Available online: April 30, 2020

Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.
Volume 69, Number 1, Pages 863–879 (2020)
DOI: 10.31801/cfsuasmas.600814
ISSN 1303–5991 E-ISSN 2618-6470

http://communications.science.ankara.edu.tr

GENERALIZED HERMITE-HADAMARD TYPE INEQUALITIES FOR PRODUCTS OF CO-ORDINATED CONVEX FUNCTIONS

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ABSTRACT. In this paper, we think products of two co-ordinated convex functions for the Hermite-Hadamard type inequalities. Using these functions we obtained Hermite-Hadamard type inequalities which are generalizations of some results given in earlier works.

1. INTRODUCTION

The following inequality discovered by C. Hermite and J. Hadamard for convex functions is well known in the literature as the Hermite–Hadamard inequality (see, e.g., [13]):

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}$$

where $f : I \subset \mathbb{R} \to \mathbb{R}$ is a convex function on the interval I and $a, b \in I$ with $a < b$.

Hermite–Hadamard inequality provides a lower and an upper estimation for the integral average of any convex function defined on a compact interval. This inequality has a notable place in mathematical analysis, optimization and so on. However, many studies have been established to demonstrate its new proofs, refinements, extensions and generalizations. A few of these studies are ([4], [9]-[11], [13]-[17], [24]-[27], [29], [34], [35], [37]) referenced works and also the references included there.

On the other hand, Hermite-Hadamard inequality is considered for convex functions on the co-ordinates in [12], [18]. If we look at the convexity of the co-ordinates, there are a lot of definitions of co-ordinated convex function. They may be stated as follows [12]:

Received by the editors: August 02, 2019; Accepted: April 24, 2020.

2010 *Mathematics Subject Classification.* Primary 26D07, 26D10; Secondary 26D15, 26B15, 26B25.

Key words and phrases. Hermite-Hadamard's inequalities, co-ordinated convex.

Definition 1. *Let us consider a bidimensional interval* $\Delta := [a,b] \times [c,d]$ *in* $\mathbb{R}^2$ *with* $a < b$ *and* $c < d$. *A function* $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ *is said to be convex on* Δ *if the following inequality satisfies*

$$f(tx + (1-t)\ z, ty + (1-t)\ w) \leq t\ f(x,y) + (1-t)\ f(z,w)$$

for all $(x,y),(z,w) \in \Delta$ *and* $t \in [0,1]$.

A modification of definition of co-ordinated convex function was defined by Dragomir [12] as follows:

Definition 2. *A function* $f : \Delta \to \mathbb{R}$ *is said to be convex on the co-ordinates on* Δ *if the partial mappings* $f_y : [a,b] \to \mathbb{R}$, $f_y(x) = f(x,y)$ *and* $f_x : [c,d] \to \mathbb{R}$, $f_x(y) = f(x,y)$ *are convex where defined for all* $x \in [a,b]$ *and* $y \in [c,d]$.

A formal definition for co-ordinated convex function may be stated as follows:

Definition 3. *A function* $f : \Delta \to \mathbb{R}$ *is called co-ordinated convex on* Δ *if the following inequality satisfies*

$$\begin{aligned} & f(tx + (1-t)\ y, su + (1-s)\ v) \qquad (1) \\ \leq\ & ts\ f(x,u) + t(1-s) f(x,v) + s(1-t) f(y,u) + (1-t)(1-s) f(y,v) \end{aligned}$$

for all $(x,u),(y,v) \in \Delta$ *and* $t,s \in [0,1]$.

The following Hermite-Hadamard type inequalities for co-ordinated convex functions were obtained by Dragomir in [12]:

Theorem 4. *Suppose that* $f : \Delta \to \mathbb{R}$ *is co-ordinated convex, then we have the following inequalities:*

$$\begin{aligned} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\leq \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right) dx + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right) dy\right] \\ &\leq \frac{1}{(b-a)(d-c)}\int_a^b \int_c^d f(x,y)\ dydx \qquad (2) \\ &\leq \frac{1}{4}\left[\frac{1}{b-a}\int_a^b f(x,c)dx + \frac{1}{b-a}\int_a^b f(x,d)dx \right. \\ &\quad \left. + \frac{1}{d-c}\int_c^d f(a,y)dy + \frac{1}{d-c}\int_c^d f(b,y)dy\right] \\ &\leq \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4} \end{aligned}$$

The above inequalities are sharp.

