Available online: April 29, 2020

Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.
Volume 69, Number 1, Pages 880–890 (2020)
DOI: 10.31801/cfsuasmas.582674
ISSN 1303–5991 E-ISSN 2618-6470
http://communications.science.ankara.edu.tr/index.php?series=A1

# A NOTE ON HYPERBOLIC $(p,q)$-FIBONACCI QUATERNIONS

TÜLAY YAĞMUR

Abstract. In this paper, we introduce a new quaternion sequence called hyperbolic $(p,q)$-Fibonacci quaternions. This new quaternion sequence includes hyperbolic Fibonacci, hyperbolic $k$-Fibonacci, hyperbolic Pell, hyperbolic $k$-Pell, hyperbolic Jacobsthal, hyperbolic $k$-Jacobsthal quaternions. We give generating function and Binet's formula for these quaternions. We also obtain some identities such as d'Ocagne's, Catalan's and Cassini's identities involving hyperbolic $(p,q)$-Fibonacci quaternions.

## 1. Introduction

Fibonacci numbers have been applied in different scientific areas such as engineering, and architecture. Recently, Fibonacci numbers have been studied and generalized by many authors in many ways. For example, one of the generalization of Fibonacci numbers is $(p,q)$-Fibonacci numbers [15,17].

For positive real numbers $p$ and $q$, the sequence of $(p,q)$-Fibonacci numbers, denoted by $\{\mathcal{F}_n\}_{n\geq 0}$, is defined by the recurrence relation

$$\mathcal{F}_n = p\mathcal{F}_{n-1} + q\mathcal{F}_{n-2}, \quad n \geq 2$$

with initial conditions $\mathcal{F}_0 = 0$ and $\mathcal{F}_1 = 1$ [17].

The $n$th term of the sequence $\{\mathcal{F}_n\}_{n\geq 0}$ is given by

$$\mathcal{F}_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \tag{1}$$

where $\alpha = \dfrac{p+\sqrt{p^2+4q}}{2}$, $\beta = \dfrac{p-\sqrt{p^2+4q}}{2}$ are the roots of the characteristic equation $t^2 - pt - q = 0$ [17].

Received by the editors: June 26, 2019; Accepted: February 24, 2020.

2010 *Mathematics Subject Classification.* Primary 11R52, 11B37; Secondary 11B39, 20G20.

*Key words and phrases.* Fibonacci numbers, $(p,q)$-Fibonacci numbers, $(p,q)$-Fibonacci quaternions, hyperbolic quaternions, hyperbolic $(p,q)$-Fibonacci quaternions.

It must be note that $\alpha + \beta = p$, $\alpha - \beta = \sqrt{p^2 + 4q}$ and $\alpha\beta = -q$. Moreover, the generating function for the sequence $\{\mathcal{F}_n\}_{n\geq 0}$ [29] is given by

$$f_{p,q}(t) = \frac{t}{1 - pt - qt^2}.$$

The $(p, q)$-Fibonacci sequence is the generalization of the familiar second-order recurrent sequences, that is, for special values of $p$ and $q$, are defined as follows:

- If $p = q = 1$, then $\{\mathcal{F}_n\}_{n\geq 0}$ is the (classical) Fibonacci sequence $\{F_n\}_{n\geq 0}$ [24].
- If $p = k$, $q = 1$, then $\{\mathcal{F}_n\}_{n\geq 0}$ is the $k$-Fibonacci sequence $\{F_{k,n}\}_{n\geq 0}$ [9].
- If $p = 2$, $q = 1$ , then $\{\mathcal{F}_n\}_{n\geq 0}$ is the Pell sequence $\{P_n\}_{n\geq 0}$ [18].
- If $p = 2$, $q = k$ then $\{\mathcal{F}_n\}_{n\geq 0}$ is the $k$-Pell sequence $\{P_{k,n}\}_{n\geq 0}$ [7].
- If $p = 1$, $q = 2$ , then $\{\mathcal{F}_n\}_{n\geq 0}$ is the Jacobsthal sequence $\{J_n\}_{n\geq 0}$ [19].
- If $p = k$, $q = 2$, then $\{\mathcal{F}_n\}_{n\geq 0}$ is the $k$-Jacobsthal sequence $\{J_{k,n}\}_{n\geq 0}$ [22].

Quaternions (real quaternions), introduced by Sir William Rowan Hamilton in the mid nineteenth century, are four-dimensional hypercomplex numbers. Quaternions are widely used in high-tech areas such as computer graphics, signal processing, and robotics, see for example [1,8,10,11], among others.

