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CONVERGENCE OF NOOR, AND ABBAS AND NAZIR ITERATION PROCEDURES FOR A CLASS OF THREE NONLINEAR QUASI CONTRACTIVE MAPS IN CONVEX METRIC SPACES

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Abstract. We define Noor iteration procedure and, Abbas and Nazir iteration procedure associated with three self maps in the setting of convex metric spaces. We prove that these iterations converge strongly to a unique common fixed point of three nonlinear quasi-contractive self maps in convex metric spaces. One of our results (Theorem 2.2) extend the result of Sastry, Babu and Srinivasa Rao [10] to three self maps. Examples are provided to illustrate our results.

1. Introduction

In 1970, Takahashi [11] introduced the concept of convexity in metric spaces as follows.

Definition 1.1. *Let (X, d) be a metric space. A map $W : X \times X \times [0, 1] \to X$ is said to be a 'convex structure' on X if*

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y) \tag{1.1}$$

for $x, y, u \in X$ and $\lambda \in [0, 1]$.

A metric space (X, d) together with a convex structure W is called a *convex metric space* and we denote it by (X, d, W).

A nonempty subset K of X is said to be *'convex'* if $W(x, y, \lambda) \in K$ for $x, y \in K$ and $\lambda \in [0, 1]$.

Remark 1.1. *Every normed linear space $(X, ||.||)$ is a convex metric space with the convex structure W defined by $W(x, y, \lambda) = \lambda x + (1 - \lambda)y$ for $x, y \in X$, and*

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$\lambda \in [0,1]$. *But, there are convex metric spaces which are not normed linear spaces* [2, 7, 11].

In 1974, Ćirić [3] introduced quasi-contraction maps in the setting of metric spaces and proved that the Picard iterative sequence converges to the fixed point in complete metric spaces.

Definition 1.2. *Let (X,d) be a metric space. A self map $T : X \to X$ is said to be a quasi-contraction map if there exists a real number $0 \leq k < 1$ such that*

$$d(Tx,Ty) \leq kM(x,y) \tag{1.2}$$

where

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\} \tag{1.3}$$

for $x, y \in X$.

In 1974, Ishikawa [6] introduced an iteration procedure in the setting of normed linear spaces as follows: Let K be a nonempty convex subset of a normed linear space X and let $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ be sequences in $[0,1]$.
For $x_0 \in K$,

$$\begin{aligned} y_n &= (1-\beta_n)x_n + \beta_n T x_n \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n, \text{ for } n = 0,1,2,\dots . \end{aligned} \tag{1.4}$$

In 1988, Ding [5] considered Ishikawa iteration procedure in the setting of convex metric spaces as follows: Let K be a nonempty convex subset of a convex metric space (X,d,W), and let $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ be sequences in $[0,1]$.
For $x_0 \in K$,

$$\begin{aligned} y_n &= W(Tx_n, x_n, \beta_n) \\ x_{n+1} &= W(Ty_n, x_n, \alpha_n) \text{ for } n = 0,1,2,..., \end{aligned} \tag{1.5}$$

and proved that the Ishikawa iteration procedure (1.5) converges strongly to a unique fixed point of a quasi-contraction map in the setting of convex metric spaces, provided $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$.

In 1999, Ćirić [4] introduced a more general quasi-contraction map and proved the convergence of the Ishikawa iteration procedure to a unique fixed point in convex metric spaces and the result is the following.

Theorem 1.1. (Ćirić [4]) *Let K be a nonempty closed convex subset of a complete convex metric space X and let $T : K \to K$ be a self map satisfying*

$$d(Tx,Ty) \leq w(M(x,y)),$$

where $M(x,y)$ is defined by (1.3) for $x, y \in K$ and
$w : (0,\infty) \to (0,\infty)$ is a map which satisfies
(i) $0 < w(t) < t$ *for each* $t > 0$,
(ii) *w increases,*
(iii) $\lim\limits_{t\to\infty} (t - w(t)) = \infty$, and
(iv) either $t - w(t)$ is monotonically increasing on $(0,\infty)$ (1.6)

or

$$w(t) \text{ is strictly increasing and } \lim_{n\to\infty} w^n(t) = 0 \text{ for } t > 0. \tag{1.7}$$

Let $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ be sequences in $[0,1]$ such that $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$.

For $x_0 \in K$, the Ishikawa iteration procedure $\{x_n\}_{n=0}^{\infty}$ defined by (1.5) *converges strongly to the unique fixed point of T.*

Sastry, Babu and Srinivasa Rao [9] improved Theorem 1.1 by replacing (1.6) and (1.7) with a single condition, namely $0 < w(t^+) < t$ for each $t > 0$ and proved the following theorem.

Theorem 1.2. [9] *Let K be a nonempty closed convex subset of a complete convex metric space (X, d, W) and $T : K \to K$ be a map that satisfies*

$$d(Tx, Ty) \le w(M(x, y)) \tag{1.8}$$

where $M(x, y)$ is defined in (1.3) *for $x, y \in K$ and $w : (0, \infty) \to (0, \infty)$ is a map such that*

(*i*) *w increases,*
(*ii*) $\lim_{t\to\infty}(t - w(t)) = \infty$, *and*
(*iii*) $0 < w(t^+) < t$ *for* $t > 0$.

Let $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ be sequences in $[0, 1]$ such that $\sum_{n=0}^{\infty}\alpha_n = \infty$. Then for any $x_0 \in K$, the sequence $\{x_n\}_{n=0}^{\infty}$ generated by the iteration procedure (1.5) *converges strongly to a unique fixed point of T.*

Remark 1.2. (*i*) *and* (*iii*) *of* Theorem 1.2 *imply that $0 < w(t) < t$ for each $t > 0$.*

Remark 1.3. *If $w(t) = kt$ for $t \in (0, \infty)$ and $0 \le k < 1$ then the map T of* Theorem 1.2 *reduces to a quasi-contraction map.*

Sastry, Babu, and Srinivasa Rao [10] extended Theorem 1.2 to a pair of self maps as follows.

Theorem 1.3. [10] *Let (X, d) be a complete convex metric space with convex structure W. Let S, T be self maps of X satisfying the inequality*

$$\max\{d(Sx, Sy), d(Tx, Ty), d(Sx, Ty)\} \le w(M'(x, y)) \text{ for all } x, y \in X$$

where $M'(x, y) = \max\{d(x, y), d(x, Sx), d(x, Sy), d(y, Sx), d(x, Tx), d(y, Ty), d(y, Sy), d(x, Ty), d(y, Tx), d(Sx, Tx), d(Sy, Ty)\}$ *and* $w : (0, \infty) \to (0, \infty)$ *is a map such that*

(*i*) *w is increasing on* $(0, \infty)$,
(*ii*) $\lim_{t\to\infty}(t - w(t)) = \infty$, *and*
(*iii*) $0 < w(t^+) < t$ *for each* $t > 0$.

