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q-QUASINORMAL OPERATORS AND ITS EXTENDED EIGENVALUES

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Abstract. In this paper, the relation between q-deformed quasinormal operators and q-quasinormal operator classes is investigated. Moreover, we prove that these are same. Also, we consider the extended eigenvalue problems for bounded q-quasinormal operators.

1. Introduction

Let q be a positive number not equal to one and A be a closed operator with dense domain on a separable Hilbert space H. If A satisfies

$$AA^* = qA^*A,$$

then A is said to be a deformed normal operator with deformation parameter q or a q-normal operator. A nonzero q-normal operator is always unbounded [17, 18]. Also, if A is a closed operator with dense domain in H and its polar decomposition $A = U|A|$ such that

$$U|A| \subset \sqrt{q}|A|U,$$

then A is called a deformed quasinormal operator with deformation parameter q or a q-quasinormal operator. Every nonzero q-quasinormal operator is unbounded [1]. The basic properties for q-deformed operators can be found in [1, 2, 3, 4, 5].

Moreover, S. Lohaj defined that the bounded operator A is a q-quasinormal operator, if the equation

$$AA^*A = qA^*AA$$

is hold [6]. He showed that if any invertible operator is q-quasinormal then $q = 1$ [6]. It is clear that a bounded deformed at quasinormal with deformation parameter q operator is q-quasinormal.

A complex number λ is said to be an extended eigenvalue of a bounded operator A if there exists an operator $X \neq 0$ such that

$$XA = \lambda AX.$$

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X is called a λ eigenoperator for A and the set of extended eigenvalues is denoted by $\sigma_{ext}(A)$ [8]. Also, the extended spectrum of bounded operators has been studied by many authors such as [8, 9, 10, 11, 12, 13, 14, 15, 16]. Biswas and Petrovic proved the result

$$\sigma_{ext}(A) \subset \{\lambda \in \mathbb{C} : \sigma(A) \cap \sigma(\lambda A) \neq \emptyset\}$$

where $\sigma(A)$ is the set of spectrum of A [9].

2. q-Quasinormal Operators and Its Extended Eigenvalues

In this paper, all operators are assumed to be linear. Let us denote by H a complex separable Hilbert space. For an operator A in H, the range and the kernel of A are denoted by $R(A)$ and $KerA$, respectively.

Lemma 2.1. *$A : H \to H$ is a q-quasinormal operator if and only if the equation $U|A|^2 = q|A|^2U$ is hold.*

Proof. Let $A : H \to H$ be a q-quasinormal operator. By the q-quasinormal definition, the equation

$$A(A^*A) = q(A^*A)A$$

is satisfied. Since its polar decomposition is $A = U|A|$, then the equation

$$U|A|^3 - q|A|^2U|A| = (U|A|^2 - q|A|^2U)|A| = 0$$

is hold. When $H = Ker|A| \oplus R(|A|)$ and $Ker(|A|) = KerU$ are hold, then

$$U|A|^2 = q|A|^2U$$

is satisfied. □

Corollary 2.2. *Let A be a closed operator with dense domain in H and $A = U|A|$ be the polar decomposition. The following statements are equivalent.*
i) A is q-quasinormal.
ii)For all $a \in \mathbb{R}$,

$$Ue^{ia|A|^2} = e^{iqa|A|^2}U, i = \sqrt{-1}$$

iii)For all $\lambda \in \mathbb{C}$ with $Im\lambda \neq 0$,

$$U(\lambda - |A|^2)^{-1} = (\lambda - q|A|^2)^{-1}U.$$

iv) For all Borel sets M,

$$E(q^{-1}M)U = UE(M),$$

where $E(.)$ is the spectral measure of $|A|$.

Every q-quasinormal operator A satisfies the relation

$$Ug(|A|^2) = g(q|A|^2)U$$

for any Borel function g.

Proof. It can be proved by using the method as in [1]. □

Corollary 2.3. *If A is a bounded q-quasinormal operator in a Hilbert space iff A is a deformed at quasinormal with deformation parameter q.*

Theorem 2.4. *Suppose that $A : H \to H$ is a q-quasinormal operator, in this case $\sigma_{ext}(A) = \mathbb{C}$ is hold.*

Proof. Let $A = U|A|$ where U is a partial isometry and $|A|$ is the square root of A^*A such that $KerU = Ker\,|A|$, be the polar decomposition of A . Since is a q-quasinormal operator, $U\,|A| = \sqrt{q}\,|A|\,U$,for $q > 1$ is true by Corollary 2.3.