Quaternions form a four-dimensional non-commutative associative algebra over the real numbers, are defined as follows:

$$\mathbf{H} = \{q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} \quad | \quad q_0, q_1, q_2, q_3 \in \mathbb{R}\},$$

where $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is a basis of $\mathbf{H}$, and the imaginary units $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ satisfy the following equalities

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1, \quad \mathbf{ij} = \mathbf{k} = -\mathbf{ji}, \quad \mathbf{jk} = \mathbf{i} = -\mathbf{kj}, \quad \mathbf{ki} = \mathbf{j} = -\mathbf{ik}.$$

For more details on quaternions, one can see, for example [14,32].

Horadam [16] defined the Fibonacci quaternions as

$$QF_n = F_n + F_{n+1}\mathbf{i} + F_{n+2}\mathbf{j} + F_{n+3}\mathbf{k},$$

where $F_n$ is the $n$th Fibonacci number.

Fibonacci quaternions have been studied and generalized by many authors, some of which can be found in [2-6,12,13,20,21,26-28,30,31], among others. One of the generalization for Fibonacci quaternions is done by Ipek. In [20], Ipek introduced the $(p, q)$-Fibonacci quaternions as

$$Q\mathcal{F}_n = \mathcal{F}_n + \mathcal{F}_{n+1}\mathbf{i} + \mathcal{F}_{n+2}\mathbf{j} + \mathcal{F}_{n+3}\mathbf{k},$$

where $\mathcal{F}_n$ is the $n$th $(p, q)$-Fibonacci number.

The author also defined the $(p, q)$-Fibonacci quaternions recursively by the relation

$$Q\mathcal{F}_n = pQ\mathcal{F}_{n-1} + qQ\mathcal{F}_{n-2}, \quad n \geq 2.$$

Moreover, Patel and Ray [26] investigated some properties of $(p,q)$-Fibonacci and $(p,q)$-Lucas quaternions.

Alexander Mac-Farlane first described hyperbolic quaternions in 1891, and these numbers are not associative. Kurt Godel used the name of these quaternions in 1949, but the author actually implied split quaternions in his definition. Hyperbolic quaternions [25], just like real quaternions, are a generalization of complex numbers by four real numbers. Moreover, just like real quaternions, hyperbolic quaternions are not commutative. But hyperbolic quaternions have zero divisors.

In [23], Kosal studied on hyperbolic quaternions and their algebraic properties. In [5], Aydin defined the hyperbolic $k$-Fibonacci quaternions. The author also investigated some algebraic properties of the hyperbolic $k$-Fibonacci quaternions.

Hyperbolic quaternions are defined as

$$\mathbf{K}=\{q=q_0+\mathbf{i}q_1+\mathbf{j}q_2+\mathbf{k}q_3 \quad | \quad q_0,q_1,q_2,q_3\in\mathbb{R}\},$$

where

$$\mathbf{i}^2=\mathbf{j}^2=\mathbf{k}^2=\mathbf{ijk}=1,\quad \mathbf{ij}=\mathbf{k}=-\mathbf{ji},\quad \mathbf{jk}=\mathbf{i}=-\mathbf{kj},\quad \mathbf{ki}=\mathbf{j}=-\mathbf{ik}.$$

Let $p=p_0+\mathbf{i}p_1+\mathbf{j}p_2+\mathbf{k}p_3$ and $q=q_0+\mathbf{i}q_1+\mathbf{j}q_2+\mathbf{k}q_3$ be two hyperbolic quaternions. Then the addition and subtraction of two hyperbolic quaternions are defined as

$$p\pm q=(p_0\pm q_0)+\mathbf{i}(p_1\pm q_1)+\mathbf{j}(p_2\pm q_2)+\mathbf{k}(p_3\pm q_3).$$

The multiplication of a hyperbolic quaternion by a real scalar $\lambda$ is defined as

$$\lambda p=\lambda p_0+\mathbf{i}\lambda p_1+\mathbf{j}\lambda p_2+\mathbf{k}\lambda p_3.$$

The multiplication of two hyperbolic quaternions is defined as

$$\begin{aligned}pq&=(p_0q_0+p_1q_1+p_2q_2+p_3q_3)+\mathbf{i}(p_0q_1+p_1q_0+p_2q_3-p_3q_2)\\&\quad+\mathbf{j}(p_0q_2-p_1q_3+p_2q_0+p_3q_1)+\mathbf{k}(p_0q_3+p_1q_2-p_2q_1+p_3q_0).\end{aligned}$$

The conjugate of a hyperbolic quaternion $q$ is denoted by $\overline{q}$ and defined by

$$\overline{q}=q_0-\mathbf{i}q_1-\mathbf{j}q_2-\mathbf{k}q_3.$$

Moreover, the norm of the hyperbolic quaternion $q$ is

$$N(q)=q\overline{q}={q_0}^2-{q_1}^2-{q_2}^2-{q_3}^2.$$

The main objective of this paper is to introduce hyperbolic $(p,q)$-Fibonacci quaternions. We then give the generating function and Binet's formula for the hyperbolic $(p,q)$-Fibonacci quaternions. In addition, we obtain some well-known identities involving these quaternions.