For $x_0 \in X$, define the Ishikawa iteration procedure associated with S and T by

$$\begin{aligned} y_n &= W(Tx_n, x_n, \beta_n) \\ x_{n+1} &= W(Sy_n, x_n, \alpha_n) \end{aligned} \tag{1.9}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequences in $(0, 1)$ with $\sum \alpha_n = \infty$. Then the sequence $\{x_n\}$ converges, $\lim_{n\to\infty} x_n = z$ (say)*, $z \in X$ and z is the unique common fixed point of S and T.*

In 2000, Noor [8] introduced a three step iteration procedure in the setting of Banach spaces as follows: For $x_0 \in K$,

$$\begin{aligned} z_n &= (1 - \gamma_n)x_n + \gamma_n Tx_n \\ y_n &= (1 - \beta_n)x_n + \beta_n Tz_n \\ x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n Ty_n \end{aligned} \tag{1.10}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are sequences in $[0,1]$.

Noor iteration procedure (1.10) in convex metric spaces is as follows:
For $x_0 \in K$,

$$\begin{aligned} z_n &= W(Tx_n, x_n, \gamma_n) \\ y_n &= W(Tz_n, x_n, \beta_n) \\ x_{n+1} &= W(Ty_n, x_n, \alpha_n) \end{aligned} \tag{1.11}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are sequences in $[0,1]$.

We call the iteration $\{x_n\}$ defined by (1.11), a *'modified Noor iteration procedure'.*

In 2014, Abbas and Nazir [1] introduced the following iteration procedure in normed linear spaces.
For $x_0 \in K$,

$$\begin{aligned} z_n &= (1-\gamma_n)x_n + \gamma_n Tx_n \\ y_n &= (1-\beta_n)Tx_n + \beta_n Tz_n \\ x_{n+1} &= (1-\alpha_n)Ty_n + \alpha_n Tz_n, \end{aligned} \tag{1.12}$$

for $n = 0, 1, 2, \ldots$.

Abbas and Nazir iteration procedure in the setting of convex metric spaces as follows: For $x_0 \in K$,

$$\begin{aligned} z_n &= W(Tx_n, x_n, \gamma_n) \\ y_n &= W(Tz_n, Tx_n, \beta_n) \\ x_{n+1} &= W(Tz_n, Ty_n, \alpha_n) \end{aligned} \tag{1.13}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are sequences in $[0,1]$.

We call the iteration $\{x_n\}$ defined by (1.13), a *'modified Abbas and Nazir iteration procedure'.*

Inspired and motivated by the results of Ćirić [4], and Sastry, Babu and Srinivasa Rao [9, 10], we define Noor iteration procedure associated with three self maps in Section 2, and prove the convergence of this iteration procedure to the common fixed point of three self maps in convex metric spaces under certain hypotheses.
In Section 3, we extend it to Abbas and Nazir iteration procedure. One of our results (Theorem 2.2) extends the result of [10] to three self maps.

2. Convergence of Noor iteration procedure

We begin this section by defining an iteration procedure in convex metric spaces as follows.

Let (X, d, W) be a convex metric space, K a nonempty convex subset of X.
Let $T_1, T_2, T_3 : K \to K$ be three self maps. For $x_0 \in K$,

$$\begin{aligned} z_n &= W(T_1x_n, x_n, \gamma_n) \\ y_n &= W(T_2z_n, x_n, \beta_n) \\ x_{n+1} &= W(T_3y_n, x_n, \alpha_n) \end{aligned} \tag{2.1}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are sequences in $[0,1]$.

We call the iteration $\{x_n\}$ defined by (2.1), a Noor iteration procedure associated with T_1, T_2 and T_3 in convex metric spaces.

Lemma 2.1. *Let (X, d, W) be a convex metric space and K be a nonempty convex subset of X. Let $T_1, T_2, T_3 : K \to K$ be three self maps satisfying the inequality*

$$\max_{i,j=1,2,3}\{d(T_ix, T_jy)\} \leq w(M_1(x,y)) \text{ for } x, y \in K \text{with } M_1(x,y) > 0, \tag{2.2}$$

where

$$M_1(x,y) = \max_{1\le i,j\le 3, i\ne j}\{d(x,y), d(x,T_ix), d(y,T_iy), d(x,T_iy), d(y,T_ix), d(T_ix,T_jx), d(T_iy,T_jy)\}, \tag{2.3}$$

$w:(0,\infty)\to(0,\infty)$ *is a map such that*

$$w \text{ increases}, \tag{2.4}$$

$$\lim_{t\to\infty}(t-w(t)) = \infty, \tag{2.5}$$

and

$$0 < w(t^+) < t \text{ for } t > 0. \tag{2.6}$$

For any $x_0 \in K$, *let* $\{x_n\}, \{y_n\}$ *and* $\{z_n\}$ *be the sequences generated by Noor iteration procedure* (2.1) *associated with three self maps* T_1, T_2, *and* T_3.

Then the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, $\{T_ix_n\}, \{T_iy_n\}$, *and* $\{T_iz_n\}$ *for* $i = 1,2,3$ *are bounded.*

Proof. For each positive integer n, we define

$A_n = \{x_k\}_{k=0}^n \cup \{y_k\}_{k=0}^n \cup \{z_k\}_{k=0}^n \cup \bigcup_{i=1}^{3}(\{T_ix_k\}_{k=0}^n \cup \{T_iy_k\}_{k=0}^n \cup \{T_iz_k\}_{k=0}^n)$ and we denote the diameter of A_n by a_n.

Let $b_n = \max_{i=1,2,3}\{\sup_{0\le k\le n} d(x_0,T_ix_k), \sup_{0\le k\le n} d(x_0,T_iy_k), \sup_{0\le k\le n} d(x_0,T_iz_k)\}$ for $n = 1,2,3...$.

We now prove that $a_n = b_n$ for $n = 1,2,...$.

Clearly, $b_n \le a_n$ for $n = 1,2,...$.

Without loss of generality, we assume that $a_n > 0$ for $n = 1,2,...$.

<u>*Case* (*i*)</u> : $a_n = d(T_ix_k, T_jx_l)$ for $0 \le k,l \le n$ and $i,j = 1,2,3$.

Since $a_n > 0$, we have $M_1(x_k,x_l) > 0$.

Therefore from the inequality (2.2) and Remark 1.2, we have

$a_n = d(T_ix_k,T_jx_l) \le w(M_1(x_k,x_l)) \le w(a_n) < a_n$,

a contradiction.

Therefore $a_n \ne d(T_ix_k,T_jx_l)$.