The following Hermite-Hadamard type inequalities for products of two co-ordinated convex functions were given by Latif and Alomari in [18]:

Theorem 5. *Let $f, g : \Delta \to [0, \infty)$ be co-ordinated convex functions on Δ, then we have the following Hermite-Hadamard type inequalities*

$$\begin{aligned} &\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)g(x,y)dxdy \qquad (3) \\ \leq\ & \frac{1}{9}K(a,b,c,d) + \frac{1}{18}\left[L(a,b,c,d) + M(a,b,c,d)\right] + \frac{1}{36}N(a,b,c,d) \end{aligned}$$

and

$$\begin{aligned} &4f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) g\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \qquad (4) \\ \leq\ & \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)g(x,y)dxdy + \frac{5}{36}K(a,b,c,d) \\ & + \frac{7}{36}\left[L(a,b,c,d) + M(a,b,c,d)\right] + \frac{2}{9}N(a,b,c,d) \end{aligned}$$

where

$$\begin{aligned} K(a,b,c,d) &= f(a,c)g(a,c) + f(b,c)g(b,c) + f(a,d)g(a,d) + f(b,d)g(b,d), \\ L(a,b,c,d) &= f(a,c)g(b,c) + f(b,c)g(a,c) + f(a,d)g(b,d) + f(b,d)g(a,d), \\ M(a,b,c,d) &= f(a,c)g(a,d) + f(b,c)g(b,d) + f(a,d)g(a,c) + f(b,d)g(b,c) \end{aligned}$$

and

$$N(a,b,c,d) = f(a,c)g(b,d) + f(b,c)g(a,d) + f(a,d)g(b,c) + f(b,d)g(a,c).$$

Now, we give the definitions of Riemann-Liouville fractional integrals for two variable functions:

Definition 6. [28] *Let $f \in L_1([a,b] \times [c,d])$. The Riemann-Liouville fractional integrals $J_{a+,c+}^{\alpha,\beta}$, $J_{a+,d-}^{\alpha,\beta}$, $J_{b-,c+}^{\alpha,\beta}$ and $J_{b-,d-}^{\alpha,\beta}$ are defined by*

$$J_{a+,c+}^{\alpha,\beta} f(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x \int_c^y (x-t)^{\alpha-1} (y-s)^{\beta-1} f(t,s)dsdt, \quad x > a,\ y > c,$$

$$J_{a+,d-}^{\alpha,\beta} f(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x \int_y^d (x-t)^{\alpha-1} (s-y)^{\beta-1} f(t,s)dsdt, \quad x > a,\ y < d,$$

$$J_{b-,c+}^{\alpha,\beta} f(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_x^b \int_c^y (t-x)^{\alpha-1} (y-s)^{\beta-1} f(t,s)dsdt, \quad x < b,\ y > c$$

and

$$J_{b-,d-}^{\alpha,\beta}f(x,y)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_x^b\int_y^d (t-x)^{\alpha-1}(s-y)^{\beta-1}f(t,s)dsdt,\quad x<b,\ y<d.$$

The following Hermite-Hadamard type inequality utilizing co-ordinated convex functions was proved by Sarikaya in [28]:

Theorem 7. *Let $f,g:\Delta:=[a,b]\times[c,d]\to[0,\infty)$ be two co-ordinated convex on Δ with $0\le a<b$ and $0\le c<d$ and $f\in L(\Delta)$. Then for $\alpha,\beta>0$ we have the following Hermite-Hadamard type inequality*

$$\begin{aligned}
& f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \qquad (5)\\
\le\ & \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}}\\
& \times\Big[J_{a+,c+}^{\alpha,\beta}f(b,d)g(b,d)+J_{a+,d-}^{\alpha,\beta}f(b,c)g(b,c)\\
& \quad +J_{b-,c+}^{\alpha,\beta}f(a,d)g(a,d)+J_{b-,d-}^{\alpha,\beta}f(a,c)g(a,c)\Big]\\
\le\ & \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}.
\end{aligned}$$

Now, let's give the notations $A_k(x;m,n)$ and $B_k(x;m,n)$ used throughout the study:

$$A_k(x;m,n)=\int_m^n (n-x)^2w_k(x)dx,\qquad B_k(x;m,n)=\int_m^n (n-x)(x-m)w_k(x)dx$$

for $k=1,2$.