## 2. The Hyperbolic $(p,q)$-Fibonacci Quaternions

In this section, we first give the definition of the hyperbolic $(p,q)$-Fibonacci quaternions. We then investigate some properties of these quaternions.

**Definition 1.** *For positive real numbers $p$ and $q$, hyperbolic $(p,q)$-Fibonacci quaternions are defined by the relation*

$$HQ\mathcal{F}_n = \mathcal{F}_n + \mathbf{i}\mathcal{F}_{n+1} + \mathbf{j}\mathcal{F}_{n+2} + \mathbf{k}\mathcal{F}_{n+3}, \tag{2}$$

*where $\mathcal{F}_n$ is the nth $(p,q)$-Fibonacci number, and $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ satisfy the equalities*

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = 1, \quad \mathbf{ij} = \mathbf{k} = -\mathbf{ji}, \quad \mathbf{jk} = \mathbf{i} = -\mathbf{kj}, \quad \mathbf{ki} = \mathbf{j} = -\mathbf{ik}. \tag{3}$$

Let $HQ\mathcal{F}_n$ be the $n$th $(p,q)$-Fibonacci number. Then, after some necessary calculations, one can obtain the following recurrence relation:

$$HQ\mathcal{F}_n = pHQ\mathcal{F}_{n-1} + qHQ\mathcal{F}_{n-2}, \quad n \geq 2, \tag{4}$$

with initial conditions

$$HQ\mathcal{F}_0 = \mathbf{i} + \mathbf{j}p + \mathbf{k}(p^2 + q), \tag{5}$$

$$HQ\mathcal{F}_1 = 1 + \mathbf{i}p + \mathbf{j}(p^2 + q) + \mathbf{k}(p^3 + 2pq). \tag{6}$$

Particular cases of Definition 1 are

- Hyperbolic Fibonacci quaternions are

$$HQF_n = F_n + \mathbf{i}F_{n+1} + \mathbf{j}F_{n+2} + \mathbf{k}F_{n+3},$$

  where $F_n$ is the $n$th Fibonacci number, with initial conditions

$$HQF_0 = \mathbf{i} + \mathbf{j} + \mathbf{k}2,$$
$$HQF_1 = 1 + \mathbf{i} + \mathbf{j}2 + \mathbf{k}3.$$

- Hyperbolic $k$-Fibonacci quaternions [5] are

$$HQF_{k,n} = F_{k,n} + \mathbf{i}F_{k,n+1} + \mathbf{j}F_{k,n+2} + \mathbf{k}F_{k,n+3},$$

  where $F_{k,n}$ is the $n$th $k$-Fibonacci number, with initial conditions

$$HQF_{k,0} = \mathbf{i} + \mathbf{j}k + \mathbf{k}(k^2 + 1),$$
$$HQF_{k,1} = 1 + \mathbf{i}k + \mathbf{j}(k^2 + 1) + \mathbf{k}(k^3 + 2k).$$

- Hyperbolic Pell quaternions are

$$HQP_n = P_n + \mathbf{i}P_{n+1} + \mathbf{j}P_{n+2} + \mathbf{k}P_{n+3},$$

  where $P_n$ is the $n$th Pell number, with initial conditions

$$HQP_0 = \mathbf{i} + 2\mathbf{j} + \mathbf{k}5,$$
$$HQP_1 = 1 + \mathbf{i}2 + \mathbf{j}5 + \mathbf{k}12.$$

- Hyperbolic $k$-Pell quaternions are
$$HQP_{k,n} = P_{k,n} + \mathbf{i}P_{k,n+1} + \mathbf{j}P_{k,n+2} + \mathbf{k}P_{k,n+3},$$
where $P_{k,n}$ is the $n$th $k$-Pell number, with initial conditions
$$\begin{aligned} HQP_{k,0} &= \mathbf{i} + \mathbf{j}2 + \mathbf{k}(4+k), \\ HQP_{k,1} &= 1 + \mathbf{i}2 + \mathbf{j}(4+k) + \mathbf{k}(8+4k). \end{aligned}$$
- Hyperbolic Jacobsthal quaternions are
$$HQJ_n = J_n + \mathbf{i}J_{n+1} + \mathbf{j}J_{n+2} + \mathbf{k}J_{n+3},$$
where $J_n$ is the $n$th Jacobsthal number, with initial conditions
$$\begin{aligned} HQJ_0 &= \mathbf{i} + \mathbf{j} + \mathbf{k}3, \\ HQJ_1 &= 1 + \mathbf{i} + \mathbf{j}3 + \mathbf{k}5. \end{aligned}$$
- Hyperbolic $k$-Jacobsthal quaternions are
$$HQJ_{k,n} = J_{k,n} + \mathbf{i}J_{k,n+1} + \mathbf{j}J_{k,n+2} + \mathbf{k}J_{k,n+3},$$
where $J_{k,n}$ is the $n$th $k$-Jacobsthal number, with initial conditions
$$\begin{aligned} HQJ_{k,0} &= \mathbf{i} + \mathbf{j}k + \mathbf{k}(k^2+2), \\ HQJ_{k,1} &= 1 + \mathbf{i}k + \mathbf{j}(k^2+2) + \mathbf{k}(k^3+4k). \end{aligned}$$