<u>*Case* (*ii*)</u> : By proceeding as in *Case* (*i*), it is easy to see that $a_n \ne d(T_ix_k,T_jy_l)$, $a_n \ne d(T_ix_k,T_jz_l)$, $a_n \ne d(T_iy_k,T_jy_l)$, $a_n \ne d(T_iy_k,T_jz_l)$, and $a_n \ne d(T_iz_k,T_jz_l)$ for $0 \le k,l \le n$ and $i,j = 1,2,3$.

<u>*Case* (*iii*)</u> : $a_n = d(x_k,T_iy_l)$ for $0 \le k,l \le n$ and $i = 1,2,3$.

If $k > 0$ then from the inequality (1.1), we have

$$\begin{aligned} a_n = d(x_k,T_iy_l) &= d(W(T_3y_{k-1},x_{k-1},\alpha_{k-1}),T_iy_l) \\ &\le \alpha_{k-1}d(T_3y_{k-1},T_iy_l) + (1-\alpha_{k-1})d(x_{k-1},T_iy_l) \\ &\le \max\{d(T_3y_{k-1},T_iy_l), d(x_{k-1},T_iy_l)\} \le a_n \text{ so that} \end{aligned}$$

$a_n = d(T_3y_{k-1},T_iy_l)$ or $a_n = d(x_{k-1},T_iy_l)$.

By *Case* (*ii*), $a_n \ne d(T_3y_{k-1},T_iy_l)$ and hence we have $a_n = d(x_{k-1},T_iy_l)$.

On continuing this process, we have $a_n = d(x_0,T_iy_l)$ so that $a_n \le b_n$.

<u>*Case* (*iv*)</u> : Either $a_n = d(x_k,T_ix_l)$ or $a_n = d(x_k,T_iz_l)$ for $0 \le k,l \le n$ and $i = 1,2,3$.

By proceeding as in *Case* (*iii*), it follows that $a_n \le b_n$.

<u>*Case* (*v*)</u> : $a_n = d(x_k,x_l)$ for $0 \le k,l \le n$.

Since $a_n > 0$, we have $k \ne l$. So, without loss of generality, we assume that $k < l$.

Therefore

$a_n = d(x_k, W(T_3y_{l-1},x_{l-1},\alpha_{l-1})) \le \alpha_{l-1}d(x_k,T_3y_{l-1}) + (1-\alpha_{l-1})d(x_k,x_{l-1})$

$\leq \max\{d(x_k, T_3y_{l-1}), d(x_k, x_{l-1})\} \leq a_n$ so that
either $a_n = d(x_k, T_3y_{l-1})$ or $a_n = d(x_k, x_{l-1})$.
If $a_n = d(x_k, x_{l-m})$ for every $1 \leq m \leq l-k$ then $a_n = 0$,
a contradiction.
Therefore $a_n = d(x_k, T_3y_{l-m})$ for some $1 \leq m \leq l-k$ and hence
$a_n \leq b_n$ follows from *Case* (*iii*).

<u>*Case* (*vi*)</u> : $a_n = d(x_k, y_l)$ for some $0 \leq k, l \leq n$.

$$a_n = d(x_k, W(T_2z_l, x_l, \beta_l)) \leq \beta_l d(x_k, T_2z_l) + (1-\beta_l)d(x_k, x_l)$$
$$\leq \max\{d(x_k, T_2z_l), d(x_k, x_l)\} \leq a_n \text{ so that}$$
$a_n = d(x_k, T_2z_l)$ or $a_n = d(x_k, x_l)$.
Now by *Case* (*iv*) and *Case* (*v*), it follows that $a_n \leq b_n$.

<u>*Case* (*vii*)</u> : $a_n = d(x_k, z_l)$ for some $0 \leq k, l \leq n$.

$$a_n = d(x_k, W(T_1x_l, x_l, \gamma_l)) \leq \gamma_l d(x_k, T_1x_l) + (1-\gamma_l)d(x_k, x_l)$$
$$\leq \max\{d(x_k, T_1x_l), d(x_k, x_l)\} \leq a_n \text{ so that}$$
either $a_n = d(x_k, T_1x_l)$ or $a_n = d(x_k, x_l)$.
Therefore by *Case* (*iv*) and *Case* (*v*), we have $a_n \leq b_n$.

<u>*Case* (*viii*)</u>: $a_n = d(y_k, T_ix_l)$ for $0 \leq k, l \leq n$ and $i = 1, 2, 3$.

$$a_n = d(W(T_2z_k, x_k, \beta_k), T_ix_l) \leq \beta_k d(T_2z_k, T_ix_l) + (1-\beta_k)d(x_k, T_ix_l)$$
$$\leq \max\{d(T_2z_k, T_ix_l), d(x_k, T_ix_l)\} \leq a_n \text{ so that}$$
$a_n = d(T_2z_k, T_ix_l)$ or $a_n = d(x_k, T_ix_l)$.
Hence by *Case* (*ii*) and *Case* (*iv*), we have $a_n \leq b_n$.

<u>*Case* (*ix*)</u> : Either $a_n = d(y_k, T_iy_l)$ or $a_n = d(y_k, T_iz_l)$ for $0 \leq k, l \leq n$ and $i = 1, 2, 3$.
By proceeding as in *Case* (*viii*), it is easy to see that $a_n \leq b_n$.

<u>*Case* (*x*)</u> : $a_n = d(y_k, y_l)$ for $0 \leq k, l \leq n$.

$$a_n = d(y_k, y_l) = d(y_k, W(T_2z_l, x_l, \beta_l)) \leq \beta_l d(y_k, T_2z_l) + (1-\beta_l)d(y_k, x_l)$$
$$\leq \max\{d(y_k, T_2z_l), d(y_k, x_l)\} \leq a_n \text{ so that}$$
either $a_n = d(y_k, T_2z_l)$ or $a_n = d(x_l, y_k)$.
Hence $a_n \leq b_n$ follows from *Case* (*ix*) and *Case* (*vi*).

<u>*Case* (*xi*)</u> : $a_n = d(y_k, z_l)$ for $0 \leq k, l \leq n$.

$$a_n = d(y_k, W(T_1x_l, x_l, \gamma_l)) \leq \gamma_l d(y_k, T_1x_l) + (1-\gamma_l)d(y_k, x_l)$$
$$\leq \max\{d(y_k, T_1x_l), d(y_k, x_l)\} \leq a_n \text{ so that}$$
either $a_n = d(y_k, T_1x_l)$ or $a_n = d(x_l, y_k)$.
By *Case* (*viii*) and *Case* (*vi*), we have $a_n \leq b_n$.

