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Degree distance and Gutman index of two graph products

Research Article

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Abstract: The degree distance was introduced by Dobrynin, Kochetova and Gutman as a weighted version of the Wiener index. In this paper, we investigate the degree distance and Gutman index of complete, and strong product graphs by using the adjacency and distance matrices of a graph.

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1. Introduction

All graphs in this paper are assumed to be undirected, finite and simple. We refer to [2] for graph theoretical notation and terminology not specified here. For a graph G, let $V(G)$, $E(G)$ and $\overline{G}$ denote the set of vertices, the set of edges and the complement of G, respectively. If G is a connected graph and $u, v \in V(G)$, then the *distance* $d(u, v)$ between u and v is the length of a shortest path connecting u and v. If v is a vertex of a connected graph G, then the *eccentricity* $e(v)$ of v is defined by $e(v) = \max\{d(u, v) \mid u \in V(G)\}$. Furthermore, the *diameter* $diam(G)$ of G is defined by $diam(G) = \max\{e(v) \mid v \in V(G)\}$.

Let G be a finite, simple, connected, undirected graph with p vertices and q edges. In what follows, we say that G is an *(p, q)-graph*. Let $V(G) = \{v_1, v_2, \ldots, v_p\}$ and $E(G) = \{e_1, e_2, \ldots, e_q\}$ be the vertex set and edge set of G, respectively. The *adjacency matrix* of G is the $p \times p$ matrix $A = A(G)$ whose (i, j) entry, denoted by a_{ij}, is defined by

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0 & \text{otherwise.} \end{cases}$$

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The *distance matrix* of G is the $p \times p$ matrix D_G whose (i,j) entry, denoted by d_{ij}, is defined by

$$d_{ij} = \begin{cases} d_G(v_i, v_j) & \text{if } v_i \neq v_j \\ 0 & \text{otherwise,} \end{cases}$$

where $d_G(v_i, v_j)$ is the length of a shortest directed path in G from v_i to v_j.

The vertex u is said to be a *neighbor* of v if they are adjacent. The *neighborhood* of a vertex v, denoted by $N_G(v)$, is the set of all neighbors of v. The *degree* of a vertex v in a graph G, denoted by $d_v = d_G(v)$, is the number of vertices in its neighborhood, that is, $d_G(v) = |N(v)|$. The *common neighborhood graph* $con(G)$ (in short congraph) of a graph G is defined as the graph with $V(con(G)) = V(G)$ and two vertices in $con(G)$ are adjacent if they have a common neighbor in G. For every $x, y \in V(G)$,

$$xy \in E(con(G)) \text{ if and only if } N_G(x) \cap N_G(y) \neq \emptyset.$$

Some basic properties of congraphs have been established; see [1, 3].

The oldest and most studied degree-based structure descriptors are the *first and second Zagreb indices* [15], defined as

$$M_1(G) = \sum_{v \in V(G)} (d_G(v))^2 \qquad \text{and} \qquad M_2(G) = \sum_{uv \in E(G)} (d_G(u))\,(d_G(v)) \ .$$

It has been shown that the first Zagreb index obeys the identity [10]

$$M_1(G) = \sum_{uv \in E(G)} (d_G(u) + d_G(v)) \ .$$

The first investigation of the sum of distance between all pairs of vertices of a (connected) graph was done by Harold Wiener in 1947, who realized that there exists a correlation between the boiling points of paraffins and this sum [20]. Eventually, the distance–based graph invariant,

$$W(G) = \sum_{\{u,v\} \subseteq V(G)} d(u,v).$$

For more details, we refer to [8, 11, 13, 19].

The degree distance was introduced by Dobrynin and Kochetova [9] and Gutman [14] as a weighted version of the Wiener index. The *degree distance* $DD(G)$ of a graph G is defined as

$$DD(G) = \sum_{\{u,v\} \subseteq V(G)} d_G(u,v)[d_G(u) + d_G(v)] = \frac{1}{2} \sum_{u,v \in V(G)} d_G(u,v)[d_G(u) + d_G(v)]$$

with the summation runs over all pairs of vertices of G. The degree distance is also known as the Schultz index in chemical literature; see [21]. In [14], Gutman showed that if G is a tree on n vertices, then $DD(G) = 4W(G) - n(n-1)$; see [5, 6] and [9]. In [7], *Gutman index* $Gut(G)$ of a graph G is defined as

$$Gut(G) = \sum_{\{u,v\} \subseteq V(G)} d_G(u) d_G(v) d(u,v).$$

For more details on Gutman index, we refer to [4, 7, 12].

The relations between the degree distance, Gutman index and Wiener index are shown in the following Table 1.

The join and strong products are defined as follows.

The *join* or *complete product* $G \vee H$ of two disjoint graphs G and H, is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{uv \,|\, u \in V(G), v \in V(H)\}$.

Table 1. Three distance parameters

Wiener index	$W(G) = \sum_{\{u,v\}\subseteq V(G)} d_G(u,v)$
Degree distance	$DD(G) = \sum_{\{u,v\}\subseteq V(G)} d_G(u,v)[d_G(u) + d_G(v)]$
Gutman index	$Gut(G) = \sum_{\{u,v\}\subseteq V(G)} d_G(u,v) d_G(u) d_G(v)$

The *strong product* $G \boxtimes H$ of graphs G and H has the vertex set $V(G) \times V(H)$. Two vertices (u,v) and (u',v') are adjacent whenever $uu' \in E(G)$ and $v = v'$, or $u = u'$ and $vv' \in E(H)$, or $uu' \in E(G)$ and $vv' \in E(H)$.

Paulraja and Agnes [16] studied the degree distance of Cartesian and lexicographic products. Later, they [17] investigated the Gutman index of Cartesian and lexicographic products. In this paper, we investigate the degree distance and Gutman index of strong and complete product graphs.

2. Preliminary

We define,

$$N_1(G) = \sum_{v\in V(G)} d_G(v)\, d_{con(G)}(v) \quad \text{and} \quad N_2(G) = \sum_{uv\in E(con(G))} d_G(u)\, d_G(v).$$

Definition 2.1. *Let $A = [a_{ij}]_{m\times n}$. Then, we define*

$$S(A) = \sum_{1\le i\le m,\ 1\le j\le n} a_{ij}.$$

The following lemma is immediate.

Lemma 2.2. *Let $A = [a_{ij}]_{n\times n}$ and $B = [b_{ij}]_{n\times n}$. Then*

(1) $S(A^T) = S(A)$ *and* $S(\alpha\, A) = \alpha\, S(A)$ *for every* $\alpha \in \mathbb{R}$*;*

(2) $S(A + B) = S(A) + S(B)$.

Lemma 2.3. *Let G be a (p,q)-graph, and let $con(G)$ be a (p,q')-graph. Let A, B, K be the adjacency matrices of $G, con(G), K_p$, respectively. Then*

(1) $S(A) = 2q$;

(2) $S(A^2) = M_1(G)$;

(3) $S(A\,B) = N_1(G)$;

(4) $S(A^3) = 2M_2(G)$;

(5) $S(A\,K) = 2q(p-1)$.

(6) $S(A\,B\,A) = 2N_2(G)$.