Let $HQ\mathcal{F}_n = \mathcal{F}_n + \mathbf{i}\mathcal{F}_{n+1} + \mathbf{j}\mathcal{F}_{n+2} + \mathbf{k}\mathcal{F}_{n+3}$ and $HQ\mathcal{F}_m = \mathcal{F}_m + \mathbf{i}\mathcal{F}_{m+1} + \mathbf{j}\mathcal{F}_{m+2} + \mathbf{k}\mathcal{F}_{m+3}$ be two hyperbolic $(p,q)$-Fibonacci quaternions. Then the addition and subtraction of two hyperbolic $(p,q)$-Fibonacci quaternions are defined by
$$\begin{aligned} HQ\mathcal{F}_n \pm HQ\mathcal{F}_m = (\mathcal{F}_n \pm \mathcal{F}_m) &+ \mathbf{i}(\mathcal{F}_{n+1} \pm \mathcal{F}_{m+1}) + \mathbf{j}(\mathcal{F}_{n+2} \pm \mathcal{F}_{m+2}) \\ &+ \mathbf{k}(\mathcal{F}_{n+3} \pm \mathcal{F}_{m+3}). \end{aligned} \tag{7}$$
The multiplication of a hyperbolic $(p,q)$-Fibonacci quaternion by a real scalar $\lambda$ is defined by
$$\lambda HQ\mathcal{F}_n = \lambda\mathcal{F}_n + \mathbf{i}\lambda\mathcal{F}_{n+1} + \mathbf{j}\lambda\mathcal{F}_{n+2} + \mathbf{k}\lambda\mathcal{F}_{n+3}. \tag{8}$$
The multiplication of two hyperbolic $(p,q)$-Fibonacci quaternions is defined by
$$\begin{aligned} &HQ\mathcal{F}_n \times HQ\mathcal{F}_m \\ &\quad = (\mathcal{F}_n\mathcal{F}_m + \mathcal{F}_{n+1}\mathcal{F}_{m+1} + \mathcal{F}_{n+2}\mathcal{F}_{m+2} + \mathcal{F}_{n+3}\mathcal{F}_{m+3}) \\ &\quad + \mathbf{i}(\mathcal{F}_n\mathcal{F}_{m+1} + \mathcal{F}_{n+1}\mathcal{F}_m + \mathcal{F}_{n+2}\mathcal{F}_{m+3} - \mathcal{F}_{n+3}\mathcal{F}_{m+2}) \\ &\quad + \mathbf{j}(\mathcal{F}_n\mathcal{F}_{m+2} - \mathcal{F}_{n+1}\mathcal{F}_{m+3} + \mathcal{F}_{n+2}\mathcal{F}_m + \mathcal{F}_{n+3}\mathcal{F}_{m+1}) \\ &\quad + \mathbf{k}(\mathcal{F}_n\mathcal{F}_{m+3} + \mathcal{F}_{n+1}\mathcal{F}_{m+2} - \mathcal{F}_{n+2}\mathcal{F}_{m+1} + \mathcal{F}_{n+3}\mathcal{F}_m). \end{aligned} \tag{9}$$

The generating function for the hyperbolic $(p,q)$-Fibonacci quaternions is given in the following theorem.

**Theorem 2.** *The generating function for the hyperbolic $(p,q)$-Fibonacci quaternions is given by*

$$G_{p,q}(t)=\frac{t+\mathbf{i}(1+t-pt)+\mathbf{j}(1+2t-pt)+\mathbf{k}(2+3t-2pt)}{1-pt-qt^2}.$$

*Proof.* Let $G_{p,q}(t)$ be the generating function for the hyperbolic $(p,q)$-Fibonacci quaternions. Then we write

$$G_{p,q}(t)=\sum_{n=0}^{\infty}HQ\mathcal{F}_n t^n=HQ\mathcal{F}_0+HQ\mathcal{F}_1 t+\ldots+HQ\mathcal{F}_n t^n+\ldots \quad . \tag{10}$$