<u>*Case* (*xii*)</u> : $a_n = d(z_k, T_ix_l)$ for $0 \leq k, l \leq n$ and $i = 1, 2, 3$.

$$a_n = d(z_k, T_ix_l) = d(W(T_1x_k, x_k, \gamma_k), T_ix_l) \leq \gamma_k d(T_1x_k, T_ix_l) + (1-\gamma_k)d(x_k, T_ix_l)$$
$$\leq \max\{d(T_1x_k, T_ix_l), d(x_k, T_ix_l)\} \leq a_n$$
so that either $a_n = d(T_1x_k, T_ix_l)$ or $d(x_k, T_ix_l)$.
Therefore by using *Case* (*i*) we have $a_n \neq d(T_1x_k, T_ix_l)$ and hence $a_n = d(x_k, T_ix_l)$.
Now by *Case* (*iv*), it follows that $a_n \leq b_n$.

<u>*Case* (*xiii*)</u> : Either $a_n = d(z_k, T_iy_l)$ or $a_n = d(z_k, T_iz_l)$ for $0 \leq k, l \leq n$ and $i = 1, 2, 3$.
By proceeding as in *Case* (*xii*), it is easy to see that $a_n \leq b_n$.

<u>*Case* (*xiv*)</u> : $a_n = d(z_k, z_l)$ for $0 \leq k, l \leq n$.

$$a_n = d(z_k, z_l) = d(z_k, W(T_1x_l, x_l, \gamma_l)) \leq \gamma_l d(z_k, T_1x_l) + (1-\gamma_l)d(z_k, x_l)$$
$$\leq \max\{d(z_k, T_1x_l), d(z_k, x_l)\} \leq a_n$$
so that $a_n = d(z_k, T_1x_l)$ or $a_n = d(z_k, x_l)$.

By *Case* (xii) and *Case* (vii), we have $a_n \leq b_n$.
Hence by considering all the above cases, we have $a_n = b_n$ for $n = 1, 2, 3...$.

We write $A = \max_{i=1,2,3}\{d(x_0, T_i x_0)\}$. Without loss of generality, we assume that $A > 0$. Now by using the inequality (2.2), we have
$d(x_0, T_i x_k) \leq d(x_0, T_i x_0) + d(T_i x_0, T_i x_k) \leq A + w(a_n)$ for $0 \leq k \leq n$ and $i = 1, 2, 3$.
Therefore $\sup_{0\leq k\leq n}\{d(x_0, T_i x_k)\} \leq A + w(a_n)$ for $i = 1, 2, 3$.
Similarly, we have $\sup_{0\leq k\leq n}\{d(x_0, T_i y_k)\} \leq A + w(a_n)$ and
$\sup_{0\leq k\leq n}\{d(x_0, T_i z_k)\} \leq A + w(a_n)$ for $i = 1, 2, 3$ so that

$$b_n \leq A + w(a_n).$$

Since $a_n = b_n$, we have

$$a_n - w(a_n) \leq A \text{ for } n = 1, 2, ... \tag{2.7}$$

If the sequence $\{a_n\}$ is not bounded then $\lim_{n\to\infty} a_n = \infty$ and hence it follows from (2.5) that $\lim_{n\to\infty}(a_n - w(a_n)) = \infty$ which contradicts (2.7).

Therefore the sequence $\{a_n\}$ is bounded and hence the conclusion of the lemma follows. □

Theorem 2.2. *Let (X, d, W) be a complete convex metric space and K be a nonempty closed convex subset of X. Let $T_1, T_2, T_3 : K \to K$ be self maps satisfying the inequality*

$$\max_{i,j=1,2,3}\{d(T_i x, T_j y)\} \leq w(M_1(x, y)) \textit{ for } x, y \in K \textit{ with } M_1(x, y) > 0,$$

where $M_1(x, y)$ is defined by (2.3) and let $w : (0, \infty) \to (0, \infty)$ be a map that satisfies the relations (2.4), (2.5), and (2.6). Let $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, and $\{\gamma_n\}_{n=0}^{\infty}$ be sequences in $[0, 1]$ such that $\sum_{n=0}^{\infty}\alpha_n = \infty$. Then the sequence $\{x_n\}$ generated by the Noor iteration procedure associated with three self maps (2.1) converges strongly to a unique common fixed point of T_1, T_2 and T_3.

Proof. Without loss of generality, we assume that $x_n \neq T_i x_n$ for any $n = 0, 1, 2, ...$ and $i = 1, 2, 3$.
For every integer $n \geq 0$, we define a set C_n by
$C_n = \{x_k\}_{k=n}^{\infty} \cup \{y_k\}_{k=n}^{\infty} \cup \{z_k\}_{k=n}^{\infty} \cup \bigcup_{i=1}^{3}(\{T_i x_k\}_{k=n}^{\infty} \cup \{T_i y_k\}_{k=n}^{\infty} \cup \{T_i z_k\}_{k=n}^{\infty})$, and we define c_n to be the diameter of C_n.
By Lemma 2.1, we have the sequence $\{c_n\}$ is bounded.
Let $d_n = \max_{i=1,2,3}\{\sup_{k\geq n} d(x_n, T_i x_k), \sup_{k\geq n} d(x_n, T_i y_k), \sup_{k\geq n} d(x_n, T_i z_k)\}$ for $n = 0, 1, 2, ...$.

Now, we prove that $c_n = d_n$ for $n = 0, 1, ...$.
Without loss of generality, we assume that $c_n > 0$.
By using the same technique discussed in Lemma 2.1, it is easy to see that $c_n \leq d_n$.
Therefore

$$c_n = d_n \textit{ for } n = 0, 1, 2, ...$$

Since $\{c_n\}$ is a decreasing sequence of nonnegative real numbers, we have $\lim_{n\to\infty} c_n = c$ for some $c \geq 0$.

Now we prove that $c = 0$. On the contrary, we assume that $c > 0$.
Therefore $c_n > 0$ for $n = 0, 1, 2, \ldots$.
Let n be a positive integer and $k \geq n$. For $i = 1, 2, 3$, we have

$$\begin{aligned} d(x_n, T_i x_k) &= d(W(T_3 y_{n-1}, x_{n-1}, \alpha_{n-1}), T_i x_k) \\ &\leq \alpha_{n-1} d(T_3 y_{n-1}, T_i x_k) + (1 - \alpha_{n-1}) d(x_{n-1}, T_i x_k) \\ &\leq \alpha_{n-1} w(M_1(y_{n-1}, x_k)) + (1 - \alpha_{n-1}) d(x_{n-1}, T_i x_k) \\ &\qquad \text{(since } M_1(y_{n-1}, x_k) > 0) \\ &\leq \alpha_{n-1} w(c_{n-1}) + (1 - \alpha_{n-1}) c_{n-1} \text{ so that} \end{aligned}$$