In [7], Budak gave the following inequalities which are used the main results:

Theorem 8. *Suppose that $w_1:[a,b]\to\mathbb{R}$ is non-negative, integrable and symmetric about $x=\frac{a+b}{2}$ (i.e. $w_1(x)=w_1(a+b-x)$). If $f,g:I\to\mathbb{R}$ are two real-valued, non-negative and convex functions on I, then for any $a,b\in I$, we have*

$$\int_a^b f(x)g(x)w_1(x)dx\le\frac{M(a,b)}{(b-a)^2}A_1(x;a,b)+\frac{N(a,b)}{(b-a)^2}B_1(x;a,b)\qquad (6)$$

where

$$M(a,b)=f(a)g(a)+f(b)g(b)\ \text{and}\ N(a,b)=f(a)g(b)+f(b)g(a).$$

Theorem 9. *Suppose that conditions of Theorem 8 hold, then we have the following inequality*

$$2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\int_a^b w_1(x)dx \tag{7}$$

$$\leq \int_a^b f(x)g(x)w_1(x)dx+\frac{M(a,b)}{(b-a)^2}B_1(x;a,b)+\frac{N(a,b)}{(b-a)^2}A_1(x;a,b)$$

where $M(a,b)$ and $N(a,b)$ are defined as in Theorem 8.

Many convexity is defined on co-ordinates and several inequalities are done by using these definitions. For example, Alomari and Darus proved Hadamard type inequalities for the $s-$convex functions and $\log-$convex functions on the co-ordinates in a rectangle from the plane $\mathbb{R}^2$ in [2] and [3] respectively. In [23] Ozdemir *et al.* gave Hadamard type inequalities for $h-$convex functions on the co-ordinates. For the others, please refer to ([1]-[3], [5]-[8], [12], [18]-[23],[28],[30]-[33], [36]).

The aim of this paper is to establish Hermite-Hadamard type inequalities for product of co-ordinated convex functions. The results presented in this paper provide extensions of those given in [6] and [18]

2. Main Results

Theorem 10. *Let $f,g:\Delta\subset\mathbb{R}^2\to[0,\infty)$ be co-ordinated convex functions on Δ. Also, $w_1:[a,b]\to\mathbb{R}$ is non-negative, integrable and symmetric about $x=\frac{a+b}{2}$ (i.e. $w_1(x)=w_1(a+b-x)$) and $w_2:[c,d]\to\mathbb{R}$ is non-negative, integrable and symmetric about $y=\frac{c+d}{2}$ (i.e. $w_2(y)=w_2(c+d-y)$). Then, we have the following Hermite-Hadamard type inequality*

$$\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)w_1(x)w_2(y)dydx$$

$$\leq \frac{A_2(y;c,d)}{(b-a)^3(d-c)^3}\left[K(a,b,c,d)A_1(x;a,b)+L(a,b,c,d)B_1(x;a,b)\right]$$

$$+\frac{B_2(y;c,d)}{(b-a)^3(d-c)^3}\left[M(a,b,c,d)A_1(x;a,b)+N(a,b,c,d)B_1(x;a,b)\right]$$

where $K(a,b,c,d)$, $L(a,b,c,d)$, $M(a,b,c,d)$ and $N(a,b,c,d)$ defined by as in Theorem 5.

Proof. Since f and g are co-ordinated convex functions on Δ, the functions f_x and g_x are convex on $[c,d]$. If the inequality (6) is applied for the functions f_x and g_x, then we obtain

$$\frac{1}{d-c}\int_c^d f_x(y)g_x(y)w_2(y)dy \le \frac{A_2(y;c,d)}{(d-c)^3}\left[f_x(c)g_x(c)+f_x(d)g_x(d)\right] \tag{8}$$

$$+\frac{B_2(y;c,d)}{(d-c)^3}\left[f_x(c)g_x(d)+f_x(d)g_x(c)\right].$$

That is,

$$\frac{1}{d-c}\int_c^d f(x,y)g(x,y)w_2(y)dy \le \frac{A_2(y;c,d)}{(d-c)^3}\left[f(x,c)g(x,c)+f(x,d)g(x,d)\right] \tag{9}$$

$$+\frac{B_2(y;c,d)}{(d-c)^3}\left[f(x,c)g(x,d)+f(x,d)g(x,c)\right].$$

Multiplying the inequality (9) by $\frac{w_1(x)}{(b-a)}$ and then integrating respect to x from a to b, we get

$$\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)w_1(x)w_2(y)dydx \tag{10}$$

$$\le \frac{A_2(y;c,d)}{(b-a)(d-c)^3}\int_a^b \left[f(x,c)g(x,c)+f(x,d)g(x,d)\right]w_1(x)dx$$

$$+\frac{B_2(y;c,d)}{(b-a)(d-c)^3}\int_a^b \left[f(x,c)g(x,d)+f(x,d)g(x,c)\right]w_1(x)dx.$$