Proof. For (1), we have

$$S(A) \;=\; \sum_{1\le i,j\le p} a_{ij} = \sum_{i=1}^{p}\sum_{j=1}^{p} a_{ij} = \sum_{i=1}^{p} d_{v_i} = 2q.$$

For (2), we have

$$\begin{aligned} S(A^2) &= \sum_{1\leq i,j\leq p} a_{ij}^{(2)} = \sum_{1\leq i,j\leq p}\sum_{k=1}^{p} a_{ik}\, a_{kj} \\ &= \sum_{k=1}^{p}\sum_{i=1}^{p} a_{ik} \sum_{j=1}^{p} a_{kj} = \sum_{k=1}^{p} d_{v_k}\, d_{v_k} = M_1(G). \end{aligned}$$

For (3), we have

$$\begin{aligned} S(A\,B) &= \sum_{1\leq i,j\leq p}\sum_{k=1}^{p} a_{ik}\, b_{kj} \\ &= \sum_{k=1}^{p}\sum_{i=1}^{p} a_{ik} \sum_{j=1}^{p} b_{kj} = \sum_{k=1}^{p} d_{v_k}\, d_{conG} v_k = N_1(G). \end{aligned}$$

For (4), we have

$$\begin{aligned} M_2(G) &= \sum_{v_iv_j\in E(G)} d_{v_i}\, d_{v_j} = \frac{1}{2}\sum_{1\leq i,j\leq p} d_{v_i}\, d_{v_j}\, a_{ij} \\ &= \frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p}\left(\sum_{k=1}^{p} a_{ki}\right)\left(\sum_{s=1}^{p} a_{js}\right) a_{ij} \\ &= \frac{1}{2}\sum_{k=1}^{p}\sum_{j=1}^{p}\sum_{s=1}^{p} a_{js} \sum_{i=1}^{p} a_{ki}a_{ij}. \end{aligned}$$

Since $\sum_{i=1}^{p} a_{ki}a_{ij}$ is the entry t_{kj} of matrix A^2, it follows that

$$\begin{aligned} M_2(G) &= \frac{1}{2}\sum_{k=1}^{p}\sum_{j=1}^{p}\sum_{s=1}^{p} a_{js} \sum_{i=1}^{p} a_{ki}a_{ij} \\ &= \frac{1}{2}\sum_{k=1}^{p}\sum_{s=1}^{p}\sum_{j=1}^{p} t_{kj}a_{js} = \frac{1}{2}\sum_{k=1}^{p}\sum_{s=1}^{p} a_{ks}^{(3)} = \frac{1}{2}S(A^3). \end{aligned}$$

For (5), we have

$$\begin{aligned} S(A\,K) &= \sum_{1\leq i,j\leq p}\sum_{r=1}^{p} a_{ir}\, k_{rj} \\ &= \sum_{r=1}^{p}\sum_{i=1}^{p} a_{ir} \sum_{j=1}^{p} k_{rj} = \sum_{r=1}^{p} d_{v_r}\,(p-1) = 2q(p-1). \end{aligned}$$

For (6), we have

$$\begin{aligned} N_2(G) &= \sum_{v_iv_j\in E(conG)} d_{v_i}\, d_{v_j} = \frac{1}{2}\sum_{1\leq i,j\leq p} d_{v_i}\, d_{v_j}\, b_{ij} \\ &= \frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p}\left(\sum_{k=1}^{p} a_{ki}\right)\left(\sum_{s=1}^{p} a_{sj}\right) b_{ij} \\ &= \frac{1}{2}\sum_{k=1}^{p}\sum_{s=1}^{p}\sum_{j=1}^{p} a_{sj} \sum_{i=1}^{p} a_{ki}b_{ij}. \end{aligned}$$

Since, $\sum_{i=1}^{p} a_{ki}b_{ij}$ is the entry t_{kj} of matrix AB, hence

$$\begin{aligned} N_2(G) &= \frac{1}{2}\sum_{k=1}^{p}\sum_{s=1}^{p}\sum_{j=1}^{p} a_{sj}\sum_{i=1}^{p} a_{ki}b_{ij} \\ &= \frac{1}{2}\sum_{k=1}^{p}\sum_{s=1}^{p}\sum_{j=1}^{p} t_{kj}a_{js} = \frac{1}{2}\sum_{k=1}^{p}\sum_{s=1}^{p} f_{ks} = \frac{1}{2}S(A\,B\,A), \end{aligned}$$

where $\sum_{i=1}^{p} t_{kj}a_{js}$ is the entry f_{ks} of matrix $A\,B\,A$. □

The following result for classical distance are from the book [18].

Lemma 2.4. [18] *Let (u,v) and (u',v') be two vertices of $G_1 \boxtimes G_2$. Then*

$$d_{G_1\boxtimes G_2}((u,v),(u',v')) = \max\{d_{G_1}(u,u'), d_{G_2}(v,v')\}.$$

For $G_2 = K_p$, the following result is immediate.

Corollary 2.5. *Let K_p be a complete graph, and let (u,v) and (u',v') be two vertices of $G \boxtimes K_p$. Then*

$$d_{G\boxtimes K_p}((u,v),(u',v')) = \begin{cases} d_G(u,u') & \text{if } u \neq u', \\ 1 & \text{if } u = u' \text{ and } v \neq v', \\ 0 & \text{if } u = u' \text{ and } v = v'. \end{cases}$$

3. Main results

In this section, we give our main results and their proofs.

3.1. Relation between degree distance and Gutman index

We first define a matrix, which will be used later.

Definition 3.1. *Let $G(V,E)$ be a graph with order n and m edges. For $k = 1, 2, \cdots, \alpha$ where α denotes the diameter of graph G, we define*

$$A_k = [a^k_{ij}]_{n\times n},$$

where $a^k_{ij} = \begin{cases} 1 & d(v_i,v_j) = k \\ 0 & \textit{otherwise.} \end{cases}$

The following results are easily seen.

Observation 3.1. *Let A and D_G be the adjacency matrix and the distance matrix of a graph G, respectively. Then*

(1) $A_1 = A$;

(2) $D_G = A_1 + 2A_2 + \cdots + \alpha A_\alpha$;

(3) $A_1 + A_2 + \cdots + A_\alpha = K$, *where K is the adjacency matrix complete graph K_n;*

(4) *if $diam(G) = 2$ then $D_G = A + 2\overline{A}$.*

Lemma 3.2. *Let G be a graph containing no triangles, and let A, B, K be the adjacency matrix of $G, con(G), K_n$, respectively. Then*

(1) *for every $u, v \in V(G)$, $d_G(u,v) = 2$ if and only if $uv \in E(con(G))$;*

(2) *if $diam(G) = 3$ then $D_G = 3K - 2A - B$.*

Proof. (1) Suppose $d_G(u,v) = 2$. Then there exists a vertex $x \in V(G)$ such that $x \notin \{u,v\}$ and $ux, xv \in E(G)$, and hence $N(u) \cap N(v) \neq \emptyset$. Therefore, we have $uv \in E(con(G))$. Conversely, we suppose $uv \in E(con(G))$. Then $N(u) \cap N(v) \neq \emptyset$, and hence there exists a vertex $x \in N(u) \cap N(v)$. Note that $ux, xv \in E(G)$. Therefore, $d_(u,v) \leq 2$. If $d(u,v) = 1$, then we have a triangle, a contradiction. So $d_G(u,v) = 2$, as desired.

(2) From Observation 3.1, we have $D_G = A_1 + 2A_2 + 3A_3$. Note that $A_1 = A$, $A_2 = B$ and $A + B + A_3 = K$. Therefore, $D_G = 3K - 2A - B$. □

Lemma 3.3. *Let $G(p,q)$ be a graph, and let A, D_G be the adjacency matrix and the distance matrix of a graph G, respectively. Then*

(1) *$S(A\,D_G) = DD(G)$;*

(2) *if $diam(G) = 2$ then $DD(G) = 4(p-1)q - M_1(G)$;*

(3) *if $diam(G) = 3$ and G has no triangles, then*

$$DD(G) = 6q(p-1) - 2M_1(G) - N_1(G).$$

Proof. (1) Since

$$\begin{aligned} S(A\,D_G) &= \sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{k=1}^{p} a_{ik}d_{kj} = \sum_{1\leq j,k\leq p}\sum_{i=1}^{p} a_{ik}\, d(v_k, v_j) \\ &= \sum_{1\leq j,k\leq p} d(v_k)\, d(v_k, v_j) \end{aligned}$$

and

$$\begin{aligned} S(D_G\,A) &= \sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{k=1}^{p} d_{ik}a_{kj} = \sum_{1\leq i,k\leq p} d(v_i, v_k)\sum_{j=1}^{p} a_{kj} \\ &= \sum_{1\leq i,k\leq p} d(v_i, v_k)\, d(v_k) = \sum_{1\leq j,k\leq p} d(v_k, v_j)\, d(v_j), \end{aligned}$$

it follows that

$$\begin{aligned} 2S(A\,D_G) &= S(A\,D_G) + S((A\,D_G)^T) = S(A\,D_G) + S(D_G\,A) \\ &= \sum_{1\leq j,k\leq p} d(v_k, v_j)[d(v_k) + d(v_j)] \\ &= 2\sum_{\{v_k,v_j\}\subseteq V(G)} d(v_k, v_j)[d(v_k) + d(v_j)] = 2DD(G). \end{aligned}$$