$\sup_{k \geq n} d(x_n, T_i x_k) \leq \alpha_{n-1} w(c_{n-1}) + (1 - \alpha_{n-1}) c_{n-1}$.
Similarly, we can show that $\sup_{k \geq n} d(x_n, T_i y_k) \leq \alpha_{n-1} w(c_{n-1}) + (1 - \alpha_{n-1}) c_{n-1}$ and $\sup_{k \geq n} d(x_n, T_i z_k) \leq \alpha_{n-1} w(c_{n-1}) + (1 - \alpha_{n-1}) c_{n-1}$.
Therefore

$$d_n \leq \alpha_{n-1} w(c_{n-1}) + (1 - \alpha_{n-1}) c_{n-1} \text{ for } n = 1, 2, \ldots .$$

Since $c_n = d_n$, we have

$$\alpha_{n-1}(c_{n-1} - w(c_{n-1})) \leq c_{n-1} - c_n \text{ for } n = 1, 2, \ldots . \tag{2.8}$$

Let $s = \inf\{c_n - w(c_n) : n \geq 0\}$. If $s = 0$ then there exists a subsequence $\{c_{n(k)}\}$ of the sequence $\{c_n\}$ such that $\lim_{k \to \infty} (c_{n(k)} - w(c_{n(k)})) = 0$, i.e., $c - w(c^+) = 0$ which is absurd due to (2.6).

Hence $s > 0$ and $c_n - w(c_n) \geq s$ for $n = 0, 1, 2, \ldots$.

It follows from the inequality (2.8) that $s\alpha_{n-1} \leq c_{n-1} - c_n$ for $n = 1, 2, \ldots$.

Now by applying the comparison test, it follows that the series $\sum \alpha_n < \infty$, a contradiction.
Therefore $c = 0$ so that the sequence $\{x_n\}$ is Cauchy and hence by the completeness of X, there exists $x \in K$ such that $\lim_{n \to \infty} x_n = x$.
Since $c = 0$, we have $\lim_{n \to \infty} d(x_n, T_i x_n) = 0$ so that $\lim_{n \to \infty} T_i x_n = x$ for $i = 1, 2, 3$.

We now prove that x is a common fixed point of T_1, T_2 and T_3. For this purpose, we let $B = \max_{i=1,2,3} \{d(x, T_i x)\}$. Suppose that $B > 0$ so that $M_1(x_n, x) > 0$ for all n.
Now, $d(T_i x_n, T_i x) \leq \max_{i,j=1,2,3} \{d(T_i x_n, T_j x)\} \leq w(M_1(x_n, x))$ for $i = 1, 2, 3$.
On letting $n \to \infty$, we have $d(x, T_i x) \leq w(B^+)$ for $i = 1, 2, 3$ so that $B \leq w(B^+)$, a contradiction.
Therefore $B = 0$ so that x is a common fixed point of T_1, T_2 and T_3.

Clearly, the uniqueness of common fixed point of T_1, T_2, T_3 follows from Remark 1.2. □

If $T_1 = T_2 = T_3$ in Theorem 2.2 then we have the following corollary.

Corollary 2.3. *Let (X, d, W) be a complete convex metric space and K be a nonempty closed convex subset of X. Let $T : K \to K$ be a map that satisfies*

$$d(Tx, Ty) \leq w(M(x, y)) \textit{ for } x, y \in K \textit{ with } M(x, y) > 0,$$

where $M(x, y)$ is defined by (1.3) and $w : (0, \infty) \to (0, \infty)$ be a map that satisfy the relations (2.4), (2.5) and (2.6). Let $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, and $\{\gamma_n\}_{n=0}^{\infty}$ be sequences in $[0, 1]$ such that $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then the sequence $\{x_n\}$ generated by the 'modified Noor iteration procedure (1.11)' converges strongly to a unique fixed point of T.

The following is an easy consequence of Corollary 2.3 and Remark 1.3.

Corollary 2.4. *Let $(X, ||.||)$ be a Banach space and K be a nonempty closed convex subset of X. Let $T : K \to K$ be a quasi-contraction map, i.e., T satisfies the inequality (1.2). Let $\{\alpha_n\}_{n=o}^{\infty}$, $\{\beta_n\}_{n=o}^{\infty}$ and $\{\gamma_n\}_{n=o}^{\infty}$ be sequences in $[0,1]$ such that $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$. Then for any $x_0 \in K$, the sequence $\{x_n\}_{n=0}^{\infty}$ generated by the Noor iteration procedure (1.10) converges strongly to a unique fixed point of T.*

The following is an example in support of Theorem 2.2.

Example 2.1. *Let $X = [0, 2]$ be equipped with the usual norm on the set of all real numbers. We define $W : X \times X \times [0,1] \to X$ by $W(x, y, \lambda) = (1-\lambda)y + \lambda x$ for $x, y \in X$ so that (X, d, W) is a complete convex metric space. Let $K = [\frac{7}{12}, \frac{95}{84}]$ so that K is a closed convex subset of X and we define $T_1, T_2, T_3 : K \to K$ by*

$$T_1 x = \begin{cases} \frac{1}{x} - x & \text{if } x \in [\frac{7}{12}, \frac{1}{\sqrt{2}}] \\ \frac{1}{\sqrt{2}} & \text{if } x \in (\frac{1}{\sqrt{2}}, \frac{95}{84}], \end{cases}$$

$$T_2 x = \begin{cases} \frac{3}{5} + \frac{\frac{3}{5}\sqrt{2}-1}{95\sqrt{2}-84}(84x - 95) & \text{if } x \in [\frac{7}{12}, \frac{1}{\sqrt{2}}] \\ \frac{1}{\sqrt{2}} & \text{if } x \in (\frac{1}{\sqrt{2}}, \frac{95}{84}], \text{ and} \end{cases}$$

$$T_3 x = \begin{cases} \frac{7}{10} + \frac{\frac{7}{10}\sqrt{2}-1}{95\sqrt{2}-84}(84x - 95) & \text{if } x \in [\frac{7}{12}, \frac{1}{\sqrt{2}}] \\ \frac{1}{\sqrt{2}} & \text{if } x \in (\frac{1}{\sqrt{2}}, \frac{95}{84}]. \end{cases}$$

Here, we note that $T_1 x \geq T_2 x \geq T_3 x \geq \frac{1}{\sqrt{2}}$ for $x \in [\frac{7}{12}, \frac{95}{84}]$, $F = \bigcap\limits_{i=1}^{3} F(T_i) = \{\frac{1}{\sqrt{2}}\}$, and T_1, T_2 and T_3 are decreasing functions on $[\frac{7}{12}, \frac{95}{84}]$.