Applying the inequality (6) to each integrals in (10), we have

$$\int_a^b f(x,c)g(x,c)w_1(x)dx \le \frac{A_1(x;a,b)}{(b-a)^2}\left[f(a,c)g(a,c)+f(b,c)g(b,c)\right] \tag{11}$$

$$+\frac{B_1(x;a,b)}{(b-a)^2}\left[f(a,c)g(b,c)+f(b,c)g(a,c)\right],$$

$$\int_a^b f(x,d)g(x,d)w_1(x)dx \le \frac{A_1(x;a,b)}{(b-a)^2}\left[f(a,d)g(a,d)+f(b,d)g(b,d)\right] \tag{12}$$

$$+\frac{B_1(x;a,b)}{(b-a)^2}\left[f(a,d)g(b,d)+f(b,d)g(a,d)\right],$$

$$\int_a^b f(x,c)g(x,d)w_1(x)dx \leq \frac{A_1(x;a,b)}{(b-a)^2}\left[f(a,c)g(a,d)+f(b,c)g(b,d)\right] \tag{13}$$

$$+\frac{B_1(x;a,b)}{(b-a)^2}\left[f(a,c)g(b,d)+f(b,c)g(a,d)\right]$$

and

$$\int_a^b f(x,d)g(x,c)w_1(x)dx \leq \frac{A_1(x;a,b)}{(b-a)^2}\left[f(a,d)g(a,c)+f(b,d)g(b,c)\right] \tag{14}$$

$$+\frac{B_1(x;a,b)}{(b-a)^2}\left[f(a,d)g(b,c)+f(b,d)g(a,c)\right].$$

Substituting the inequalities (11)-(14) in the inequality (10) and then arranging the result obtained, we get desired result. On the other hand, the same result is obtained by using the convexity of functions f_y and g_y. □

Theorem 11. *Let $f,g:\Delta\subset\mathbb{R}^2\to[0,\infty)$ be co-ordinated convex functions on Δ with $a<b, c<d$. Also, $w_1:[a,b]\to\mathbb{R}$ is non-negative, integrable and symmetric about $x=\frac{a+b}{2}$ (i.e. $w_1(x)=w_1(a+b-x)$) and $w_2:[c,d]\to\mathbb{R}$ is non-negative, integrable and symmetric about $y=\frac{c+d}{2}$ (i.e. $w_2(y)=w_2(c+d-y)$). Then, we have the following Hermite-Hadamard type inequality*

$$
\begin{aligned}
&4\int_a^b\int_c^d f\left(\frac{a+b}{2},\frac{c+d}{2}\right)g\left(\frac{a+b}{2},\frac{c+d}{2}\right)w_1(x)w_2(y)dydx\\
\leq\ & \int_a^b\int_c^d f(x,y)\,g(x,y)\,w_1(x)w_2(y)dydx\\
&+\frac{K(a,b,c,d)}{(b-a)^2(d-c)^2}\left[B_1(x;a,b)A_2(y;c,d)+B_2(y;c,d)A_1(x;a,b)+B_1(x;a,b)B_2(y;c,d)\right]\\
&+\frac{L(a,b,c,d)}{(b-a)^2(d-c)^2}\left[B_2(y;c,d)B_1(x;a,b)+A_2(y;c,d)A_1(x;a,b)+A_1(x;a,b)B_2(y;c,d)\right]\\
&+\frac{M(a,b,c,d)}{(b-a)^2(d-c)^2}\left[B_2(y;c,d)B_1(x;a,b)+A_2(y;c,d)A_1(x;a,b)+B_1(x;a,b)A_2(y;c,d)\right]
\end{aligned}
$$

$$+\frac{N(a,b,c,d)}{(b-a)^2(d-c)^2}\left[A_1(x;a,b)B_2(y;c,d)+A_2(y;c,d)B_1(x;a,b)+A_2(y;c,d)A_1(x;a,b)\right].$$