For (2), we have

$$\begin{aligned} DD(G) &= S(A\,D_G) = S(A(A + 2\overline{A})) \\ &= 2S(A(A + \overline{A})) - S(A^2) \\ &= 2S(A\,K) - S(A^2) = 4(p-1)q - M_1(G). \end{aligned}$$

For (3), we have

$$\begin{aligned} DD(G) &= S(A\,D_G) \\ &= S(A(3K-2A-B)) = 3S(A\,K) - 2S(A^2) - S(A\,B) \\ &= 6q(p-1) - 2M_1(G) - N_1(G). \end{aligned}$$

□

Lemma 3.4. *Let G be a (p,q)-graph and A be the adjacency matrix of G. Then $Gut(G) = \frac{1}{2}S(A\,D_G\,A)$*

Proof.

$$\begin{aligned} S(A\,D_G\,A) &= \sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{k=1}^{p} a_{ik}\sum_{s=1}^{p} d_{ks}a_{sj} \\ &= \sum_{k=1}^{p}\sum_{s=1}^{p}\sum_{i=1}^{p} a_{ik}\sum_{j=1}^{p} a_{sj}d_{ks} \\ &= \sum_{k=1}^{p}\sum_{s=1}^{p} d_G(v_k)\cdot d_G(v_s)\cdot d(v_k,v_s) \\ &= 2\sum_{\{v_k,v_s\}\subseteq V} d_G(v_k)d_G(v_s)d(v_k,v_s) = 2Gut(G). \end{aligned}$$

□

Corollary 3.5. *Let $G(p,q)$ be a graph, then*

$$\frac{\delta}{2} \le \frac{Gut(G)}{DD(G)} \le \frac{\Delta}{2}.$$

Proof. Since, $S(A\,D_G) = DD(G)$ and $Gut(G) = \frac{1}{2}S(A\,D_G\,A)$, hence

$$\begin{aligned} 2Gut(G) - DD(G) &= S(A\,D_G\,A) - S(A\,D_G) \\ &= S(A\,D_G(A-I)) \\ &= \sum_{1\le i,j\le p}\sum_{k=1}^{p} t_{ik}(a_{kj} - 1_{kj}) \\ &= \sum_{1\le i,k\le p} t_{ik}(d_G(v_k) - 1). \end{aligned}$$

Therefore,

$$(\delta-1)\sum_{1\le i,k\le p} t_{ik} \le 2Gut(G) - DD(G) \le (\Delta-1)\sum_{1\le i,k\le p} t_{ik}.$$

Hence,

$$(\delta-1)S(A\,D_G) \le 2Gut(G) - DD(G) \le (\Delta-1)S(A\,D_G).$$

Thus,

$$\delta DD(G) \le 2Gut(G) \le \Delta DD(G),$$

that is

$$\frac{\delta}{2} \le \frac{Gut(G)}{DD(G)} \le \frac{\Delta}{2}.$$

□

3.2. For degree distance

In this subsection, we study the degree distance of strong product graphs. We first begin with an easy case.

Theorem 3.6. *Let G be a connected graph with p_1 vertices and q_1 edges, and K_p be a complete graph with order p. Then*

$$DD(G \boxtimes K_p) = p^3 DD(G) + 2p^2(p-1)[W(G) + q_1] + p_1 p(p-1)^2.$$

Proof. Let $V(G) = V_1$ and $V(K_p) = V_2$. From the definition of strong product and Corollary 2.5, we have

$$\begin{aligned}
& DD(G \boxtimes K_p) \\
&= \sum_{\{(a,b),(c,d)\} \subseteq V_1 \times V_2} [d_{G\boxtimes K_p}(a,b) + d_{G\boxtimes K_p}(c,d)] d_{G\boxtimes K_p}[(a,b),(c,d)] \\
&= \sum_{\{(a,b),(c,d)\} \subseteq V_1 \times V_2} [d_G(a) + d_{K_p}(b) + d_G(a)d_{K_p}(b) + d_G(c) + d_{K_p}(d) + d_G(c)d_{K_p}(d)]
\end{aligned}$$

$$\begin{aligned}
&= \sum_{\{(a,b),(c,d)\} \subseteq V_1 \times V_2, a\neq c} [d_G(a) + p - 1 + d_G(a)(p-1) + d_G(c) + p - 1 + d_G(c)(p-1)] \cdot d_G(a,c) \\
&\quad + \sum_{\{(a,b),(c,d)\} \subseteq V_1 \times V_2, a=c} [d_G(a) + p - 1 + d_G(a)(p-1) + d_G(c) + p - 1 + d_G(c)(p-1)] \cdot 1 \\
&= \sum_{\{(a,b),(c,d)\} \subseteq V_1 \times V_2, a\neq c} [p(d_G(a) + d_G(c)) + 2(p-1)] d_G(a,c) \\
&\quad + \sum_{\{(a,b),(c,d)\} \subseteq V_1 \times V_2, a=c} [2pd_G(a) + 2(p-1)] \\
&= p \sum_{\{(a,b),(c,d)\} \subseteq V_1 \times V_2, a\neq c} [d_G(a) + d_G(c)] d_G(a,c) + 2(p-1) \sum_{\{(a,b),(c,d)\} \subseteq V_1 \times V_2, a\neq c} d_G(a,c) \\
&\quad + 2p \sum_{\{(a,b),(c,d)\} \subseteq V_1 \times V_2, a=c} d_G(a) + 2(p-1) \sum_{\{(a,b),(c,d)\} \subseteq V_1 \times V_2, a=c} 1 \\
&= p^3 DD(G) + 2p^2(p-1)W(G) + 2p \cdot \frac{p(p-1)}{2} \cdot 2q_1 + 2p_1(p-1) \cdot \frac{p(p-1)}{2} \\
&= p^3 DD(G) + 2p^2(p-1)[W(G) + q_1] + p_1 p(p-1)^2.
\end{aligned}$$

□

For the strong product of two general graphs, we have the following.

Theorem 3.7. *Let G_1 be a connected graph with p_1 vertices and q_1 edges, and G_2 be a connected graph* *with p_2 vertices and q_2 edges. Then*

$$\begin{aligned}
&\max\Big\{DD(G_1)[2p_2q_2+p_2^2]+4p_2q_2W(G_1)+2p_2(p_2-1)q_1W(G_2)+DD(G_2)(2q_1+p_1),\\
&DD(G_2)[2p_1q_1+p_1^2]+4p_1q_1W(G_2)+2p_1(p_1-1)q_2W(G_1)+DD(G_1)(2q_2+p_2)\Big\}\\
\leq\ & DD(G_1\boxtimes G_2)\leq(2q_2+p_2)(p_2+1)DD(G_1)+q_2(4p_2+2p_1^2-2p_1)W(G_1)\\
&+(2q_1+p_1)(p_1+1)DD(G_2)+q_1(4p_1+2p_2^2-2p_2)W(G_2).
\end{aligned}$$

Moreover, the lower bound is sharp.