We define $w : (0, \infty) \to (0, \infty)$ by $w(t) = \frac{9t}{10}$ so that w satisfies the relations (2.4), (2.5) and (2.6). In the following, we show that the inequality (2.2) holds. For this purpose, we consider the following three cases.

<u>Case (i)</u> : $\frac{7}{12} \leq x < y \leq \frac{1}{\sqrt{2}}$.

In this case, $M_1(x, y) = d(x, T_1 x) = \frac{1}{x} - 2x$ and

$$\max_{i,j=1,2,3} \{d(T_i x, T_j y)\} = d(T_1 x, T_3 y) \leq d(T_1 x, \tfrac{1}{\sqrt{2}}) = \tfrac{1}{x} - x - \tfrac{1}{\sqrt{2}} \leq \tfrac{9}{10}(\tfrac{1}{x} - 2x) = w(M_1(x, y)).$$

<u>Case (ii)</u> : $\frac{7}{12} \leq x < \frac{1}{\sqrt{2}} \leq y \leq \frac{95}{84}$.

Here, we have $\max\limits_{i,j=1,2,3} \{d(T_i x, T_j y)\} = d(T_1 x, \frac{1}{\sqrt{2}}) = \frac{1}{x} - x - \frac{1}{\sqrt{2}}$ and

$$M_1(x, y) = \begin{cases} \frac{1}{x} - 2x & \text{if } T_1 x \geq y \\ y - x & \text{if } T_1 x \leq y. \end{cases}$$

If $T_1 x \leq y$ then $$\max_{i,j=1,2,3} \{d(T_i x, T_j y)\} = \tfrac{1}{x} - x - \tfrac{1}{\sqrt{2}} \leq \tfrac{9}{10}(\tfrac{1}{x} - 2x) \leq \tfrac{9}{10}(y - x) = w(M_1(x, y)).$$

Similarly, we show that the inequality (2.2) is true if $T_1 x \leq y$.

<u>Case (iii)</u> : $\frac{1}{\sqrt{2}} \leq x < y \leq \frac{95}{84}$.

In this case, $\max\limits_{i,j=1,2,3} \{d(T_i x, T_j y)\} = 0$ and hence the inequality (2.2) trivially holds.

We choose $\beta_n = \gamma_n = \frac{1}{2}$ and $\alpha_n = \frac{1}{n+2}$ for $n = 0, 1, 2, ...$ so that $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$.

Let $x_0 \in [\frac{7}{12}, \frac{95}{84}]$ *be arbitrary, and let* $\{x_n\}_{n=0}^{\infty}$ *be the sequence generated by Noor iteration procedure associated with* T_1, T_2, *and* T_3, *i.e, the sequence* $\{x_n\}_{n=0}^{\infty}$ *is defined by* (2.1) *so that* $z_n = W(T_1x_n, x_n, \gamma_n) = \frac{x_n+T_1x_n}{2}$, $y_n = W(T_2z_n, x_n, \beta_n) = \frac{x_n+T_2z_n}{2}$ *and* $x_{n+1} = W(T_3y_n, x_n, \alpha_n) = \frac{n+1}{n+2}x_n + \frac{1}{n+2}T_3y_n$ *for* $n = 0, 1, 2, \ldots$.

We now show that the sequence $\{x_n\}_{n=0}^{\infty}$ *converges to* $\frac{1}{\sqrt{2}}$ *which is the common fixed point of* T_1, T_2 *and* T_3.

Case (i) : $\frac{7}{12} \le x_0 < \frac{1}{\sqrt{2}}$.

By induction on n, *we show that*

$x_{n+1} - \frac{1}{\sqrt{2}} = (\frac{n+1}{n+2} + \frac{42(\frac{7}{10}\sqrt{2}-1)}{(n+2)(95\sqrt{2}-84)})(x_n - \frac{1}{\sqrt{2}})$ *and* $x_n < \frac{1}{\sqrt{2}}$ *for all* $n \ge 0$.

We assume that $x_n < \frac{1}{\sqrt{2}}$ *for some* $n \ge 0$ *so that* $z_n = \frac{x_n+T_1x_n}{2} = \frac{1}{2x_n} > \frac{1}{\sqrt{2}}$, $y_n = \frac{x_n+T_2z_n}{2} = \frac{1}{2}(x_n + \frac{1}{\sqrt{2}}) < \frac{1}{\sqrt{2}}$ *and*

$$\begin{aligned} x_{n+1} &= \tfrac{n+1}{n+2}x_n + \tfrac{1}{n+2}T_3y_n = \tfrac{n+1}{n+2}x_n + \tfrac{1}{n+2}(\tfrac{7}{10} + \tfrac{\frac{7}{10}\sqrt{2}-1}{95\sqrt{2}-84}(84y_n - 95)) \\ &= \tfrac{n+1}{n+2}x_n + \tfrac{1}{n+2}(\tfrac{7}{10} + \tfrac{\frac{7}{10}\sqrt{2}-1}{95\sqrt{2}-84}(42(x_n + \tfrac{1}{\sqrt{2}}) - 95)) \\ &= (\tfrac{n+1}{n+2} + \tfrac{42(\frac{7}{10}\sqrt{2}-1)}{(n+2)(95\sqrt{2}-84)})x_n + \tfrac{1}{n+2}(\tfrac{7}{10} + \tfrac{42(\frac{7}{10}\sqrt{2}-1)}{\sqrt{2}(95\sqrt{2}-84)} - \tfrac{95(\frac{7}{10}\sqrt{2}-1)}{95\sqrt{2}-84}) \\ &= (\tfrac{n+1}{n+2} + \tfrac{42(\frac{7}{10}\sqrt{2}-1)}{(n+2)(95\sqrt{2}-84)})(x_n - \tfrac{1}{\sqrt{2}}) + \tfrac{1}{\sqrt{2}} + \tfrac{1}{n+2}B_n \end{aligned}$$

where $B_n = -\frac{1}{\sqrt{2}} + \frac{42(\frac{7}{10}\sqrt{2}-1)}{\sqrt{2}(95\sqrt{2}-84)} + \frac{7}{10} + \frac{42(\frac{7}{10}\sqrt{2}-1)}{\sqrt{2}(95\sqrt{2}-84)} - \frac{95(\frac{7}{10}\sqrt{2}-1)}{(95\sqrt{2}-84)} = 0$ *so that*

$$x_{n+1} - \frac{1}{\sqrt{2}} = A_n(x_n - \frac{1}{\sqrt{2}}) \tag{2.9}$$

where $A_n = (\frac{n+1}{n+2} + \frac{42(\frac{7}{10}\sqrt{2}-1)}{(n+2)(95\sqrt{2}-84)})$.

Since $0 < A_n < 1$, *we have* $x_{n+1} < \frac{1}{\sqrt{2}}$.