Multiplying the Eq. (10) with $pt$ and $qt^2$ respectively, we get

$$ptG_{p,q}(t)=pHQ\mathcal{F}_0 t+pHQ\mathcal{F}_1 t^2+\ldots+pHQ\mathcal{F}_{n-1}t^n+\ldots$$

and

$$qt^2G_{p,q}(t)=qHQ\mathcal{F}_0 t^2+qHQ\mathcal{F}_1 t^3+\ldots+qHQ\mathcal{F}_{n-2}t^n+\ldots \quad .$$

Then we have

$$\begin{aligned}(1-pt-qt^2)G_{p,q}(t)&=HQ\mathcal{F}_0+(HQ\mathcal{F}_1-pHQ\mathcal{F}_0)t\\&\quad+\sum_{n=2}^{\infty}(HQ\mathcal{F}_n-pHQ\mathcal{F}_{n-1}-qHQ\mathcal{F}_{n-2})t^n\\&=HQ\mathcal{F}_0+(HQ\mathcal{F}_1-pHQ\mathcal{F}_0)t.\end{aligned}$$

By the Eqs. (5) and (6), we get

$$(1-pt-qt^2)G_{p,q}(t)=t+\mathbf{i}(1+t-pt)+\mathbf{j}(1+2t-pt)+\mathbf{k}(2+3t-2pt)$$

which is the desired result. □

Particular cases of Theorem 2 are

- The generating function of the hyperbolic (classical) Fibonacci quaternions is
$$f(t)=\frac{t+\mathbf{i}+\mathbf{j}(1+t)+\mathbf{k}(2+t)}{1-t-t^2}.$$
- The generating function of the hyperbolic $k$-Fibonacci quaternions is
$$f_k(t)=\frac{t+\mathbf{i}(1+t(1-k))+\mathbf{j}(1+t(2-k))+\mathbf{k}(2+t(3-2k))}{1-kt-t^2}.$$
- The generating function of the hyperbolic Pell quaternions is
$$g(t)=\frac{t+\mathbf{i}(1-t)+\mathbf{j}+\mathbf{k}(2-t)}{1-2t-t^2}.$$
- The generating function of the hyperbolic $k$-Pell quaternions is
$$g_k(t)=\frac{t+\mathbf{i}(1-t)+\mathbf{j}+\mathbf{k}(2-t)}{1-2t-kt^2}.$$

- The generating function of the hyperbolic Jacobsthal quaternions is

$$h(t) = \frac{t + \mathbf{i} + \mathbf{j}(1+t) + \mathbf{k}(2+t)}{1 - t - 2t^2}.$$

- The generating function of the hyperbolic $k$-Jacobsthal quaternions is

$$h_k(t) = \frac{t + \mathbf{i}(1 + t(1-k)) + \mathbf{j}(1 + t(2-k)) + \mathbf{k}(2 + t(3-2k))}{1 - kt - 2t^2}.$$

The following theorem gives the Binet's formula for the hyperbolic $(p,q)$-Fibonacci quaternions.

**Theorem 3.** *The nth term of the hyperbolic $(p,q)$-Fibonacci quaternion is given by*

$$HQ\mathcal{F}_n = \frac{\alpha^* \alpha^n - \beta^* \beta^n}{\alpha - \beta},$$

*where* $\alpha^* = 1 + \mathbf{i}\alpha + \mathbf{j}\alpha^2 + \mathbf{k}\alpha^3$, $\alpha = \frac{p+\sqrt{p^2+4q}}{2}$ *and* $\beta^* = 1 + \mathbf{i}\beta + \mathbf{j}\beta^2 + \mathbf{k}\beta^3$, $\beta = \frac{p-\sqrt{p^2+4q}}{2}$.

*Proof.* Using the definition of the hyperbolic $(p,q)$-Fibonacci quaternions and the Binet's formula of the $(p,q)$-Fibonacci numbers, we have

$$\begin{aligned} HQ\mathcal{F}_n &= \mathcal{F}_n + \mathbf{i}\mathcal{F}_{n+1} + \mathbf{j}\mathcal{F}_{n+2} + \mathbf{k}\mathcal{F}_{n+3} \\ &= \frac{\alpha^n - \beta^n}{\alpha - \beta} + \mathbf{i}\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} + \mathbf{j}\frac{\alpha^{n+2} - \beta^{n+2}}{\alpha - \beta} + \mathbf{k}\frac{\alpha^{n+3} - \beta^{n+3}}{\alpha - \beta} \\ &= \frac{\alpha^n(1 + \mathbf{i}\alpha + \mathbf{j}\alpha^2 + \mathbf{k}\alpha^3) - \beta^n(1 + \mathbf{i}\beta + \mathbf{j}\beta^2 + \mathbf{k}\beta^3)}{\alpha - \beta}. \end{aligned}$$

