosen later and $0<\sigma<(p-q)/p$.
Taking the derivative of (3.7) and using equation (1.1), we get

$$\begin{aligned}\Psi'(t) &= (1-\sigma)H^{-\sigma}(t)H'(t)+\varepsilon\left(\int_\Omega uu_t dx+\int_\Omega \nabla u\nabla u_t dx\right)\\ &= (1-\sigma)H^{-\sigma}(t)H'(t)+2(\gamma+1)\varepsilon H(t)+2(\gamma+1)\varepsilon E(t)\\ &\quad -\varepsilon\|\nabla u\|^2-\varepsilon\|\nabla u\|^{2(\gamma+1)}+\varepsilon\|u\|_p^p-\varepsilon\int_\Omega |u|^{q-2}uu_t dx\\ &= (1-\sigma)H^{-\sigma}(t)H'(t)+2(\gamma+1)\varepsilon H(t)\\ &\quad +\varepsilon\left(1-\frac{2(\gamma+1)}{p}\right)\|u\|_p^p+\varepsilon\gamma\|\nabla u\|^2-\varepsilon\int_\Omega |u|^{q-2}uu_t dx.\end{aligned} \tag{3.8}$$

We just need to estimate the last term of the right-hand terms of (3.8).
By using the following Young's inequality

$$XY\le \delta^{-1}X^2+\delta Y^2, \tag{3.9}$$

for $\delta>0$, with $X=|u|^{\frac{q-2}{2}}u_t$ and $Y=|u|^{\frac{q-2}{2}}u$, we find

$$\begin{aligned}\int_\Omega |u|^{q-2}uu_t dx &\le \int_\Omega |u|^{\frac{q-2}{2}}u_t|u|^{\frac{q-2}{2}}udx\\ &\le \delta^{-1}\int_\Omega |u|^{q-2}u_t^2dx+\delta\int_\Omega |u|^q dx.\end{aligned} \tag{3.10}$$

and therefore, (3.8) becomes

$$\begin{aligned}\Psi'(t) &\ge (1-\sigma)H^{-\sigma}(t)H'(t)+2(\gamma+1)\varepsilon H(t)+\varepsilon\left(1-\frac{2(\gamma+1)}{p}\right)\|u\|_p^p\\ &\quad +\varepsilon\gamma\|\nabla u\|^2-\varepsilon\delta\|u\|_q^q-\varepsilon\delta^{-1}\int_\Omega |u|^{q-2}u_t^2dx.\end{aligned} \tag{3.11}$$

By taking δ such that $\delta^{-1}=\lambda H^{-\sigma}(t)$ for λ enough large constants to be fixed later, and by using (2.5), we have

$$\begin{aligned}\Psi'(t) &\ge (1-\sigma-\lambda\varepsilon)H^{-\sigma}(t)H'(t)+\varepsilon\gamma\|\nabla u\|^2+2(\gamma+1)\varepsilon H(t)\\ &\quad +\varepsilon\left(1-\frac{2(\gamma+1)}{p}\right)\|u\|_p^p-\varepsilon\lambda^{-1}H^{\sigma}(t)\|u\|_q^q.\end{aligned} \tag{3.12}$$

By the embeddings $L^p(\Omega)\hookrightarrow L^q(\Omega)\hookrightarrow L^2(\Omega)$ (since $p>q>2$), taking into account (3.6), we obtain

$$\begin{aligned}H^{\sigma}(t)\|u\|_q^q &\le c_1\|u\|_p^{p\sigma}\|u\|_q^q\\ &\le c_2\|u\|_p^{p\sigma+q},\end{aligned} \tag{3.13}$$

where c_1 and c_2 are positive constants.
Since $0<\frac{q}{p}<1$, now applying the following algebraic inequality

$$x^l\le (x+1)\le (1+\frac{1}{z})(x+z),\quad \forall x\ge 0,\ 0\le l\le 1,\ z>0, \tag{3.14}$$

especially, by the selection of σ, taking $x=\|u\|_p^p$, $l=(p\sigma+q)/p$, $z=H(0)$, and by using (3.6), we get

$$\begin{aligned}\|u\|_p^{p\sigma+q} &\le (1+\frac{1}{H(0)})(\|u\|_p^p+H(0))\\ &\le c_3\|u\|_p^p.\end{aligned} \tag{3.15}$$

Taking into account (3.12) and (3.15), we have

$$\begin{aligned}\Psi'(t) &\ge (1-\sigma-\lambda\varepsilon)H^{-\sigma}(t)H'(t)+\varepsilon\gamma\|\nabla u\|^2+2(\gamma+1)\varepsilon H(t)\\ &\quad +\varepsilon(1-\frac{2(\gamma+1)}{p}-c_3\lambda^{-1})\|u\|_p^p.\end{aligned} \tag{3.16}$$

