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Twelve Kinds of Graphs of Lattice Implication Algebras Based on Filter and LI- Ideal

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Abstract: In this paper, at first we introduce the concepts of filter- annihilator, LI- ideal- annihilator, right- filter- annihilator, left- filter- annihilator, right- LI- ideal- annihilator, and left- LI- ideal- annihilator. Then by using of these concepts, are constructed six new types of graphs in a lattice implication algebra$(L, \vee, \wedge, ', \rightarrow, 0, I)$ which are denoted by $\Phi_F(L), \Phi_A(L), \Delta_F(L), \Sigma_F(L), \Delta_A(L)$, and $\Sigma_A(L)$, respectively. Then basic properties of graph theory such as connectivity, regularity, and planarity on the structure of these graphs are investigated. Secondly, by utilizing of binary operations $\oplus$ and $\otimes$ we construct graphs $\Psi_F(L)$ and $\Psi_A(L)$, respectively. Thirdly, via the binary operations $\oplus$ and $\otimes$, concept of annihilator we construct graphs $\Omega_F(L)$ and $\Omega_A(L)$, respectively. Finally, by utilizing of binary operations $\wedge$ and $\vee$, we construct graphs $\Upsilon_F(L)$ and $\Upsilon_A(L)$, respectively, some their interesting properties are presented.

Keywords: Lattice implication algebra, Diameter, Chromatic number, Euler graph.

1. Introduction

Algebraic combinatorics is an area of mathematics that employs methods of abstract algebra in various combinatorial contexts and vice versa. Associating a graph to an algebraic structure is a research subject in this area and has attracted considerable attention. In fact, the research in this subject aims at exposing the relationship between algebra and graph theory and at advancing the application of one to the other. The story goes back to a paper of Beck [1] in 1998, where he introduced the idea of a zero-divisor graph of a commutative ring with identity. He defined $\Gamma(R)$ to be the graph whose vertices are elements of R and in which two vertices x and y are adjacent if and only if $xy = 0$. Recently, Halas and Jukl in [2] introduced the zero divisor graphs of posets. The study of the zero-divisor graphs of posets was then

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continued by Xue and Liu in [3], Maimani et al. in [4]. More recently, a different method of associating a zero-divisor graph to a poset P was proposed by Lu and Wu in [5]. In order to research the logical system whose propositional value is given in a lattice, Xu [6] proposed the concept of lattice implication algebras, and discussed some of their properties. Xu and Qin [7] introduced the notions of filter in a lattice implication algebra, and investigated their properties. In [8], Y. B. Jun et al. proposed the concept of an LI- ideal of a lattice implication algebra. In this paper, we deal with zero-divisor graphs of lattice implication algebras based on filter and LI- ideal. Jun and Lee [9] defined the concept of associated graph of BCK-algebra and verified some properties of this graph. Zahiri and Borzooei [10] associated a new graph to a BCI-algebra which is denoted by $G(X)$, this definition is based on branches of X. The study of graphs of BCI/ BCK- algebras was then continued by Tahmasbpour such that in [11, 12] studied chordality of graph defined by Zahiri and Borzooei, introduced four types of graphs of BCK- algebras which are constructed by equivalence classes determined by ideal I and dual ideal $I^{\vee}$. Also, in [13, 14] introduced two new graphs of lattice implication algebras based on LI-ideal. Furthermore, in [15, 16] introduced two new graphs of BCK- algebras based on fuzzy ideal μ_I and fuzzy dual ideal $\mu_{I^{\vee}}$, two new graphs of lattice implication algebras based on fuzzy filter μ_F and fuzzy LI- ideal μ_A. In this paper, the graphs defined are slightly different from the graphs defined in [11, 12, 13, 14, 15, 16]. Also, this paper is divided into eight parts.

In Section 2, we recall some concepts of graph theory such as connected graph, planar graph, outerplanar graph, Eulerian graph, and chromatic number, among others.

Section 3, is an introduction to a general theory of lattice implication algebras. We will first give the notions of lattice implication algebras, and investigate their elementary and fundamental properties, and then deal with a number of basic concepts, such as filter, and LI-ideal, among others.

In Section 4, inspired by ideas from Behzadi et al. [17], we study the graphs of lattice implication algebras which are constructed from filter-annihilator and LI- ideal-annihilator, denoted by $\Phi_F(L)$ and $\Phi_A(L)$.

In Section 5, inspired by ideas from Behzadi et al. [17], we study the graphs of lattice implication algebras which are constructed from right- filter- annihilator, left- filter-annihilator, right- LI- ideal-annihilator, left- LI- ideal- annihilator, denoted by $\Delta_F(L), \Sigma_F(L), \Delta_A(L)$, and $\Sigma_A(L)$, respectively.

In Section 6,we introduce the associated graphs $\Psi_F(L)$ and $\Psi_A(L)$ which are constructed from binary operations $\oplus$ and $\otimes$, respectively.

In Section 7,we introduce the associated graphs $\Omega_F(L)$ and $\Omega_A(L)$ which are constructed from concept annihilator, binary operations $\oplus$ and $\otimes$, respectively.

In Section 8, inspired by ideas from Alizadeh et al. [18], we introduce the associated graphs $\Upsilon_F(L)$ and $\Upsilon_A(L)$, which are constructed from binary operations $\vee$ and $\wedge$, respectively.