Proof. Since f and g are co-ordinated convex functions on Δ, the functions f_x, g_x, f_y and g_y are convex. Applying the inequality (7) for the functions $f_{\frac{c+d}{2}}$ and $g_{\frac{c+d}{2}}$ with $y=\frac{c+d}{2}$ and then multiplying both sides of the result obtained by $2\int\limits_c^d w_2(y)dy$, we get

$$4f\left(\frac{a+b}{2},\frac{c+d}{2}\right)g\left(\frac{a+b}{2},\frac{c+d}{2}\right)\int\limits_a^b\int\limits_c^d w_1(x)w_2(y)dydx \tag{15}$$

$$\leq 2\int\limits_a^b\int\limits_c^d f\left(x,\frac{c+d}{2}\right)g\left(x,\frac{c+d}{2}\right)w_1(x)w_2(y)dydx$$

$$+\left\{2\int\limits_c^d\left[\frac{f\left(a,\frac{c+d}{2}\right)g\left(a,\frac{c+d}{2}\right)+f\left(b,\frac{c+d}{2}\right)g\left(b,\frac{c+d}{2}\right)}{(b-a)^2}\right]w_2(y)dy\right\}B_1(x;a,b)$$

$$+\left\{2\int\limits_c^d\left[\frac{f\left(a,\frac{c+d}{2}\right)g\left(b,\frac{c+d}{2}\right)+f\left(b,\frac{c+d}{2}\right)g\left(a,\frac{c+d}{2}\right)}{(b-a)^2}\right]w_2(y)dy\right\}A_1(x;a,b).$$

Similarly, if we apply the inequality (7) for the functions $f_{\frac{a+b}{2}}$ and $g_{\frac{a+b}{2}}$ with $x=\frac{a+b}{2}$ and then multiply both sides of the result obtained by $2\int\limits_a^b w_1(x)dx$, we get

$$4f\left(\frac{a+b}{2},\frac{c+d}{2}\right)g\left(\frac{a+b}{2},\frac{c+d}{2}\right)\int\limits_a^b\int\limits_c^d w_1(x)w_2(y)dydx \tag{16}$$

$$\leq 2\int\limits_a^b\int\limits_c^d f\left(\frac{a+b}{2},y\right)g\left(\frac{a+b}{2},y\right)w_1(x)w_2(y)dydx$$

$$+\left\{2\int\limits_a^b\left[\frac{f\left(\frac{a+b}{2},c\right)g\left(\frac{a+b}{2},c\right)+f\left(\frac{a+b}{2},d\right)g\left(\frac{a+b}{2},d\right)}{(d-c)^2}\right]w_1(x)dx\right\}B_2(y;c,d)$$

$$+\left\{2\int\limits_a^b\left[\frac{f\left(\frac{a+b}{2},c\right)g\left(\frac{a+b}{2},d\right)+f\left(\frac{a+b}{2},d\right)g\left(\frac{a+b}{2},c\right)}{(d-c)^2}\right]w_1(x)dx\right\}A_2(y;c,d).$$

Using the inequality (7) for each integrals in inequalities (15) and (16), we have

$$2f\left(a,\frac{c+d}{2}\right)g\left(a,\frac{c+d}{2}\right)\int_c^d w_2(y)dy \leq \int_c^d f(a,y)g(a,y)w_2(y)dy \tag{17}$$

$$+\left[\frac{f(a,c)g(a,c)+f(a,d)g(a,d)}{(d-c)^2}\right]B_1(y;c,d)$$

$$+\left[\frac{f(a,c)g(a,d)+f(a,d)g(a,c)}{(d-c)^2}\right]A_1(y;c,d),$$

$$2f\left(b,\frac{c+d}{2}\right)g\left(b,\frac{c+d}{2}\right)\int_c^d w_2(y)dy \leq \int_c^d f(b,y)g(b,y)w_2(y)dy \tag{18}$$