Thus, by induction on n, *we have* $x_n < \frac{1}{\sqrt{2}}$ *and the equation* (2.9) *is true for* $n = 0, 1, 2, \ldots$.

By (2.9), *we have*

$$|x_{n+1} - \frac{1}{\sqrt{2}}| = (\prod_{i=0}^{n} A_i)|x_0 - \frac{1}{\sqrt{2}}| \text{ for } n = 0, 1, 2, \ldots . \tag{2.10}$$

Since $1 - A_n = \frac{1}{n+2} - \frac{42(\frac{7}{10}\sqrt{2}-1)}{(n+2)(95\sqrt{2}-84)} > \frac{1}{n+2}$ *for* $n = 0, 1, 2, \ldots$, *we have the series* $\sum_{n=0}^{\infty}(1 - A_n) = \infty$ *so that* $\lim_{n\to\infty} \prod_{i=0}^{n} A_i = 0$ *and hence* $\lim_{n\to\infty} x_n = \frac{1}{\sqrt{2}}$.

Case (ii) : $\frac{1}{\sqrt{2}} < x_0 \le \frac{95}{84}$.

In this case, we show that $x_{n+1} - \frac{1}{\sqrt{2}} = \frac{n+1}{n+2}(x_n - \frac{1}{\sqrt{2}})$ *and* $x_n > \frac{1}{\sqrt{2}}$ *for all* $n \ge 0$.

We assume that $x_n > \frac{1}{\sqrt{2}}$ *for some* $n \ge 0$ *so that*

$z_n = \frac{x_n+T_1x_n}{2} = \frac{1}{2}(x_n + \frac{1}{\sqrt{2}}) > \frac{1}{\sqrt{2}}$, $y_n = \frac{x_n+T_2y_n}{2} = \frac{1}{2}(x_n + \frac{1}{\sqrt{2}}) > \frac{1}{\sqrt{2}}$, *and*

$$\begin{aligned} x_{n+1} &= (1 - \tfrac{1}{n+2})x_n + \tfrac{1}{n+2}T_3y_n \\ &= \tfrac{n+1}{n+2}x_n + \tfrac{1}{n+2}\tfrac{1}{\sqrt{2}} = \tfrac{n+1}{n+2}(x_n - \tfrac{1}{\sqrt{2}}) + \tfrac{n+1}{n+2}\tfrac{1}{\sqrt{2}} + \tfrac{1}{(n+2)}\tfrac{1}{\sqrt{2}}. \\ &= \tfrac{n+1}{n+2}(x_n - \tfrac{1}{\sqrt{2}}) + \tfrac{1}{\sqrt{2}} \text{ so that} \end{aligned}$$

$$x_{n+1} - \frac{1}{\sqrt{2}} = \frac{n+1}{n+2}(x_n - \frac{1}{\sqrt{2}}) \tag{2.11}$$

and hence $x_{n+1} > \frac{1}{\sqrt{2}}$.

Therefore, by induction on n, *we have* $x_{n+1}-\frac{1}{\sqrt{2}}=\frac{n+1}{n+2}(x_n-\frac{1}{\sqrt{2}})$ *and* $x_n>\frac{1}{\sqrt{2}}$ *for* $n=0,1,2,...$ *so that* $|x_{n+1}-\frac{1}{\sqrt{2}}|=\frac{1}{n+2}|x_0-\frac{1}{\sqrt{2}}|$ *for* $n=0,1,2,...$ *and hence the sequence* $\{x_n\}_{n=0}^{\infty}$ *converges to* $\frac{1}{\sqrt{2}}$.

Hence the maps T_1,T_2 *and* T_3 *satisfy all the hypotheses of* Theorem 2.2 *and for any* $x_0\in[\frac{7}{12},\frac{95}{84}]$, *the Noor iteration procedure associated with* T_1,T_2 *and* T_3, *converges to the unique common fixed point* $\frac{1}{\sqrt{2}}$ *of* T_1,T_2 *and* T_3.

We use MATLAB 13 software to find out the number of iterations at which the sequence $\{x_n\}_{n=0}^{\infty}$ *converges to the common fixed point* $\frac{1}{\sqrt{2}}$ *of* T_1,T_2 *and* T_3.

Table 1. $x_0=0.6$, $\alpha_n=\frac{1}{n+2},\beta_n=\frac{1}{2}=\gamma_n$

No. of iterations (n)	x_n	y_n	z_n
0	0.6	0.65355391	.8333333333
1	0.636001577	0.671554179	0.786161573
50	0.703107130	0.705106956	0.711129184
5000	0.707066766	0.707086774	0.707146798
50000	0.707102855	0.707104818	0.707110708
100000	0.707104829	0.707105805	0.707108733
150000	0.707105484	0.707106133	0.707108078
194105	**0.707105781**	0.707106281	0.707107781

The 194105^{th} *iteration has got the value of* $x_n=0.707105781$ *which approximates the common fixed point* $\frac{1}{\sqrt{2}}$ *of* T_1,T_2 *and* T_3 *with an error less than* 10^{-5}.

Remark 2.1. *If we choose* $\gamma_n\equiv 0$, *and* $T_1=T_2$ *in Theorem 2.2 then Theorem 1.3 follows as a corollary to Theorem 2.2. Hence our result (Theorem 2.2) extends Theorem 1.3 to three self maps.*

3. Convergence of Abbas and Nazir iteration

We now define Abbas and Nazir iteration procedure associated with three self maps T_1,T_2 and T_3 in convex metric spaces as follows: For any $x_0\in K$,

$$\begin{aligned} z_n&=W(T_1x_n,x_n,\gamma_n)\\ y_n&=W(T_2z_n,T_2x_n,\beta_n)\\ x_{n+1}&=W(T_3z_n,T_3y_n,\alpha_n)\end{aligned}\tag{3.1}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are sequences in $[0,1]$.

Theorem 3.1. *Let* (X,d,W) *be a complete convex metric space and* K, *a nonempty closed convex subset of* X. *Let* $T_1,T_2,T_3:K\to K$ *be self maps of* K *that satisfy*

$$\max_{i,j=1,2,3}\{d(T_ix,T_jy)\}\le w(M_1(x,y))\text{ for }x,y\in K\text{ with }M_1(x,y)>0,$$

where $M_1(x,y)$ *is defined by* (2.3) *and* $w:(0,\infty)\to(0,\infty)$ *is a map that satisfies the relations* (2.4), (2.5), *and* (2.6). *Let* $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, *and* $\{\gamma_n\}_{n=0}^{\infty}$ *be sequences in* $[0,1]$. *Then the sequence* $\{x_n\}$ *generated by Abbas and Nazir iteration procedure associated with three self maps* (3.1) *converges strongly to a unique common fixed point of* T_1,T_2 *and* T_3.