If we take $\alpha^* = 1 + \mathbf{i}\alpha + \mathbf{j}\alpha^2 + \mathbf{k}\alpha^3$ and $\beta^* = 1 + \mathbf{i}\beta + \mathbf{j}\beta^2 + \mathbf{k}\beta^3$, we obtain the desired result. □

Particular cases of Therorem 3 are

- The Binet's formula of the $n$th hyperbolic (classical) Fibonacci quaternion is

$$HQF_n = \frac{1}{\sqrt{5}}(\alpha^* \alpha^n - \beta^* \beta^n),$$

where $\alpha^* = 1 + \mathbf{i}\alpha + \mathbf{j}\alpha^2 + \mathbf{k}\alpha^3$, $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta^* = 1 + \mathbf{i}\beta + \mathbf{j}\beta^2 + \mathbf{k}\beta^3$, $\beta = \frac{1-\sqrt{5}}{2}$.

- The Binet's formula of the $n$th hyperbolic $k$-Fibonacci quaternion [5] is

$$HQF_{k,n} = \frac{1}{\sqrt{k^2+4}}({r_1}^* {r_1}^n - {r_2}^* {r_2}^n),$$

where ${r_1}^* = 1+\mathbf{i}r_1+\mathbf{j}{r_1}^2+\mathbf{k}{r_1}^3$, $r_1 = \frac{k+\sqrt{k^2+4}}{2}$ and ${r_2}^* = 1+\mathbf{i}r_2+\mathbf{j}{r_2}^2+\mathbf{k}{r_2}^3$, $r_2 = \frac{k-\sqrt{k^2+4}}{2}$.

- The Binet's formula of the $n$th hyperbolic Pell quaternion is
$$HQP_n = \frac{1}{2\sqrt{2}}(x_1^* {x_1}^n - {x_2}^* x_2^n),$$
where ${x_1}^* = 1+\mathbf{i}x_1+\mathbf{j}{x_1}^2+\mathbf{k}{x_1}^3$, $x_1 = 1+\sqrt{2}$ and ${x_2}^* = 1+\mathbf{i}x_2+\mathbf{j}{x_2}^2+\mathbf{k}{x_2}^3$, $x_2 = 1-\sqrt{2}$.
- The Binet's formula of the $n$th hyperbolic $k$-Pell quaternion is
$$HQP_{k,n} = \frac{1}{2\sqrt{1+k}}(y_1^* {y_1}^n - {y_2}^* y_2^n),$$
where ${y_1}^* = 1+\mathbf{i}y_1+\mathbf{j}{y_1}^2+\mathbf{k}{y_1}^3$, $y_1 = 1+\sqrt{1+k}$ and ${y_2}^* = 1+\mathbf{i}y_2+\mathbf{j}{y_2}^2+\mathbf{k}{y_2}^3$, $y_2 = 1-\sqrt{1+k}$.
- The Binet's formula of the $n$th hyperbolic Jacobsthal quaternion is
$$HQJ_n = \frac{2^* 2^n - (-1)^* (-1)^n}{3},$$
where $2^* = 1+\mathbf{i}2+\mathbf{j}4+\mathbf{k}8$ and $(-1)^* = 1-\mathbf{i}+\mathbf{j}-\mathbf{k}$.
- The Binet's formula of the $n$th hyperbolic $k$-Jacobsthal quaternion is
$$HQJ_{k,n} = \frac{1}{\sqrt{k^2+8}}(w_1^* {w_1}^n - {w_2}^* w_2^n),$$
where ${w_1}^* = 1+\mathbf{i}w_1+\mathbf{j}{w_1}^2+\mathbf{k}{w_1}^3$, $w_1 = \frac{k+\sqrt{k^2+8}}{2}$ and ${w_2}^* = 1+\mathbf{i}w_2+\mathbf{j}{w_2}^2+\mathbf{k}{w_2}^3$, $w_2 = \frac{k-\sqrt{k^2+8}}{2}$.

The d'Ocagne's identity involving the hyperbolic $(p,q)$-Fibonacci quaternions is given in the following theorem.

**Theorem 4.** *Let $m$ and $n$ be two positive integers, such that $n \leq m$. Then we have*
$$HQ\mathcal{F}_m \times HQ\mathcal{F}_{n+1} - HQ\mathcal{F}_{m+1} \times HQ\mathcal{F}_n = \frac{(-q)^n}{\sqrt{p^2+4q}}(\alpha^*\beta^*\alpha^{m-n} - \beta^*\alpha^*\beta^{m-n}).$$