In particular, if G be a connected graph with p vertices and q edges, then

$$\begin{aligned}
&(2q+p)(p+1)DD(G)+2pq(p+1)W(G)\\
\leq\ & DD(G\boxtimes G)\leq 2\big\{(2q+p)(p+1)DD(G)+2pq(p+1)W(G)\big\}.
\end{aligned}$$

Proof. From Lemma 2.4 and the definition of degree distance, we have

$$DD(G_1\boxtimes G_2) = \sum_{\{(a,b),(c,d)\}\subseteq V_1\times V_2}[d_{G_1\boxtimes G_2}(a,b)+d_{G_1\boxtimes G_2}(c,d)]d_{G_1\boxtimes G_2}[(a,b),(c,d)]$$

$$= \sum_{\{(a,b),(c,d)\}\subseteq V_1\times V_2}[d_{G_1}(a)+d_{G_2}(b)+d_{G_1}(a)d_{G_2}(b)+d_{G_1}(c)+d_{G_2}(d)+d_{G_1}(c)d_{G_2}(d)]$$

$$\cdot\max\{d_{G_1}(a,c),d_{G_2}(b,d)\}$$

$$\begin{aligned}
\geq\ \max\Bigg\{&\sum_{\substack{\{(a,b),(c,d)\}\subseteq V_1\times V_2\\ a\neq c}}[d_{G_1}(a)+d_{G_2}(b)+d_{G_1}(a)d_{G_2}(b)+d_{G_1}(c)+d_{G_2}(d)+d_{G_1}(c)d_{G_2}(d)]d_{G_1}(a,c)\\
&+\sum_{\substack{\{(a,b),(c,d)\}\subseteq V_1\times V_2\\ a=c}}[d_{G_1}(a)+d_{G_2}(b)+d_{G_1}(a)d_{G_2}(b)+d_{G_1}(c)+d_{G_2}(d)+d_{G_1}(c)d_{G_2}(d)]d_{G_2}(b,d),\\
&\sum_{\substack{\{(a,b),(c,d)\}\subseteq V_1\times V_2\\ b\neq d}}[d_{G_1}(a)+d_{G_2}(b)+d_{G_1}(a)d_{G_2}(b)+d_{G_1}(c)+d_{G_2}(d)+d_{G_1}(c)d_{G_2}(d)]d_{G_2}(b,d)\\
&+\sum_{\substack{\{(a,b),(c,d)\}\subseteq V_1\times V_2\\ b=d}}[d_{G_1}(a)+d_{G_2}(b)+d_{G_1}(a)d_{G_2}(b)+d_{G_1}(c)+d_{G_2}(d)+d_{G_1}(c)d_{G_2}(d)]d_{G_1}(a,c)\Bigg\}
\end{aligned}$$

$$= \max\Bigg\{ \sum_{\substack{\{(a,b),(c,d)\}\subseteq V_1\times V_2 \\ a\neq c}} [d_{G_1}(a)+d_{G_1}(c)]d_{G_1}(a,c) + \sum_{\substack{\{(a,b),(c,d)\}\subseteq V_1\times V_2 \\ a\neq c}} [d_{G_2}(b)+d_{G_2}(d)]d_{G_1}(a,c)$$

$$+ \sum_{\substack{\{(a,b),(c,d)\}\subseteq V_1\times V_2 \\ a\neq c}} [d_{G_1}(a)d_{G_2}(b)+d_{G_1}(c)d_{G_2}(d)]d_{G_1}(a,c)$$

$$+ \sum_{\{(a,b),(a,d)\}\subseteq V_1\times V_2} [2d_{G_1}(a)+(d_{G_1}(a)+1)(d_{G_2}(b)+d_{G_2}(d))]d_{G_2}(b,d),$$

$$\sum_{\substack{\{(a,b),(c,d)\}\subseteq V_1\times V_2 \\ b\neq d}} [d_{G_1}(a)+d_{G_1}(c)]d_{G_2}(b,d) + \sum_{\substack{\{(a,b),(c,d)\}\subseteq V_1\times V_2 \\ b\neq d}} [d_{G_2}(b)+d_{G_2}(d)]d_{G_2}(b,d)$$

$$+ \sum_{\substack{\{(a,b),(c,d)\}\subseteq V_1\times V_2 \\ b\neq d}} [d_{G_1}(a)d_{G_2}(b)+d_{G_1}(c)d_{G_2}(d)]d_{G_2}(b,d)$$

$$+ \sum_{\{(a,b),(c,b)\}\subseteq V_1\times V_2} [2d_{G_2}(b)+(d_{G_2}(b)+1)(d_{G_1}(a)+d_{G_1}(c))]d_{G_1}(a,c)\Bigg\}$$

$$= \max\Big\{p_2^2 DD(G_1)+4p_2q_2W(G_1)+\sum_{\{(a,b),(c,d)\}\subseteq V_1\times V_2} [d_{G_1}(a)d_{G_2}(b)+d_{G_1}(c)d_{G_2}(d)]d_{G_1}(a,c)$$

$$+2p_2(p_2-1)q_1W(G_2)+DD(G_2)(2q_1+p_1),$$

$$p_1^2 DD(G_2)+4p_1q_1W(G_2)+\sum_{\{(a,b),(c,d)\}\subseteq V_1\times V_2} [d_{G_1}(a)d_{G_2}(b)+d_{G_1}(c)d_{G_2}(d)]d_{G_2}(b,d)$$

$$+2p_1(p_1-1)q_2W(G_1)+DD(G_1)(2q_2+p_2)\Big\}$$

Since

$$\sum_{\{(a,b),(c,d)\}\subseteq V_1\times V_2} [d_{G_1}(a)d_{G_2}(b)+d_{G_1}(c)d_{G_2}(d)]d_{G_1}(a,c)$$

$$= \sum_{\{(a,b),(c,d)\}\subseteq V_1\times V_2} [d_{G_1}(a)d_{G_2}(b)\cdot d_{G_1}(a,c)] + \sum_{\{(a,b),(c,d)\}\subseteq V_1\times V_2} [d_{G_1}(c)d_{G_2}(d)\cdot d_{G_1}(a,c)]$$

$$= \sum_{\{a,c\}\subseteq V_1} d_{G_1}(a)d_{G_1}(a,c) \cdot \sum_{\{b,d\}\subseteq V_2} d_{G_2}(b) + \sum_{\{a,c\}\subseteq V_1} d_{G_1}(c)d_{G_1}(a,c) \cdot \sum_{\{b,d\}\subseteq V_2} d_{G_2}(d)$$
$$= 2p_2q_2 \cdot \sum_{\{a,c\}\subseteq V_1} d_{G_1}(a)d_{G_1}(a,c) + 2p_2q_2 \cdot \sum_{\{a,c\}\subseteq V_1} d_{G_1}(c)d_{G_1}(a,c)$$
$$= 2p_2q_2\left(\sum_{\{a,c\}\subseteq V_1} [d_{G_1}(a) + d_{G_1}(c)]d_{G_1}(a,c)\right)$$
$$= 2p_2q_2 \cdot DD(G_1)$$

and similarly

$$\sum_{\{(a,b),(c,d)\}\subseteq V_1\times V_2} [d_{G_1}(a)d_{G_2}(b) + d_{G_1}(c)d_{G_2}(d)]d_{G_2}(b,d)$$
$$= 2p_1q_1\Big(\sum_{\{b,d\}\subseteq V_2} [d_{G_2}(b) + d_{G_2}(d)]d_{G_2}(b,d)\Big) = 2p_1q_1DD(G_2),$$

it follows that

$$\begin{aligned} & DD(G_1 \boxtimes G_2) \\ \geq\ & \max\Big\{DD(G_1)[2p_2q_2 + p_2^2] + 4p_2q_2W(G_1) + 2p_2(p_2-1)q_1W(G_2) + DD(G_2)(2q_1+p_1), \\ & DD(G_2)[2p_1q_1 + p_1^2] + 4p_1q_1W(G_2) + 2p_1(p_1-1)q_2W(G_1) + DD(G_1)(2q_2+p_2)\Big\}. \end{aligned}$$

Also, we have

$$\begin{aligned} & DD(G_1 \boxtimes G_2) \\ = & \sum_{\{(a,b),(c,d)\}\subseteq V_1\times V_2} [d_{G_1\boxtimes G_2}(a,b) + d_{G_1\boxtimes G_2}(c,d)]d_{G_1\boxtimes G_2}[(a,b),(c,d)] \\ = & \sum_{\{(a,b),(c,d)\}\subseteq V_1\times V_2} [d_{G_1}(a) + d_{G_2}(b) + d_{G_1}(a)d_{G_2}(b) + d_{G_1}(c) + d_{G_2}(d) + d_{G_1}(c)d_{G_2}(d)] \\ & \cdot \max\{d_{G_1}(a,c), d_{G_2}(b,d)\} \\ \leq & \sum_{\{(a,b),(c,d)\}\subseteq V_1\times V_2} [d_{G_1}(a) + d_{G_2}(b) + d_{G_1}(a)d_{G_2}(b) + d_{G_1}(c) + d_{G_2}(d) + d_{G_1}(c)d_{G_2}(d)]d_{G_1}(a,c) \\ & + \sum_{\{(a,b),(c,d)\}\subseteq V_1\times V_2} [d_{G_1}(a) + d_{G_2}(b) + d_{G_1}(a)d_{G_2}(b) + d_{G_1}(c) + d_{G_2}(d) + d_{G_1}(c)d_{G_2}(d)]d_{G_2}(b,d) \\ = & (2q_2+p_2)(p_2+1)DD(G_1) + q_2(4p_2 + 2p_1^2 - 2p_1)W(G_1) \\ & + (2q_1+p_1)(p_1+1)DD(G_2) + q_1(4p_1 + 2p_2^2 - 2p_2)W(G_2). \end{aligned}$$

□

To show the sharpness of the lower bounds of Theorem 3.7, we consider the following example.