$$+\left[\frac{f(b,c)g(b,c)+f(b,d)g(b,d)}{(d-c)^2}\right]B_1(y;c,d)$$

$$+\left[\frac{f(b,c)g(b,d)+f(b,d)g(b,c)}{(d-c)^2}\right]A_1(y;c,d),$$

$$2f\left(a,\frac{c+d}{2}\right)g\left(b,\frac{c+d}{2}\right)\int_c^d w_2(y)dy \leq \int_c^d f(a,y)g(b,y)w_2(y)dy \tag{19}$$

$$+\left[\frac{f(a,c)g(b,c)+f(a,d)g(b,d)}{(d-c)^2}\right]B_1(y;c,d)$$

$$+\left[\frac{f(a,c)g(b,d)+f(a,d)g(b,c)}{(d-c)^2}\right]A_1(y;c,d),$$

$$2f\left(b,\frac{c+d}{2}\right)g\left(a,\frac{c+d}{2}\right)\int_c^d w_2(y)dy \leq \int_c^d f(b,y)g(a,y)w_2(y)dy \tag{20}$$

$$+\left[\frac{f(b,c)g(a,c)+f(b,d)g(a,d)}{(d-c)^2}\right]B_1(y;c,d)$$

$$+\left[\frac{f(b,c)g(a,d)+f(b,d)g(a,c)}{(d-c)^2}\right]A_1(y;c,d),$$

$$2f\left(\frac{a+b}{2},c\right)g\left(\frac{a+b}{2},c\right)\int_a^b w_1(x)dx \leq \int_a^b f(x,c)g(x,c)w_1(x)dx \tag{21}$$

$$+\left[\frac{f(a,c)g(a,c)+f(b,c)g(b,c)}{(b-a)^2}\right]B_1(x;a,b)$$

$$+\left[\frac{f(a,c)g(b,c)+f(b,c)g(a,c)}{(b-a)^2}\right]A_1(x;a,b),$$

$$2f\left(\frac{a+b}{2},d\right)g\left(\frac{a+b}{2},d\right)\int_a^b w_1(x)dx \leq \int_a^b f(x,d)g(x,d)w_1(x)dx \tag{22}$$

$$+\left[\frac{f(a,d)g(a,d)+f(b,d)g(b,d)}{(b-a)^2}\right]B_1(x;a,b)$$

$$+\left[\frac{f(a,d)g(b,d)+f(b,d)g(a,d)}{(b-a)^2}\right]A_1(x;a,b),$$

$$2f\left(\frac{a+b}{2},d\right)g\left(\frac{a+b}{2},d\right)\int\limits_a^b w_1(x)dx \le \int\limits_a^b f(x,d)g(x,d)w_1(x)dx \tag{23}$$
$$+\left[\frac{f(a,d)g(a,d)+f(b,d)g(b,d)}{(b-a)^2}\right]B_1(x;a,b)$$
$$+\left[\frac{f(a,d)g(b,d)+f(b,d)g(a,d)}{(b-a)^2}\right]A_1(x;a,b),$$

$$2f\left(\frac{a+b}{2},c\right)g\left(\frac{a+b}{2},d\right)\int\limits_a^b w_1(x)dx \le \int\limits_a^b f(x,c)g(x,d)w_1(x)dx \tag{24}$$
$$+\left[\frac{f(a,c)g(a,d)+f(b,c)g(b,d)}{(b-a)^2}\right]B_1(x;a,b)$$
$$+\left[\frac{f(a,c)g(b,d)+f(b,c)g(a,d)}{(b-a)^2}\right]A_1(x;a,b)$$

and

$$2f\left(\frac{a+b}{2},d\right)g\left(\frac{a+b}{2},c\right)\int\limits_a^b w_1(x)dx \le \int\limits_a^b f(x,d)g(x,c)w_1(x)dx \tag{25}$$
$$+\left[\frac{f(a,d)g(a,c)+f(b,d)g(b,c)}{(b-a)^2}\right]B_1(x;a,b)$$
$$+\left[\frac{f(a,d)g(b,c)+f(b,d)g(a,c)}{(b-a)^2}\right]A_1(x;a,b).$$