Firstly, we assume that $0 \in \sigma_p\,(A)$, then there exists an element y in $H\setminus\{0\}$ such that $Ay = 0$ and for every $x \in H$,

$$A\,(y \otimes y)\,x = A\,(x, y)\,y = (x, y)\,Ay = 0$$

and

$$(y \otimes y)\,Ax = (y \otimes y)\,U\,|A|\,x = (U\,|A|\,x, y)\,y = \sqrt{q}\,(x, U^*\,|A|\,y)\,y = 0.$$

Then,

$$(y \otimes y)\,U|A| = U|A|\,(y \otimes y) = 0$$

is obtained. This means that $\sigma_{ext}\,(A) = \mathbb{C}$ since $0 \in \sigma_p\,(A)$.

Now, let $A : H \to H$ be a q-quasinormal operator such that $0 \notin \sigma_p\,(A)$, in this case, the equation

$$AA^*A = qA^*AA$$

is hold. Since A is a bounded operator, we have $AA^* - qA^*A \neq 0$ and

$$(AA^* - qA^*A)A = 0A(AA^* - qA^*A).$$

Consequently, the zero is an extended spectrum of A. Because of $0 \notin \sigma_p(A)$, U is an isometry. Also, from the von Neuman-Wold decomposition the equality

$$H = \overset{\infty}{\underset{n=0}{\oplus}} U^n(KerU^*)$$

is verified and subspaces $U^n\,(KerU^*)$, n is a nonnegative integer, are invariant under $|A|$ [7].

Moreover, it is defined $T_\lambda := \sum_{n=0}^{\infty} \lambda^n P_n$ such that $0 < |\lambda| \leqslant 1$, where P_n are projection operators on $U^n(KerU^*)$ for all $n \geqslant 0$. It is clear that T_λ is a bonded operator for all $0 < |\lambda| \leqslant 1$. Also, the following equations

$$T_\lambda|A| = |A|T_\lambda$$

$$T_\lambda U = \lambda U T_\lambda$$

are satisfied, so

$$U^nT_\lambda A = (U^nT_\lambda)U|A| = q^{n/2}\lambda|A|(U^nT_\lambda) = q^{n/2}\lambda AU^nT_\lambda, n \geqslant 0.$$

Since $q > 1$ and $0 < |\lambda| \leqslant 1$, $\sigma_{ext}(A) = \mathbb{C}$ is obtained. □

Example 2.1. *Let H be a separable Hilbert space. If $\{e_n\}, n \geqslant 0$ is an orthonormal basis of H, and a sequence $\{w_n\}, w_n \neq 0, n \geqslant 0$ of complex numbers such that*

$$D(S_u) = \{\sum_{n=0}^{\infty} \alpha_n e_n \in H : \sum_{n=0}^{\infty} \mid \alpha_n \mid^2 \mid w_n \mid^2 < \infty\}$$

and

$$S_u e_n = w_n e_{n+1}$$

for all $n \geqslant 0$, then S_u is called a unilateral weighted shift with weights w_n. A unilateral weighted shift S_u in H with weights w_n is q-quasinormal if and only if

$$\mid w_n \mid = \left(\frac{1}{\sqrt{q}}\right)^n \mid w_0 \mid$$

for all $n \geqslant 0$ [1]. In particular, for $q > 1$ a unilateral weighted shift is a bounded q-quasinormal and $\sigma_p(S_u) = \emptyset$ [1]. Then, from Theorem 2.4 $\sigma_{ext}(S_u) = \mathbb{C}$.

Corollary 2.5. *Let $A : H \to H$ be a q-quasinormal operator, then for every $n \in \mathbb{N}$, $q^{n/2} \in \sigma_{ext}(|A|)$.*

Corollary 2.6. *Let $A : H \to H$ be a q-quasinormal operator, if $\lambda \in \sigma_{ext}(|A|)$, then for every $n \in \mathbb{N}$, $q^{\frac{n}{2}}\lambda \in \sigma_{ext}(|A|)$.*

Corollary 2.7. *If A is a bounded q-quasinormal operator and $0 \notin \sigma_p(A)$, then $0 \in \sigma_c(|A|)$.*

Proof. Let $A : H \to H$ be a q-quasinormal operator, then for every $n \in \mathbb{N}$, $\sigma(|A|) \cap \sigma(q^{n/2}|A|) \neq \emptyset$. □

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