Example 1. Let G be a complete graph of order n. If $n = 2$, then $G = K_2$ and $G \boxtimes G = K_4$, and hence $DD(G \boxtimes G) = 36 = (2q+p)(p+1)DD(G) + 2pq(p+1)W(G)$. If $n = 3$, then $G = K_3$ and $G \boxtimes G = K_9$, and hence $DD(G \boxtimes G) = 576 = (2q+p)(p+1)DD(G) + 2pq(p+1)W(G)$. From the proof of Theorem 3.7, one can check that $K_n \boxtimes K_n$ is an sharp example of the lower bound.

3.3. For Gutman index

In this subsection, we study the Gutman index of strong product graphs. We first begin with an easy case.

Theorem 3.8. *Let G be a connected graph with p_1 vertices and q_1 edges, and K_p be a complete graph with p vertices. Then*

$$\begin{aligned} Gut(G \boxtimes K_p) &= p^4 Gut(G) + p^3(p-1)DD(G) + p^2(p-1)^2 \cdot W(G) \\ &+ \frac{p^3(p-1)}{2} M_1(G) + \frac{p(p-1)^3}{2} p_1 + 2p^2(p-1)^2 q_1. \end{aligned}$$

Proof. Let $V(G) = V_1$ and $V(K_p) = V_2$. From the definition of strong product and Corollary 2.5, we have

$$\begin{aligned} & Gut(G \boxtimes K_p) \\ = & \sum_{\{(a,b),(c,d)\} \subseteq V_1 \times V_2} d_{G\boxtimes K_p}(a,b) \cdot d_{G\boxtimes K_p}(c,d) \cdot d_{G\boxtimes K_p}[(a,b),(c,d)] \\ = & \sum_{\{(a,b),(c,d)\} \subseteq V_1 \times V_2} [d_G(a) + d_{K_p}(b) + d_G(a)d_{K_p}(b)] \cdot [d_G(c) + d_{K_p}(d) + d_G(c)d_{K_p}(d)] \\ & \cdot d_{G\boxtimes K_p}[(a,b),(c,d)] \\ = & \sum_{\{(a,b),(c,d)\} \subseteq V_1 \times V_2, a \neq c} [pd_G(a) + p - 1] \cdot [pd_G(c) + p - 1] \cdot d_G(a,c) \\ & + \sum_{\{(a,b),(c,d)\} \subseteq V_1 \times V_2, a = c} [pd_G(a) + p - 1] \cdot [pd_G(a) + (p-1)] \cdot 1 \\ = & \; p^2 \sum_{\{(a,b),(c,d)\} \subseteq V_1 \times V_2, a \neq c} d_G(a)d_G(c)d_G(a,c) \\ & + p(p-1) \sum_{\{(a,b),(c,d)\} \subseteq V_1 \times V_2, a \neq c} [d_G(a) + d_G(c)]d_G(a,c) \\ & + (p-1)^2 \sum_{\{(a,b),(c,d)\} \subseteq V_1 \times V_2, a \neq c} d_G(a,c) + p^2 \sum_{\{(a,b),(c,d)\} \subseteq V_1 \times V_2, a = c} d_G^2(a) \\ & + \sum_{\{(a,b),(c,d)\} \subseteq V_1 \times V_2, a = c} (p-1)^2 + 2p(p-1) \sum_{\{(a,b),(c,d)\} \subseteq V_1 \times V_2, a = c} d_G(a) \\ = & \; p^4 Gut(G) + p^3(p-1)DD(G) + p^2(p-1)^2 \cdot W(G) \\ & + \frac{p^3(p-1)}{2} M_1(G) + \frac{p(p-1)^3}{2} p_1 + 2p^2(p-1)^2 q_1. \end{aligned}$$

□

For the strong product of two general graphs, we have the following.

Theorem 3.9. *Let G_1 be a connected graph with p_1 vertices and q_1 edges, and G_2 be a connected graph with p_2 vertices and q_1 edges. Then*

$$\begin{aligned}
&\max\Big\{Gut(G_1)(p_2^2+4p_2q_2+4q_2^2)+(2p_2q_2+4q_2^2)DD(G_1)+4q_2^2W(G_1)\\
&\quad+M_1(G_1)W(G_2)+[2q_1+M_1(G_1)]DD(G_2)+[p_1+4q_1+M_1(G_1)]Gut(G_2),\\
&\quad Gut(G_2)(p_1^2+4p_1q_1+4q_1^2)+(2p_1q_1+4q_1^2)DD(G_2)+4q_1^2W(G_2)\\
&+M_2(G_2)W(G_1)+[2q_2+M_2(G_2)]DD(G_1)+[p_2+4q_2+M_2(G_2)]Gut(G_1)\Big\}\\
\leq\ & Gut(G_1\boxtimes G_2)\\
\leq\ & Gut(G_1)[p_2^2+4p_2q_2+4q_2^2+p_2+4q_2+M_2(G_2)]+[2p_2q_2+4q_2^2+2q_2+M_2(G_2)]DD(G_1)\\
&+[4q_2^2+M_2(G_2)]W(G_1)+Gut(G_2)[p_1^2+4p_1q_1+4q_1^2+p_1+4q_1+M_1(G_1)]\\
&+[2p_1q_1+4q_1^2+2q_1+M_1(G_1)]DD(G_2)+[4q_1^2+M_1(G_1)]W(G_2).
\end{aligned}$$

In particular, if G be a connected graph with p vertices and q edges, then

$$\begin{aligned}
&Gut(G)[p^2+4pq+4q^2+p+4q]+[2pq+4q^2+2q+M_1(G)]DD(G)+[4q^2+M_1(G)]W(G)\\
\leq\ & Gut(G\boxtimes G)\\
\leq\ & 2\big\{Gut(G)[p^2+4pq+4q^2+p+4q]+[2pq+4q^2+2q+M_1(G)]DD(G)+[4q^2+M_1(G)]W(G)\big\}
\end{aligned}$$

Proof. Let $V(G_1)=V_1$ and $V(G_2)=V_2$. From the definition of strong product and Lemma 2.4, we have

$$\begin{aligned}
&Gut(G_1\boxtimes G_2)\\
=\ & \sum_{\{(a,b),(c,d)\}\subseteq V_1\times V_2} d_{G_1\boxtimes G_2}(a,b)\cdot d_{G_1\boxtimes G_2}(c,d)\cdot d_{G_1\boxtimes G_2}[(a,b),(c,d)]\\
=\ & \sum_{\{(a,b),(c,d)\}\subseteq V_1\times V_2} [d_{G_1}(a)+d_{G_2}(b)+d_{G_1}(a)d_{G_2}(b)]\cdot[d_{G_1}(c)+d_{G_2}(d)+d_{G_1}(c)d_{G_2}(d)]\\
&\qquad\cdot\max\{d_{G_1}(a,c),d_{G_2}(b,d)\}\\
\geq\ & \max\Bigg\{\sum_{\substack{\{(a,b),(c,d)\}\subseteq V_1\times V_2\\ a\neq c}} [d_{G_1}(a)+d_{G_2}(b)+d_{G_1}(a)d_{G_2}(b)]\cdot[d_{G_1}(c)+d_{G_2}(d)+d_{G_1}(c)d_{G_2}(d)]d_{G_1}(a,c)\\
&+\sum_{\substack{\{(a,b),(c,d)\}\subseteq V_1\times V_2\\ a=c}} [d_{G_1}(a)+d_{G_2}(b)+d_{G_1}(a)d_{G_2}(b)]\cdot[d_{G_1}(c)+d_{G_2}(d)+d_{G_1}(c)d_{G_2}(d)]d_{G_2}(b,d),\\
&\sum_{\substack{\{(a,b),(c,d)\}\subseteq V_1\times V_2\\ b\neq d}} [d_{G_1}(a)+d_{G_2}(b)+d_{G_1}(a)d_{G_2}(b)]\cdot[d_{G_1}(c)+d_{G_2}(d)+d_{G_1}(c)d_{G_2}(d)]d_{G_2}(b,d)\\
&+\sum_{\substack{\{(a,b),(c,d)\}\subseteq V_1\times V_2\\ b=d}} [d_{G_1}(a)+d_{G_2}(b)+d_{G_1}(a)d_{G_2}(b)]\cdot[d_{G_1}(c)+d_{G_2}(d)+d_{G_1}(c)d_{G_2}(d)]d_{G_1}(a,c)\Bigg\}
\end{aligned}$$