Proof. By using the same technique discussed in Lemma 2.1 and Theorem 2.2 of Section 2, it is easy to see that the diameter c_n of the set $C_n = \{x_k\}_{k\geq n}\cup\{y_k\}_{k\geq n}\cup\{z_k\}_{k\geq n}\cup\bigcup_{i=1}^{3}(\{T_ix_k\}_{k\geq n}\cup\{T_iy_k\}_{k\geq n}\cup\{T_iz_k\}_{k\geq n})$ is equal to $d_n = \max_{i=1,2,3}\{\sup_{k\geq n} d(x_n, T_ix_k), \sup_{k\geq n} d(x_n, T_iy_k), \sup_{k\geq n} d(x_n, T_iz_k)\}$ for $n = 0, 1, 2, ...$ and $\lim_{n\to\infty} c_n = c$ for some $c \geq 0$.

We now prove that $c = 0$. On the contrary, we suppose that $c > 0$ so that $c_n > 0$ for $n = 0, 1, 2, ...$.

For a positive integer n, let $k \geq n$. Then for $i = 1, 2, 3$ we have

$$d(x_n, T_ix_k) = d(W(T_3z_{n-1}, T_3y_{n-1}, \alpha_{n-1}), T_ix_k)$$
$$\leq \alpha_{n-1}d(T_3z_{n-1}, T_ix_k) + (1-\alpha_{n-1})d(T_3y_{n-1}, T_ix_k) \leq w(c_{n-1}).$$

Therefore $\sup_{k\geq n} d(x_n, T_ix_k) \leq w(c_{n-1})$ for $i = 1, 2, 3$ and $n = 1, 2, 3, ...$.

Similarly, $\sup_{k\geq n} d(x_n, T_iy_k) \leq w(c_{n-1})$ and $\sup_{k\geq n} d(x_n, T_iz_k) \leq w(c_{n-1})$ for $i = 1, 2, 3$ and $n = 1, 2, ...$ so that

$$c_n = d_n \leq w(c_{n-1}).$$

On letting $n \to \infty$, we have $c \leq w(c^+)$,
a contradiction.

Therefore $c = 0$ and hence the conclusion of the theorem follows from the lines of the proof of Theorem 2.2. □

Corollary 3.2. *Let (X, d, W) be a complete convex metric space and K be a nonempty closed convex subset of X. Let $T : K \to K$ be a map such that*

$$d(Tx, Ty) \leq w(M(x, y)) \text{ for } x, y \in K \text{ with } M(x, y) > 0,$$

where $M(x, y)$ is defined by (1.3) *and $w : (0,\infty) \to (0,\infty)$ is a map that satisfies the relations* (2.4), (2.5), *and* (2.6). *Let $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, and $\{\gamma_n\}_{n=0}^{\infty}$ be sequences in* $[0, 1]$. *Then the sequence $\{x_n\}$ generated by the modified Abbas and Nazir iteration procedure* (1.13) *converges strongly to a unique fixed point of T.*

Corollary 3.3. *Let $(X, ||.||)$ be a Banach space and K be a nonempty closed convex subset of X. Let $T : K \to K$ be a quasi-contraction map, i.e., T satisfies the inequality* (1.2). *Let $\{\alpha_n\}_{n=o}^{\infty}$, $\{\beta_n\}_{n=o}^{\infty}$ and $\{\gamma_n\}_{n=o}^{\infty}$ be sequences in* $[0, 1]$. *Then for any $x_0 \in K$, the sequence $\{x_n\}_{n=0}^{\infty}$ generated by Abbas and Nazir iteration procedure* (1.12) *converges strongly to a unique fixed point of T.*

The following example is in support of Theorem 3.1.

Example 3.1. *Let X, K, T_1, T_2 and T_3 be as in Example 2.1. Let $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ be arbitrary sequences in* $[0, 1]$. *Let $x_0 \in K$ and $\{x_n\}_{n=0}^{\infty}$ be the sequence generated by* (3.1) *so that $z_n = (1-\gamma_n)x_n + \gamma_nT_1x_n$,*
$y_n = (1-\beta_n)T_2x_n + \beta_nT_2z_n$ and $x_{n+1} = (1-\alpha_n)T_3y_n + \alpha_nT_3z_n$ for $n = 0, 1, 2, ...$.
Here we note that T_1, T_2 and T_3 satisfy all the hypotheses of Theorem 3.1. *Further, it is easy to see that $x_1 \geq \frac{1}{\sqrt{2}}$ and $x_n = \frac{1}{\sqrt{2}}$ for $n = 2, 3, ...$ so that the sequence $\{x_n\}_{n=0}^{\infty}$ converges to the common fixed point $\frac{1}{\sqrt{2}}$ of T_1, T_2 and T_3.*

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References

[1] M. Abbas and T. Nazir, A new faster iteration process applied to constrained minimization and feasibility problems, Mat. Vesnik., 66(2) (2014), 223-234.

[2] M. Bridson and A. Haefliger, Metric spaces of non-positive curvature, Springer-Verlag, Berlin, Heidelberg, New York, 1999.

[3] L. B. Ćirić, A generalization of Banach's contraction principle, Proc. Amer. Math. Soc., 45(2) (1974), 267-273.

[4] L. B. Ćirić, Convergence theorems for a sequence of Ishikawa iterations for nonlinear quasi-contractive mappings, Indian J. Pure Appl. Math., 30(4) (1999), 425-433.

[5] X. P. Ding, Iteration processes for nonlinear mappings in convex metric spaces, J. Math. Anal. Appl., 132(1) (1988), 114-122.

[6] S. Ishikawa, Fixed points by a new iteration method, Proc. Amer. Math. Soc., 44(1) (1974), 147-150.

[7] M. Moosaei, Fixed point theorems in convex metric spaces, Fixed Point Theory and Appl., Vol. 2012(2012), Article 164, 6 pages.

[8] M. A. Noor, New approximation schemes for general variational inequalities. J. Math. Anal. Appl., 251(1) (2000), 217-229.

[9] K. P. R. Sastry, G. V. R. Babu and Ch. Srinivasa Rao, Convergence of an Ishikawa iteration scheme for nonlinear quasi-contractive mappings in convex metric spaces, Tamkang J. Math., 32(2) (2001), 117-126.

[10] K. P. R. Sastry, G. V. R. Babu and Ch. Srinivasa Rao, Convergence of an Ishikawa iteration scheme for a nonlinear quasi-contractive pair of selfmaps in convex metric spaces, Indian J. Pure Appl. Math., 33(2) (2002), 203-214.

[11] W. Takahashi, A convexity in metric space and nonexpansive mappings, I, Kodai Math. Sem. Rep., 22(2) (1970), 142-149.

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