For large λ such that $1-\frac{2(\gamma+1)}{p}-c_3\lambda^{-1}=c_4>0$, once λ is fixed, we choose ε small enough such that $1-\sigma-\lambda\varepsilon>0$, then there exist $c_5>0$ such that (3.16) become

$$\Psi'(t)\ge c_5\left(H(t)+\|u\|_p^p+\|\nabla u\|^2\right). \tag{3.17}$$

Then we have

$$\Psi(t)\ge\Psi(0)>0,\ \forall t\ge 0. \tag{3.18}$$

We now estimate $\Psi(t)^{\frac{1}{1-\sigma}}$. By using (3.7) and (3.5), we obtain

$$\begin{aligned}\Psi(t) &= H^{1-\sigma}(t)-\varepsilon\left(\frac{1}{2(\gamma+1)}\|\nabla u\|^{2(\gamma+1)}+H(t)-\frac{1}{p}\|u\|_p^p\right)+\frac{\varepsilon}{2}\|u\|^2\\ &\leq (1-\varepsilon)H^{1-\sigma}(t)+\frac{\varepsilon}{p}\|u\|_p^p-\frac{\varepsilon}{2(\gamma+1)}\|\nabla u\|^{2(\gamma+1)}+\frac{\varepsilon}{2}\|u\|^2,\end{aligned}$$

where we have used the fact that $H(t)\geq H^{1-\sigma}(t)$ (this can be ensured by (3.3), (3.4) and $0<\sigma<1$). By Poincare's inequality

$$\Psi(t)\leq(1-\varepsilon)H^{1-\sigma}(t)+\frac{\varepsilon}{p}\|u\|_p^p+\frac{\varepsilon}{2}\|\nabla u\|^2.$$

Now, by using algebraic inequality (3.14) for $x=\|u\|_p^{p/(1-\sigma)}$, $l=1-\sigma<1$, $z=H^{1/(1-\sigma)}(0)$, we have

$$\begin{aligned}\|u\|_p^p &\leq (1+\frac{1}{H^{1/(1-\sigma)}(0)})\left(\|u\|_p^{p/(1-\sigma)}+H^{1/(1-\sigma)}(0)\right)\\ &\leq C\|u\|_p^{p/(1-\sigma)}.\end{aligned}\tag{3.19}$$

Also, again using algebraic inequality (3.14) for $x=\|\nabla u\|_2^{2/(1-\sigma)}$, $l=1-\sigma<1$, $z=H^{1/(1-\sigma)}(0)$, we get

$$\begin{aligned}\|\nabla u\|_2^2 &\leq (1+\frac{1}{H^{1/(1-\sigma)}(0)})\left(\|\nabla u\|_2^{2/(1-\sigma)}+H^{1/(1-\sigma)}(0)\right)\\ &\leq C\|\nabla u\|_2^{2/(1-\sigma)}.\end{aligned}$$

Thus

$$\Psi^{\frac{1}{1-\sigma}}(t)\leq C\left[H(t)+\|u\|_p^p+\|\nabla u\|^2\right].\tag{3.20}$$

By combining of (3.17) and (3.20) we arrive at

$$\Psi'(t)\geq\xi\Psi^{\frac{1}{1-\sigma}}(t),\tag{3.21}$$

where $\xi>0$ is a constant.
A simple integration of (3.21) over $(0,t)$ yields

$$\Psi^{\frac{\sigma}{1-\sigma}}(t)\geq\frac{1}{\Psi^{-\frac{\sigma}{1-\sigma}}(0)-\frac{\xi\sigma t}{1-\sigma}}.\tag{3.22}$$

The estimate (3.22) shows that $\Psi(t)$ blows up in time

$$T^*\leq\frac{1-\sigma}{\xi\sigma\Psi^{\frac{\sigma}{1-\sigma}}(0)}.$$

□

4. Exponential growth of solutions

In this section, we state and prove exponential growth result. Throughout this proof, C is used to point out general positive constant.