$$= \max\Bigg\{ \sum_{\substack{\{(a,b),(c,d)\}\subseteq V_1\times V_2\\ a\neq c}} [d_{G_1}(a)+d_{G_2}(b)+d_{G_1}(a)d_{G_2}(b)]\cdot[d_{G_1}(c)+d_{G_2}(d)+d_{G_1}(c)d_{G_2}(d)]d_{G_1}(a,c)$$

$$+ \sum_{\{(a,b),(a,d)\}\subseteq V_1\times V_2} [d_{G_1}(a)+d_{G_2}(b)+d_{G_1}(a)d_{G_2}(b)]\cdot[d_{G_1}(a)+d_{G_2}(d)+d_{G_1}(a)d_{G_2}(d)]d_{G_2}(b,d),$$

$$\sum_{\substack{\{(a,b),(c,d)\}\subseteq V_1\times V_2\\ b\neq d}} [d_{G_1}(a)+d_{G_2}(b)+d_{G_1}(a)d_{G_2}(b)]\cdot[d_{G_1}(c)+d_{G_2}(d)+d_{G_1}(c)d_{G_2}(d)]d_{G_2}(b,d)$$

$$+ \sum_{\{(a,b),(c,b)\}\subseteq V_1\times V_2} [d_{G_1}(a)+d_{G_2}(b)+d_{G_1}(a)d_{G_2}(b)]\cdot[d_{G_1}(c)+d_{G_2}(b)+d_{G_1}(c)d_{G_2}(b)]d_{G_1}(a,c)\Bigg\}$$

$$= \max\{X_1+X_2, Y_1+Y_2\},$$

where

$$X_1 = \sum_{\substack{\{(a,b),(c,d)\}\subseteq V_1\times V_2\\ a\neq c}} [d_{G_1}(a)+d_{G_2}(b)+d_{G_1}(a)d_{G_2}(b)]\cdot[d_{G_1}(c)+d_{G_2}(d)+d_{G_1}(c)d_{G_2}(d)]d_{G_1}(a,c),$$

$$X_2 = \sum_{\{(a,b),(a,d)\}\subseteq V_1\times V_2} [d_{G_1}(a)+d_{G_2}(b)+d_{G_1}(a)d_{G_2}(b)]\cdot[d_{G_1}(a)+d_{G_2}(d)+d_{G_1}(a)d_{G_2}(d)]d_{G_2}(b,d),$$

$$Y_1 = \sum_{\substack{\{(a,b),(c,d)\}\subseteq V_1\times V_2\\ b\neq d}} [d_{G_1}(a)+d_{G_2}(b)+d_{G_1}(a)d_{G_2}(b)]\cdot[d_{G_1}(c)+d_{G_2}(d)+d_{G_1}(c)d_{G_2}(d)]d_{G_2}(b,d),$$

and

$$Y_2 = \sum_{\{(a,b),(c,b)\}\subseteq V_1\times V_2} [d_{G_1}(a)+d_{G_2}(b)+d_{G_1}(a)d_{G_2}(b)]\cdot[d_{G_1}(c)+d_{G_2}(b)+d_{G_1}(c)d_{G_2}(b)]d_{G_1}(a,c).$$

Note that

$$X_1 = \sum_{\substack{\{(a,b),(c,d)\}\subseteq V_1\times V_2\\ a\neq c}} d_{G_1}(a)\cdot d_{G_1}(c)\cdot d_{G_1}(a,c) + \sum_{\substack{\{(a,b),(c,d)\}\subseteq V_1\times V_2\\ a\neq c}} d_{G_1}(a)\cdot d_{G_2}(d)\cdot d_{G_1}(a,c)$$

$$+ \sum_{\substack{\{(a,b),(c,d)\}\subseteq V_1\times V_2\\ a\neq c}} d_{G_1}(a)d_{G_1}(c)d_{G_2}(d)d_{G_1}(a,c) + \sum_{\substack{\{(a,b),(c,d)\}\subseteq V_1\times V_2\\ a\neq c}} d_{G_2}(b)d_{G_1}(c)d_{G_1}(a,c)$$

$$+ \sum_{\substack{\{(a,b),(c,d)\}\subseteq V_1\times V_2\\ a\neq c}} d_{G_2}(b)d_{G_2}(d)d_{G_1}(a,c) + \sum_{\substack{\{(a,b),(c,d)\}\subseteq V_1\times V_2\\ a\neq c}} d_{G_2}(b)d_{G_1}(c)d_{G_2}(d)d_{G_1}(a,c)$$

$$+ \sum_{\substack{\{(a,b),(c,d)\}\subseteq V_1\times V_2\\ a\neq c}} d_{G_1}(a)d_{G_2}(b)d_{G_1}(c)d_{G_1}(a,c) + \sum_{\substack{\{(a,b),(c,d)\}\subseteq V_1\times V_2\\ a\neq c}} d_{G_1}(a)d_{G_2}(b)d_{G_2}(d)d_{G_1}(a,c)$$

$$+ \sum_{\substack{\{(a,b),(c,d)\}\subseteq V_1\times V_2\\ a\neq c}} d_{G_1}(a)d_{G_2}(b)d_{G_1}(c)d_{G_2}(d)d_{G_1}(a,c)$$

$$= p_2^2 Gut(G_1) + 4p_2q_2 Gut(G_1) + 4q_2^2 Gut(G_1) + 2p_2q_2 DD(G_1) + 4q_2^2 DD(G_1) + 4q_2^2 W(G_1)$$

$$= Gut(G_1)(p_2^2 + 4p_2q_2 + 4q_2^2) + (2p_2q_2 + 4q_2^2)DD(G_1) + 4q_2^2 W(G_1)$$

and

$$X_2 = \sum_{\{(a,b),(c,d)\}\subseteq V_1\times V_2} d_{G_1}(a)^2 \cdot d_{G_2}(b,d) + \sum_{\{(a,b),(c,d)\}\subseteq V_1\times V_2} d_{G_1}(a)\cdot d_{G_2}(d)\cdot d_{G_2}(b,d)$$

$$+ \sum_{\{(a,b),(c,d)\}\subseteq V_1\times V_2} d_{G_1}(a)^2 d_{G_2}(d)d_{G_2}(b,d) + \sum_{\{(a,b),(c,d)\}\subseteq V_1\times V_2} d_{G_2}(b)d_{G_1}(a)d_{G_2}(b,d)$$

$$+ \sum_{\{(a,b),(c,d)\}\subseteq V_1\times V_2} d_{G_2}(b)d_{G_2}(d)d_{G_2}(b,d) + \sum_{\{(a,b),(c,d)\}\subseteq V_1\times V_2} d_{G_2}(b)d_{G_1}(a)d_{G_2}(d)d_{G_2}(b,d)$$

$$+ \sum_{\{(a,b),(c,d)\}\subseteq V_1\times V_2} d_{G_1}(a)^2 d_{G_2}(b)d_{G_2}(b,d) + \sum_{\{(a,b),(c,d)\}\subseteq V_1\times V_2} d_{G_1}(a)d_{G_2}(b)d_{G_2}(d)d_{G_2}(b,d)$$

$$+ \sum_{\{(a,b),(c,d)\}\subseteq V_1\times V_2} d_{G_1}(a)^2 d_{G_2}(b)d_{G_2}(d)d_{G_2}(b,d)$$

$$= M_1(G_1)W(G_2) + 2q_1 DD(G_2) + M_1(G_1)DD(G_2) + p_1 Gut(G_2) + 4q_1 Gut(G_2) + M_1(G_1)Gut(G_2)$$

$$= M_1(G_1)W(G_2) + [2q_1 + M_1(G_1)]DD(G_2) + [p_1 + 4q_1 + M_1(G_1)]Gut(G_2).$$