*Proof.* Using the Binet's formula of the hyperbolic $(p,q)$-Fibonacci quaternions, we have
$$\begin{aligned}
&HQ\mathcal{F}_m \times HQ\mathcal{F}_{n+1} - HQ\mathcal{F}_{m+1} \times HQ\mathcal{F}_n \\
&= \frac{\alpha^*\alpha^m - \beta^*\beta^m}{\alpha-\beta} \times \frac{\alpha^*\alpha^{n+1} - \beta^*\beta^{n+1}}{\alpha-\beta} - \frac{\alpha^*\alpha^{m+1} - \beta^*\beta^{m+1}}{\alpha-\beta} \times \frac{\alpha^*\alpha^n - \beta^*\beta^n}{\alpha-\beta} \\
&= \frac{1}{(\alpha-\beta)^2}(\alpha^*\beta^*(\alpha^{m+1}\beta^n - \alpha^m\beta^{n+1}) + \beta^*\alpha^*(\alpha^n\beta^{m+1} - \alpha^{n+1}\beta^m))
\end{aligned}$$

$$= \frac{1}{(\alpha-\beta)}(\alpha^*\beta^*\alpha^m\beta^n - \beta^*\alpha^*\alpha^n\beta^m)$$
$$= \frac{1}{(\alpha-\beta)}(\alpha\beta)^n(\alpha^*\beta^*\alpha^{m-n} - \beta^*\alpha^*\beta^{m-n}).$$

Since $\alpha-\beta=\sqrt{p^2+4q}$ and $\alpha\beta=-q$ , we obtain the desired result. □

Note that, if we take $p=k$, $q=1$ as a special case in Theorem 4, we obtain the equivalent result for d'Ocagne's identity involving the hyperbolic $k$-Fibonacci quaternions given in [5].

The following theorem gives the Catalan's identity for the hyperbolic $(p,q)$-Fibonacci quaternions.

**Theorem 5.** *Let $n$ and $r$ be two positive integers. Then we have*

$$HQ\mathcal{F}_{n-r}\times HQ\mathcal{F}_{n+r} - HQ\mathcal{F}_n^2 = \frac{(-q)^{n-r}}{p^2+4q}(\alpha^*\beta^*\beta^r - \beta^*\alpha^*\alpha^r)(\alpha^r-\beta^r).$$

*Proof.* Using the Binet's formula of the hyperbolic $(p,q)$-Fibonacci quaternions, we have

$$HQ\mathcal{F}_{n-r}\times HQ\mathcal{F}_{n+r} - HQ\mathcal{F}_n\times HQ\mathcal{F}_n$$
$$= \frac{\alpha^*\alpha^{n-r}-\beta^*\beta^{n-r}}{\alpha-\beta}\times\frac{\alpha^*\alpha^{n+r}-\beta^*\beta^{n+r}}{\alpha-\beta} - \frac{\alpha^*\alpha^{n}-\beta^*\beta^{n}}{\alpha-\beta}\times\frac{\alpha^*\alpha^{n}-\beta^*\beta^{n}}{\alpha-\beta}$$
$$= \frac{1}{(\alpha-\beta)^2}(\alpha^*\beta^*(\alpha\beta)^{n-r}(\alpha^r\beta^r-\beta^{2r}) + \beta^*\alpha^*(\alpha\beta)^{n-r}(\alpha^r\beta^r-\alpha^{2r}))$$
$$= \frac{1}{(\alpha-\beta)^2}(\alpha\beta)^{n-r}(\alpha^*\beta^*\beta^r - \beta^*\alpha^*\alpha^r)(\alpha^r-\beta^r).$$

Since $\alpha-\beta=\sqrt{p^2+4q}$ and $\alpha\beta=-q$ , we obtain the desired result. □

Note that, if we take $p=k$, $q=1$ as a special case in Theorem 5, we obtain the equivalent result for Catalan's identity involving the hyperbolic $k$-Fibonacci quaternions given in [5].

If we take $r=1$ in Theorem 5, we obtain the Cassini's identity involving the hyperbolic $(p,q)$-Fibonacci quaternions as

$$HQ\mathcal{F}_{n-1}\times HQ\mathcal{F}_{n+1} - HQ\mathcal{F}_n^2 = \frac{(-q)^{n-1}}{\sqrt{p^2+4q}}(\alpha^*\beta^*\beta - \beta^*\alpha^*\alpha).$$

**Acknowledgements.** The author would like to thank the anonymous reviewers for their careful reading, valuable comments and suggestions who helped to improve the presentation of the paper.

## References


[1] Adler, S. L., Quaternionic quantum mechanics and quantum fields, 88. Oxford Univ. Press on Demand, 1995.

[2] Akyigit, M., Kosal, H. H. and Tosun, M., Split Fibonacci quaternions, *Adv. Appl. Clifford Algebras* 23 (2013), 535-545.

[3] Aydin, F. T., The $k$-Fibonacci dual quaternions, *Int. J. Mathematical Analysis* 12 (2018), 363-373.