Theorem 4.1. *Suppose that (2.1) hold and $p>4(\gamma+1)$, $u_0\in H_0^1(\Omega)$ and u is a solution of the system (1.1), $E(0)<0$. Then the solution of the system (1.1) grows exponentially.*

Proof. We define

$$G(t)=H(t)+\frac{\varepsilon}{2}\|u\|^2+\frac{\varepsilon}{2}\|\nabla u\|^2,\tag{4.1}$$

where $H(t)=-E(t)$. By taking the time derivative of (4.1) and by (1.1), we have

$$\begin{aligned}G'(t) &= H'(t)+\varepsilon\left(\int_\Omega uu_t dx+\int_\Omega\nabla u\nabla u_t dx\right)\\ &= \|\nabla u_t\|^2+\|u_t\|^2+\int_\Omega|u|^{q-2}u_t^2dx\\ &\quad -\varepsilon\|\nabla u\|^2-\varepsilon\|\nabla u\|^{2(\gamma+1)}+\varepsilon\|u\|_p^p-\varepsilon\int_\Omega|u|^{q-2}uu_tdx\\ &\geq \|u_t\|^2+\int_\Omega|u|^{q-2}u_t^2dx+\varepsilon\|u\|_p^p-\varepsilon\|\nabla u\|^2\\ &\quad -\varepsilon\|\nabla u\|^{2(\gamma+1)}-\varepsilon\int_\Omega|u|^{q-2}uu_tdx.\end{aligned}\tag{4.2}$$

We just need to estimate the last term of the right-hand terms of (4.2). Applying inequality (3.9), we find

$$\begin{aligned}\int_\Omega |u|^{q-2} uu_t dx &\leq \int_\Omega |u|^{\frac{q-2}{2}} u_t |u|^{\frac{q-2}{2}} udx \\ &\leq \delta^{-1} \int_\Omega |u|^{q-2} u_t^2 dx + \delta \int_\Omega |u|^q dx.\end{aligned}$$

Therefore, we have

$$\begin{aligned}G^{'}(t) \geq\ & \|u_t\|^2 - \varepsilon \|\nabla u\|^2 - \varepsilon \|\nabla u\|^{2(\gamma+1)} \\ &+\varepsilon \|u\|_p^p - \varepsilon\delta \|u\|_q^q + \left(1-\varepsilon\delta^{-1}\right) \int_\Omega |u|^{q-2} u_t^2 dx. \end{aligned} \tag{4.3}$$

By using

$$\|u\|_p^p = pH(t) + \frac{p}{2} \|\nabla u\|^2 + \frac{p}{2(\gamma+1)} \|\nabla u\|^{2(\gamma+1)}.$$

Hence, (4.3) becomes

$$\begin{aligned}G'(t) \geq\ & \|u_t\|^2 - \varepsilon \|\nabla u\|^2 - \varepsilon \|\nabla u\|^{2(\gamma+1)} \\ &+\varepsilon \left[pH(t) + \frac{p}{2} \|\nabla u\|^2 + \frac{p}{2(\gamma+1)} \|\nabla u\|^{2(\gamma+1)}\right] \\ &-\varepsilon\delta \|u\|_q^q + \left(1-\varepsilon\delta^{-1}\right) \int_\Omega |u|^{q-2} u_t^2 dx \\ \geq\ & \|u_t\|^2 + \left(1-\varepsilon\delta^{-1}\right) \int_\Omega |u|^{q-2} u_t^2 dx \\ &+\varepsilon a_1 \|\nabla u\|^{2(\gamma+1)} + \varepsilon b_1 \|\nabla u\|^2 + \varepsilon pH(t) - \varepsilon\delta \|u\|_q^q \end{aligned} \tag{4.4}$$

where $a_1 = \frac{p}{2(\gamma+1)} - 1 > 0$ and $b_1 = \frac{p}{2} - 1 > 0$.
Thanks to the embedding $L^p \hookrightarrow L^q$, $p > q$

$$\begin{aligned}\|u\|_q^q &\leq C \|u\|_p^q \\ &\leq C \left(\|u\|_p^p\right)^{\frac{q}{p}}. \end{aligned} \tag{4.5}$$

Since $0 < \frac{q}{p} < 1$, now applying algebraic inequality (3.14), in particular, taking $x = \|u\|_p^p$, $l = q/p$, $z = H(0)$, we get

$$\left(\|u\|_p^p\right)^{\frac{q}{p}} \leq (1 + \frac{1}{H(0)})(\|u\|_p^p + H(0)),$$

then from (4.5) and (3.6), we obtain

$$\begin{aligned}\|u\|_q^q &\leq C \|u\|_p^q \\ &\leq C_1 \|u\|_p^p. \end{aligned} \tag{4.6}$$

So, we have

$$\begin{aligned}G'(t) \geq\ & \|u_t\|^2 + \left(1-\varepsilon\delta^{-1}\right) \int_\Omega |u|^{q-2} u_t^2 dx + \varepsilon a_1 \|\nabla u\|^{2(\gamma+1)} \\ &+\varepsilon b_1 \|\nabla u\|^2 + \varepsilon pH(t) - \varepsilon\delta C_1 \|u\|_p^p. \end{aligned} \tag{4.7}$$