Similarly, we have

$$Y_1 = Gut(G_2)(p_1^2 + 4p_1q_1 + 4q_1^2) + (2p_1q_1 + 4q_1^2)DD(G_2) + 4q_1^2 W(G_2)$$

and

$$Y_2 = M_2(G_2)W(G_1) + [2q_2 + M_2(G_2)]DD(G_1) + [p_2 + 4q_2 + M_2(G_2)]Gut(G_1).$$

Then

$$\begin{aligned} & Gut(G_1 \boxtimes G_2) \\ \geq\ & \max\Big\{Gut(G_1)(p_2^2 + 4p_2q_2 + 4q_2^2) + (2p_2q_2 + 4q_2^2)DD(G_1) + 4q_2^2 W(G_1) \\ & + M_1(G_1)W(G_2) + [2q_1 + M_1(G_1)]DD(G_2) + [p_1 + 4q_1 + M_1(G_1)]Gut(G_2), \\ & Gut(G_2)(p_1^2 + 4p_1q_1 + 4q_1^2) + (2p_1q_1 + 4q_1^2)DD(G_2) + 4q_1^2 W(G_2) \\ & + M_2(G_2)W(G_1) + [2q_2 + M_2(G_2)]DD(G_1) + [p_2 + 4q_2 + M_2(G_2)]Gut(G_1)\Big\} \end{aligned}$$

and

$$\begin{aligned}
&Gut(G_1 \boxtimes G_2)\\
= &\sum_{\{(a,b),(c,d)\}\subseteq V_1\times V_2} d_{G_1\boxtimes G_2}(a,b)\cdot d_{G_1\boxtimes G_2}(c,d)\cdot d_{G_1\boxtimes G_2}[(a,b),(c,d)]\\
= &\sum_{\{(a,b),(c,d)\}\subseteq V_1\times V_2} [d_{G_1}(a)+d_{G_2}(b)+d_{G_1}(a)d_{G_2}(b)]\cdot[d_{G_1}(c)+d_{G_2}(d)+d_{G_1}(c)d_{G_2}(d)]\\
&\quad\cdot \max\{d_{G_1}(a,c), d_{G_2}(b,d)\}\\
\leq &\sum_{\{(a,b),(c,d)\}\subseteq V_1\times V_2} [d_{G_1}(a)+d_{G_2}(b)+d_{G_1}(a)d_{G_2}(b)]\cdot[d_{G_1}(c)+d_{G_2}(d)+d_{G_1}(c)d_{G_2}(d)]d_{G_1}(a,c)\\
&+\sum_{\{(a,b),(c,d)\}\subseteq V_1\times V_2} [d_{G_1}(a)+d_{G_2}(b)+d_{G_1}(a)d_{G_2}(b)]\cdot[d_{G_1}(c)+d_{G_2}(d)+d_{G_1}(c)d_{G_2}(d)]d_{G_2}(b,d)\\
= &X_1+X_2+Y_1+Y_2\\
\leq &Gut(G_1)[p_2^2+4p_2q_2+4q_2^2+p_2+4q_2+M_2(G_2)]+[2p_2q_2+4q_2^2+2q_2+M_2(G_2)]DD(G_1)\\
&+[4q_2^2+M_2(G_2)]W(G_1)+Gut(G_2)[p_1^2+4p_1q_1+4q_1^2+p_1+4q_1+M_1(G_1)]\\
&+[2p_1q_1+4q_1^2+2q_1+M_1(G_1)]DD(G_2)+[4q_1^2+M_1(G_1)]W(G_2).
\end{aligned}$$

□

To show the sharpness of the lower bounds of Theorem 3.9, we consider the following example.

Example 1. Let G be a complete graph of order n. If $n = 2$, then $G = K_2$ and $G \boxtimes G = K_4$, and hence $Gut(G\boxtimes G) = 54 = Gut(G)[p^2+4pq+4q^2+p+4q]+[2pq+4q^2+2q+M_1(G)]DD(G)+[4q^2+M_1(G)]W(G)$. If $n = 3$, then $G = K_3$ and $G \boxtimes G = K_9$, and hence $Gut(G \boxtimes G) = 2304 = Gut(G)[p^2 + 4pq + 4q^2 + p + 4q] + [2pq + 4q^2 + 2q + M_1(G)]DD(G) + [4q^2 + M_1(G)]W(G)$. From the proof of Theorem 3.9, one can check that $K_n \boxtimes K_n$ is an sharp example of the lower bound.

3.4. For complete product

We first give the following lemma.

Lemma 3.10. (1) *If $A = [a_{ij}]_{n\times m}$ be any matrix and $I = [1]_{p\times n}$, then $S(I\,A) = pS(A)$;*

(2) *If $A = [a_{ij}]_{m\times n}$ and $I = [1]_{n\times p}$, then $S(A\,I) = pS(A)$;*

(3) *If $A = [a_{ij}]_{p\times m}$, $I = [1]_{m\times n}$ and $B = [b_{ij}]_{n\times q}$, then $S(A\,I\,B) = S(A)\cdot S(B)$. In particular, if $A = [a_{ij}]_{n\times n}$ then $S(A\,I\,A) = S(A)^2$.*

Proof. For (1), we have

$$\begin{aligned}
S(I\,A) &= \sum_{i=1}^{p}\sum_{j=1}^{m}\sum_{k=1}^{n} 1_{ik}a_{kj} = \sum_{i=1}^{p}\sum_{k=1}^{n}\sum_{j=1}^{m} a_{kj}\\
&= \sum_{i=1}^{p} S(A) = pS(A).
\end{aligned}$$

For (2), we have

$$\begin{aligned} S(A\,I) &= \sum_{i=1}^{m}\sum_{j=1}^{p}\sum_{k=1}^{n} a_{ik}1_{kj} = \sum_{j=1}^{p}\sum_{i=1}^{m}\sum_{k=1}^{n} a_{ik} \\ &= \sum_{j=1}^{p} S(A) = pS(A). \end{aligned}$$

For (3), we have

$$\begin{aligned} S(A\,I\,B) &= \sum_{i=1}^{p}\sum_{j=1}^{q}\sum_{k=1}^{m} a_{ik}\sum_{s=1}^{n} 1_{ks}b_{sj} = \sum_{i=1}^{p}\sum_{k=1}^{m} a_{ik}\sum_{j=1}^{q}\sum_{s=1}^{n} b_{sj} \\ &= S(A)\cdot S(B). \end{aligned}$$

□

Corollary 3.11. *Let G be a (p,q)-graph and A and K be the adjacency matrix of G and K_p respectively. Let $I = [1]_{p\times p}$ and I_p be the identity matrix. Then $S(AKA) = 4q^2 - M_1(G)$.*

Proof. By Lemma 3.10, we have

$$\begin{aligned} S(A\,K\,A) &= S(A(I - I_p)A) \\ &= S(A\,I\,A) - S(A^2) \\ &= S(A)^2 - S(A^2) = 4q^2 - M_1(G). \end{aligned}$$

□

Theorem 3.12. *Let G be a (p,q)-graph, then*

(1) *If $diam(G) = 2$, then $Gut(G) = 4q^2 - M_1(G) - M_2(G)$.*

(2) *If $diam(G) = 3$ and G has no cycles of size 3 then*

$$Gut(G) = 6q^2 - \frac{3}{2}M_1(G) - 2M_2(G) - N_2(G).$$

Proof. (1) By Lemma 3.4 and Observation 3.1, we have:

$$\begin{aligned} 2Gut(G) &= S(A\,D_G\,A) = S(A(A + 2\overline{A})A) = S(A^3) + 2S(A\,\overline{A}\,A) \\ &= 2S(A(A + \overline{A})A) - S(A^3) = 2S(A\,K\,A) - S(A^3) \\ &= 8q^2 - 2M_1(G) - 2M_2(G). \end{aligned}$$