When the inequalities (17)-(25) is written in (15) and (16) and then the results obtained are added side by side and rearranged, we obtain

$$
\begin{aligned}
& 8f\left(\frac{a+b}{2},\frac{c+d}{2}\right)g\left(\frac{a+b}{2},\frac{c+d}{2}\right)\int_a^b\int_c^d w_1(x)w_2(y)dydx \qquad (26)\\
\leq\ & 2\int_a^b\int_c^d f\left(x,\frac{c+d}{2}\right)g\left(x,\frac{c+d}{2}\right)w_1(x)w_2(y)dydx\\
& +2\int_a^b\int_c^d f\left(\frac{a+b}{2},y\right)g\left(\frac{a+b}{2},y\right)w_1(x)w_2(y)dydx\\
& +\frac{B_1(x;a,b)}{(b-a)^2}\int_c^d\left[f(a,y)g(a,y)+f(b,y)g(b,y)\right]w_2(y)dy\\
& +\frac{A_1(x;a,b)}{(b-a)^2}\int_c^d\left[f(a,y)g(b,y)+f(b,y)g(a,y)\right]w_2(y)dy\\
& +\frac{B_2(y;c,d)}{(d-c)^2}\int_a^b\left[f(x,c)g(x,c)+f(x,d)g(x,d)\right]w_1(x)dx\\
& +\frac{A_2(y;c,d)}{(d-c)^2}\int_a^b\left[f(x,c)g(x,d)+f(x,d)g(x,c)\right]w_1(x)dx\\
& +\frac{2K(a,b,c,d)}{(b-a)^2(d-c)^2}B_1(x;a,b)B_2(y;c,d)\\
& +\frac{2L(a,b,c,d)}{(b-a)^2(d-c)^2}A_1(x;a,b)B_2(y;c,d)\\
& +\frac{2M(a,b,c,d)}{(b-a)^2(d-c)^2}B_1(x;a,b)A_2(y;c,d)\\
& +\frac{2N(a,b,c,d)}{(b-a)^2(d-c)^2}A_1(x;a,b)A_2(y;c,d).
\end{aligned}
$$

The inequality (7) is applied to $f\left(x,\frac{c+d}{2}\right)g\left(x,\frac{c+d}{2}\right)$ and then the result is multiplied by $w_1(x)$ and integrated over $[a,b]$,we get

$$
2\int_a^b\int_c^d f\left(x,\frac{c+d}{2}\right)g\left(x,\frac{c+d}{2}\right)w_1(x)w_2(y)dydx \qquad (27)
$$

$$\leq \int_a^b \int_c^d f(x,y)g(x,y)w_1(x)w_2(y)dydx$$

$$+\frac{B_2(y;c,d)}{(d-c)^2}\int_a^b \left[f(x,c)g(x,c)+f(x,d)g(x,d)\right]w_1(x)dx$$

$$+\frac{A_2(y;c,d)}{(d-c)^2}\int_a^b \left[f(x,c)g(x,d)+f(x,d)g(x,c)\right]w_1(x)dx.$$

Similarly, if we apply the inequality (7) to $f\left(\frac{a+b}{2},y\right)g\left(\frac{a+b}{2},y\right)$ and then the result is multiplied by $w_2(y)$ and integrated over $[c,d]$,we get

$$2\int_a^b \int_c^d f\left(\frac{a+b}{2},y\right)g\left(\frac{a+b}{2},y\right)w_1(x)w_2(y)dydx \tag{28}$$

$$\leq \int_a^b \int_c^d f(x,y)g(x,y)w_1(x)w_2(y)dydx$$

$$+\frac{B_1(x;a,b)}{(b-a)^2}\int_c^d \left[f(a,y)g(a,y)+f(b,y)g(b,y)\right]w_2(y)dy$$

$$+\frac{A_1(x;a,b)}{(b-a)^2}\int_c^d \left[f(a,y)g(b,y)+f(b,y)g(a,y)\right]w_2(y)dy.$$

Substituting the inequalities (27) and (28) in the inequality (26) and reordering the results obtained, we have

$$8f\left(\frac{a+b}{2},\frac{c+d}{2}\right)g\left(\frac{a+b}{2},\frac{c+d}{2}\right)\int_a^b \int_c^d w_1(x)w_2(y)dydx \tag{29}$$

$$\leq 2\int_a^b \int_c^d f(x,y)g(x,y)w_1(x)w_2(y)dydx$$

$$+\frac{2B_1(x;a,b)}{(b-a)^2}\int_c^d \left[f(a,y)g(a,y)+f(b,y)g(b,y)\right]w_2(y)dy$$