Taking

$$\begin{aligned}2(\gamma+1)a_2 &= a_1 > 0, \\ 2b_2 &= b_1 > 0,\end{aligned}$$

and noting that

$$\begin{aligned}2(\gamma+1)H(t) &= \frac{2(\gamma+1)}{p} \|u\|_p^p - \|\nabla u\|^{2(\gamma+1)} - (\gamma+1) \|\nabla u\|^2, \\ 2H(t) &= \frac{2}{p} \|u\|_p^p - \frac{1}{\gamma+1} \|\nabla u\|^{2(\gamma+1)} - \|\nabla u\|^2,\end{aligned}$$

we get

$$
\begin{aligned}
G'(t) &\geq \|u_t\|^2+\left(1-\varepsilon\delta^{-1}\right)\int_\Omega |u|^{q-2}u_t^2dx+\varepsilon(a_1-a_2)\|\nabla u\|^{2(\gamma+1)}+\varepsilon a_2\|\nabla u\|^{2(\gamma+1)} \\
&\quad +\varepsilon(b_1-b_2)\|\nabla u\|^2+\varepsilon b_2\|\nabla u\|^2+\varepsilon pH(t)-\varepsilon\delta C_1\|u\|_p^p \\
&\geq \|u_t\|^2+\left(1-\varepsilon\delta^{-1}\right)\int_\Omega |u|^{q-2}u_t^2dx+\varepsilon(a_1-a_2)\|\nabla u\|^{2(\gamma+1)} \\
&\quad +\varepsilon(b_1-b_2)\|\nabla u\|^2+\varepsilon pH(t)+\varepsilon a_2\left[\|\nabla u\|^{2(\gamma+1)}-\frac{2(\gamma+1)}{p}\|u\|_p^p+(\gamma+1)\|\nabla u\|^2\right] \\
&\quad +\varepsilon a_2\left[\frac{2(\gamma+1)}{p}\|u\|_p^p-(\gamma+1)\|\nabla u\|^2\right] \\
&\quad +\varepsilon b_2\left[\|\nabla u\|^2-\frac{2}{p}\|u\|_p^p+\frac{1}{\gamma+1}\|\nabla u\|^{2(\gamma+1)}\right] \\
&\quad +\varepsilon b_2\left[\frac{2}{p}\|u\|_p^p-\frac{1}{\gamma+1}\|\nabla u\|^{2(\gamma+1)}\right]-\varepsilon\delta C_1\|u\|_p^p \\
&= \|u_t\|^2+\left(1-\varepsilon\delta^{-1}\right)\int_\Omega |u|^{q-2}u_t^2dx+\varepsilon\left[(2\gamma+1)a_2-\frac{b_2}{\gamma+1}\right]\|\nabla u\|^{2(\gamma+1)} \\
&\quad +\varepsilon\left[b_2-a_2(\gamma+1)\right]\|\nabla u\|^2 \\
&\quad +\varepsilon(p-2(\gamma+1)a_2-2b_2)H(t)+\varepsilon\left(\frac{2(\gamma+1)}{p}a_2+\frac{2}{p}b_2-\delta C_1\right)\|u\|_p^p. \qquad (4.8)
\end{aligned}
$$

Taking δ small enough such that $\frac{2(\gamma+1)}{p}a_2+\frac{2}{p}b_2-\delta C_1>0$, then taking ε small enough such that $1-\varepsilon\delta^{-1}>0$, and $(2\gamma+1)a_2-\frac{b_2}{\gamma+1}>0$, $b_2-a_2(\gamma+1)>0$ and noting that

$$
\begin{aligned}
p-2(\gamma+1)a_2-2b_2 &= p-a_1-b_1 \\
&= \frac{p\gamma}{2(\gamma+1)}+2>0,
\end{aligned}
$$

then

$$G'(t)\geq C\left(\|u_t\|^2+\|\nabla u\|^{2(\gamma+1)}+\|\nabla u\|^2+\|u\|_p^p+H(t)\right). \qquad (4.9)$$

Thus, the functional $G(t)$ is strictly positive and increasing for all $t\geq 0$.
Conversely, from $G(t)$ function we obtain

$$
\begin{aligned}
G(t) &= H(t)+\frac{\varepsilon}{2}\|u\|^2+\frac{\varepsilon}{2}\|\nabla u\|^2 \\
&\leq C(\|\nabla u\|^2+H(t)) \\
&\leq C(\|\nabla u\|^2+\|\nabla u\|^{2(\gamma+1)}+\|u_t\|^2+\|u\|_p^p+H(t)). \qquad (4.10)
\end{aligned}
$$

From (4.10) and (4.9) we arrive at

$$G'(t)\geq rG(t) \qquad (4.11)$$

where $r>0$ is a constant.
Integration of (4.11) over $(0,t)$ gives us

$$G(t)\geq G(0)\exp(rt).$$

□

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