(2) By Lemma 3.4 and Observation 3.1, we have:

$$\begin{aligned} 2Gut(G) &= S(A\,D_G\,A) = S(A(3K - 2A - B)A) \\ &= 3S(A\,K\,A) - 2S(A^3) - S(A\,B\,A) \\ &= 12q^2 - 3M_1(G) - 4M_2(G) - 2N_2(G). \end{aligned}$$

□

Remark 3.13. *Let $A_1 = [a_{ij}]_{n_1 \times n_1}$ and $A_2 = [b_{ij}]_{n_2 \times n_2}$ be the adjacency matrix of G_1 and G_2, respectively. Let D_G be distance matrix of graph $G = G_1 \vee G_2$. Let $I_1 = [1]_{n_1 \times n_1}$, $I_2 = [1]_{n_2 \times n_2}$, $I_1^{'} = [1]_{n_1 \times n_2}$, $I_2^{'} = [1]_{n_2 \times n_1}$ and I_n be identity matrices. Then*

$$D_G = \begin{pmatrix} 2I_1 - A_1 - 2I_{n_1}, & I_1^{'} \\ I_2^{'}, & 2I_2 - A_2 - 2I_{n_2} \end{pmatrix}$$

is distance matrix of $G_1 \vee G_2$.

Theorem 3.14. *Let G_1 be a graph with order n_1 and m_1 edges and G_2 be a graph with order n_2 and m_2 edges. Then*

$$DD(G_1 \vee G_2) = 4(n_1 + n_2 - 1)(m_1 + m_2 + n_1 n_2) - M(G_1 \vee G_2).$$

Proof. Since $diam(G_1 \vee G_2) = 2$, it follows from Lemma 3.3 that

$$DD(G_1 \vee G_2) = 4(n_1 + n_2 - 1)(m_1 + m_2 + n_1 n_2) - M_1(G_1 \vee G_2).$$

For computing $M_1(G_1 \vee G_2)$, let A be the adjacency matrix of graph $G = G_1 \vee G_2$. Then

$$\begin{aligned} M_1(G_1 \vee G_2) &= S(A^2) = S\left[\begin{pmatrix} A_1, & I_1^{'} \\ I_2^{'}, & A_2 \end{pmatrix}\begin{pmatrix} A_1, & I_1^{'} \\ I_2^{'}, & A_2 \end{pmatrix}\right] \\ &= S\begin{pmatrix} {A_1}^2 + I_1^{'} I_2^{'}, & A_1 I_1^{'} + I_1^{'} A_2 \\ I_2^{'} A_1 + A_2 I_2^{'}, & I_2^{'} I_1^{'} + {A_2}^2 \end{pmatrix} \\ &= S({A_1}^2) + S(I_1^{'} I_2^{'}) + S(A_1 I_1^{'}) + S(I_1^{'} A_2) \\ &+ S(I_2^{'} A_1) + S(A_2 I_2^{'}) + S(I_2^{'} I_1^{'}) + S({A_2}^2) \\ &= M_1(G_1) + n_1^2 n_2 + 4n_2 m_1 + 4n_1 m_2 + n_2^2 n_1 + M_1(G_2). \end{aligned}$$

□

Theorem 3.15. *Let G_1 be an (n_1, m_1)-graph and let G_2 be an (n_2, m_2)-graph. Then*

$$Gut(G_1 \vee G_2) = 4(m_1 + m_2 + n_1 n_2)^2 - M_1(G_1 \vee G_2) - M_2(G_1 \vee G_2).$$

Proof. Let $A_1 = [a_{ij}]_{n_1 \times n_1}$ and $A_2 = [b_{ij}]_{n_2 \times n_2}$ be the adjacency matrix of G_1 and of G_2 respectively. Let D_G be distance matrix of graph $G = G_1 \vee G_2$. If we set $I_1 = [1]_{n_1 \times n_1}$, $I_2 = [1]_{n_2 \times n_2}$, $I_1^{'} = [1]_{n_1 \times n_2}$, $I_2^{'} = [1]_{n_2 \times n_1}$ and I_n be identity matrix, then it follows from Theorem 3.12 that

$$Gut(G_1 \vee G_2) = 4(m_1 + m_2 + n_1 n_2)^2 - M_1(G_1 \vee G_2) - M_2(G_1 \vee G_2),$$

since $diam(G_1 \vee G_2) = 2$.

For computing $M_2(G_1 \vee G_2)$, let A be the adjacency matrix of graph $G = G_1 \vee G_2$. Then

$$
\begin{aligned}
&2M_2(G_1 \vee G_2) \\
=\ & S(A^3) = S\left[\begin{pmatrix} A_1, & I_1^{'} \\ I_2^{'}, & A_2 \end{pmatrix}\begin{pmatrix} A_1, & I_1^{'} \\ I_2^{'}, & A_2 \end{pmatrix}\begin{pmatrix} A_1, & I_1^{'} \\ I_2^{'}, & A_2 \end{pmatrix}\right] \\
=\ & S\left[\begin{pmatrix} {A_1}^2 + I_1^{'}I_2^{'} & A_1I_1^{'} + I_1^{'}A_2 \\ I_2^{'}A_1 + A_2I_2^{'} & I_2^{'}I_1^{'} + {A_2}^2 \end{pmatrix}\begin{pmatrix} A_1, & I_1^{'} \\ I_2^{'}, & A_2 \end{pmatrix}\right] \\
=\ & S\left[\begin{pmatrix} {A_1}^3 + I_1^{'}I_2^{'}A_1 + A_1I_1^{'}I_2^{'} + I_1^{'}A_2I_2^{'} & {A_1}^2I_1^{'} + I_1^{'}I_2^{'}I_1^{'} + A_1I_1^{'}A_2 + I_1^{'}{A_2}^2 \\ I_2^{'}{A_1}^2 + A_2I_2^{'}A_1 + I_2^{'}I_1^{'}I_2^{'} + {A_2}^2I_2^{'} & I_2^{'}A_1I_1^{'} + A_2I_2^{'}I_1^{'} + I_2^{'}I_1^{'}A_2 + {A_2}^3 \end{pmatrix}\right] \\
=\ & S({A_1}^3) + S(I_1^{'}I_2^{'}A_1) + S(A_1I_1^{'}I_2^{'}) + S(I_1^{'}A_2I_2^{'}) \\
+\ & S({A_1}^2I_1^{'}) + S(I_1^{'}I_2^{'}I_1^{'}) + S(A_1I_1^{'}A_2) + S(I_1^{'}{A_2}^2) \\
+\ & S(I_2^{'}{A_1}^2) + S(A_2I_2^{'}A_1) + S(I_2^{'}I_1^{'}I_2^{'}) + S({A_2}^2I_2^{'}) \\
+\ & S(I_2^{'}A_1I_1^{'}) + S(A_2I_2^{'}I_1^{'}) + S(I_2^{'}I_1^{'}A_2) + S({A_2}^3).
\end{aligned}
$$

From Lemma 3.10, we have

$$
\begin{aligned}
2M_2(G_1 \vee G_2) \ =\ & 2M_2(G_1) + 4n_1n_2m_1 + 2n_2M_1(G_1) + 2n_1^2n_2^2 + 8m_1m_2 \\
+\ & 2n_1^2m_2 + 2n_1M_1(G_2) + 2n_2^2m_1 + 4n_1n_2m_2 + 2M_2(G_2),
\end{aligned}
$$

and hence

$$
\begin{aligned}
M_2(G_1 \vee G_2) \ =\ & M_2(G_1) + M_2(G_2) + n_2M_1(G_1) + n_1M_1(G_2) \\
+\ & 2n_1n_2m_2 + 2n_1n_2m_1 + n_1^2n_2^2 + 4m_1m_2 + n_1^2m_2 + n_2^2m_1 \\
=\ & M_2(G_1) + M_2(G_2) + n_2M_1(G_1) + n_1M_1(G_2) \\
+\ & (n_1n_2 + 2m_2)(n_1n_2 + 2m_1) + n_1^2m_2 + n_2^2m_1.
\end{aligned}
$$

□

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