[4] Aydin, F. T., Bicomplex Fibonacci quaternions, *Chaos Solitons Fractals* 106 (2018), 147-153.

[5] Aydin, F. T., Hyperbolic $k$-Fibonacci quaternions, arXiv:1812.00781v1, 2018.

[6] Bilgici, G., Tokeser, U. and Unal, Z., $k$- Fibonacci and $k$-Lucas generalized quaternions, *Konuralp J. Math.* 5 (2017), 102-113.

[7] Catarino, P., On some identities and generating functions for $k$-Pell numbers, *Int. J. Mathematical Analysis* 7 (2013), 1877-1884.

[8] Ell, T. A., Bihan, N. L. and Sangwine, S. J., Quaternion Fourier transforms for signal and image processing, John Wiley and Sons, 2014.

[9] Falcon, S. and Plaza, A., On the Fibonacci $k$-numbers, *Chaos Solitons Fractals* 32 (2007), 1615-1624.

[10] Finkelstein, D., Jauch, J. M., Schiminovich, S. and Speiser, D., Foundations of quaternion quantum mechanics, *J. Math. Phys.* 3 (1962), 207-220.

[11] Girard, P. R., The quaternion group and modern physics, *Eur. J. Phys.* 5 (1984), 25-32.

[12] Halici, S., On Fibonacci quaternions, *Adv. Appl. Clifford Algebras* 22 (2012), 321-327.

[13] Halici, S., On complex Fibonacci quaternions, *Adv. Appl. Clifford Algebras* 23 (2013), 105-112.

[14] Hamilton, W. R., Lectures on quaternions, Hodges and Smith, Dublin, 1853.

[15] Horadam, A. F., A generalized Fibonacci sequence, *The American Math. Monthly* 68 (1961), 455-459.

[16] Horadam, A. F., Complex Fibonacci numbers and Fibonacci quaternions, *The American Math. Monthly* 70 (1963), 289-291.

[17] Horadam, A. F., Basic properties of a certain generalized sequence of numbers, *The Fibonacci Quart.* 3 (1965), 161-176.

[18] Horadam, A. F., Pell identities, *The Fibonacci Quart.* 9 (1971), 245-252.

[19] Horadam, A. F., Jacobsthal representation numbers, *The Fibonacci Quart.* 34 (1996), 40-54.

[20] Ipek, A., On $(p,q)$-Fibonacci quaternions and their Binet formulas, generating functions and certain binomial sums, *Adv. Appl. Clifford Algebras* 27 (2017), 1343-1351.

[21] Iyer, M. R., A note on Fibonacci quaternions, *The Fibonacci Quart.* 7 (1969), 225-229.

[22] Jhala, D., Sisodiya, K. and Rathore, G. P. S., On some identities for $k$-Jacobsthal numbers, *Int. J. Mathematical Analysis* 7 (2013), 551-556.

[23] Kosal, I. A., A note on hyperbolic quaternions, *Universal Journal of Mathematics and Applications* 1 (2018), 155-159.

[24] Koshy, T., Fibonacci and Lucas Numbers with Applications, John Wiley and Sons, New York, 2001.

[25] Mac-Farlane, A., Hyperbolic quaternions, *Proc. Roy. Soc. Edinburg* 23 (1902), 169-180.

[26] Patel, B. K. and Ray, P. K., On the properties of $(p,q)$-Fibonacci and $(p,q)$-Lucas quaternions, *Mathematical Reports* 21 (2019), 15-25.

[27] Polatli, E., Kizilates, C. and Kesim, S., On split $k$-Fibonacci and $k$-Lucas quaternions, *Adv. Appl. Clifford Algebras* 26 (2016), 353-362.

[28] Ramirez, J. L., Some combinatorial properties of the $k$-Fibonacci and the $k$-Lucas quaternions, *An. St. Univ. Ovidius Constanta* 23 (2015), 201-212.

[29] Ribenboim, P., My Numbers, My Friends: Popular Lectures on Number Theory, Springer-Verlag, New York, 2000.

[30] Senturk, T. D., Dasdemir, A., Bilgici, G. and Unal, Z., On unrestricted Horadam generalized quaternions, *Utilitas Mathematica* 110 (2019), 89-98.
[31] Tan, E., Yilmaz, S. and Sahin, M., On a new generalization of Fibonacci quaternions, *Chaos Solitons Fractals* 82 (2016), 1-4.
[32] Ward, J. P., Quaternions and Cayley Numbers: Algebra and Applications, Kluwer, London, 1997.

*Current address*: Tülay Yağmur: Department of Mathematics, Aksaray University, 68100 Aksaray, Turkey.
*E-mail address*: tulayyagmurr@gmail.com, tulayyagmur@aksaray.edu.tr
ORCID Address: http://orcid.org/0000-0002-6224-1921