$$+\frac{2A_1(x;a,b)}{(b-a)^2}\int_c^d [f(a,y)g(b,y)+f(b,y)g(a,y)]\,w_2(y)dy$$

$$+\frac{2B_2(y;c,d)}{(d-c)^2}\int_a^b [f(x,c)g(x,c)+f(x,d)g(x,d)]\,w_1(x)dx$$

$$+\frac{2A_2(y;c,d)}{(d-c)^2}\int_a^b [f(x,c)g(x,d)+f(x,d)g(x,c)]\,w_1(x)dx$$

$$+\frac{2K(a,b,c,d)}{(b-a)^2(d-c)^2}B_1(x;a,b)B_2(y;c,d)$$

$$+\frac{2L(a,b,c,d)}{(b-a)^2(d-c)^2}A_1(x;a,b)B_2(y;c,d)$$

$$+\frac{2M(a,b,c,d)}{(b-a)^2(d-c)^2}B_1(x;a,b)A_2(y;c,d)$$

$$+\frac{2N(a,b,c,d)}{(b-a)^2(d-c)^2}A_1(x;a,b)A_2(y;c,d).$$

By applying the inequality (6) to each integral in (29) and later rearranging the results obtained, we obtain desired inequality. □

Remark 12. *If we choose $w_1(x)=1$ and $w_2(y)=1$ in Theorem 10 and Theorem 11, we get (3) and (4) respectively.*

Remark 13. *If we choose $w_1(x)=\frac{\alpha}{(b-a)^{\alpha-1}}\left[(b-x)^{\alpha-1}+(x-a)^{\alpha-1}\right]$ with $\alpha>0$ and $w_2(y)=\frac{\beta}{(d-c)^{\beta-1}}\left[(d-y)^{\beta-1}+(y-c)^{\beta-1}\right]$ with $\beta>0$ in Theorem 10 and Theorem 11, we get*

$$\begin{aligned}
&\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}}\\
&\times\Big[J^{\alpha,\beta}_{a+,c+}f(b,d)g(b,d)+J^{\alpha,\beta}_{a+,d-}f(b,c)g(b,c)\\
&\quad +J^{\alpha,\beta}_{b-,c+}f(a,d)g(a,d)+J^{\alpha,\beta}_{b-,d-}f(a,c)g(a,c)\Big]\\
\leq\ &\left[\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right]\left[\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right]K(a,b,c,d)
\end{aligned}$$

$$+\left[\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right]\left[\frac{\alpha}{(\alpha+1)(\alpha+2)}\right]L(a,b,c,d)$$
$$+\left[\frac{\beta}{(\beta+1)(\beta+2)}\right]\left[\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right]M(a,b,c,d)$$
$$+\left[\frac{\beta}{(\beta+1)(\beta+2)}\right]\left[\frac{\alpha}{(\alpha+1)(\alpha+2)}\right]N(a,b,c,d)$$

and

$$\begin{aligned}
&4f\left(\frac{a+b}{2},\frac{c+d}{2}\right)g\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\
\leq\;& \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}}\\
&\times\Big[J_{a+,c+}^{\alpha,\beta}f(b,d)g(b,d)+J_{a+,d-}^{\alpha,\beta}f(b,c)g(b,c)\\
&\quad+J_{b-,c+}^{\alpha,\beta}f(a,d)g(a,d)+J_{b-,d-}^{\alpha,\beta}f(a,c)g(a,c)\Big]\\
&+\left\{\frac{\alpha}{2(\alpha+1)(\alpha+2)}+\left[\frac{\beta}{(\beta+1)(\beta+2)}\right]\left[\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right]\right\}K(a,b,c,d)\\
&+\left\{\frac{1}{2}\left[\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right]+\left[\frac{\alpha}{(\alpha+1)(\alpha+2)}\right]\left[\frac{\beta}{(\beta+1)(\beta+2)}\right]\right\}L(a,b,c,d)\\
&+\left\{\frac{1}{2}\left[\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right]+\left[\frac{\alpha}{(\alpha+1)(\alpha+2)}\right]\left[\frac{\beta}{(\beta+1)(\beta+2)}\right]\right\}M(a,b,c,d)\\
&+\left\{\frac{1}{4}-\left[\frac{\alpha}{(\alpha+1)(\alpha+2)}\right]\left[\frac{\beta}{(\beta+1)(\beta+2)}\right]\right\}N(a,b,c,d)
\end{aligned}$$

which is proved by Budak and Sarikaya [6].

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