2. Introduction to Graph Theory

In this section, for convenience of the reader, we recall some definitions and notations concerning graphs and posets for later use.

Definition 2.1. ([18, 19]) For a graph G, we denote the set of vertices of G as $V(G)$ and the set of edges as $E(G)$. A graph G is said to be complete if every two distinct vertices are joined by exactly one edge. The greatest induced complete subgraph denotes a clique. If graph G contains a clique with n elements, and every clique has at most n elements, we say that the clique number of G is n and write $\omega(G) = n$. Also, a graph G is said to be connected if there is a path between any given pairs of vertices, otherwise the graph is disconnected. For distinct vertices x and y of G, let $d(x,y)$ be the length of the shortest path from x to y and if there is no such path we define $d(x,y) \coloneqq \infty$. The diameter of G is $diam(G) \coloneqq \sup\{d(x,y); x,y \in V(G)\}$. Also, the girth of a graph G, is denoted by $gr(G)$, is the length of the shortest cycle in G if G has a cycle; otherwise, we get $gr(G) \coloneqq \infty$. The neighborhood of a vertex x is the set $N_G(\{x\}) = \{y \in V(G); xy \in E(G)\}$. Graph H is called a subgraph of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A graph G is called regular of degree k when every vertex has precisely k neighbors. A cubic graph is a graph in which all vertices have degree three. In other words, a cubic graph is a 3- regular graph. Moreover, for distinct vertices x and y, we use the notation $x - y$ to show that is x connected to y. Let $P = (V, \leq)$ be a poset. If $x \leq y$ but $x \neq y$, then we write $x < y$. If x and y are in V, then y covers x in P if $x < y$ and there is no $z \in V$, with $x < z < y$. Two sets $\{x \in P; x\ covers\ 0\}$ and $\{x \in P; 1\ covers\ x\}$, denoted by $atom(P)$ and $coatom(P)$, respectively. Let $L \subseteq P$, we say L is a chain if for all $x, y \in L, x \leq y$ or $y \leq x$. Chain L is maximal if for all chain $L', L \subseteq L'$ implies that $L = L'$.

Definition 2.2. ([1]) If K is the smallest number of colors needed to color the vertices of G so that no two adjacent vertices share the same color, we say that the chromatic number of G is K and write $\chi(G) = K$. Moreover, we have $\chi(G) \geq \omega(G)$.

Definition 2.3. ([19]) A closed walk in a graph G containing all the edges of G is called an Euler line in G. A graph containing an Euler line is called an Euler graph. We know that a walk is always connected. Since the Euler line (which is a walk) contains all the edges of the graph, an Euler graph is connected. Euler's theorem says that the connected graph G is Eulerian if and only if all vertices of G are of even degree.

Definition 2.4. ([20]) A subdivision of a graph is any graph that can be obtained from the original graph by replacing edges by paths. Graph G is planar if it can be drawn in a plane without the edges having to cross. Proving that a graph is planar amounts to redrawing the edges in such a way that no edges will cross. One may need to move the vertices around and the edges may have to be drawn in a very indirect fashion. Kuratowski's theorem says that a finite graph is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$. The clique number of any planar graph is less than or equal to four.

Definition 2.5. ([21]) Let G be a plane graph. A face is a region bounded by edges. An undirected graph is an outerplanar graph if it can be drawn in the plane without crossing in such a way that all of the vertices belong to the unbounded face of the drawing. There is a characterization of outerplanar graphs that says a graph is outerplanar if and only if it does not contain a subdivision of K_4 or $K_{2,3}$.

Definition 2.6. ([22]) The number g is called the genus of the surface if it is homeomorphic to a sphere with g handles or equivalently holes. Also, the genus g of a graph G is the smallest genus of all surfaces in such a way that the graph G can be drawn on it without any edge-crossing. The graphs of genus zero are precisely the planar graphs since the genus of a plane is zero. The graphs that can be drawn on a torus without edge- crossing are called toroidal. They have a genus of one since the genus of a torus is one. The notation $\gamma(G)$ stands for the genus of a graph G.

Theorem 2.7. ([23]) For the positive integers m and n, we have:

$(i)\ \gamma(K_n) = \left\lceil \frac{1}{12}(n-3)(n-4) \right\rceil$ if $n \geq 3$,

$(ii)\ \gamma(K_{m,n}) = \left\lceil \frac{1}{4}(m-2)(n-2) \right\rceil$ if $m, n \geq 2$.

3. Introduction to Lattice Implication Algebras

Definition 3.1. ([24]) By a lattice implication algebra we mean a bounded lattice $(L,\vee,\wedge,0,I)$ with order-reversing involution $'$ and a binary operation $\rightarrow$ satisfying the following axioms:

$(I1) x\rightarrow(y\rightarrow z)=y\rightarrow(x\rightarrow z)$,

$(I2) x\rightarrow x=I$,

$(I3) x\rightarrow y=y'\rightarrow x'$,

$(I4) x\rightarrow y=y\rightarrow x=I\Rightarrow x=y$,

$(I5) (x\rightarrow y)\rightarrow y=(y\rightarrow x)\rightarrow x$,

$(L1) (x\vee y)\rightarrow z=(x\rightarrow z)\wedge(y\rightarrow z)$,

$(L2) (x\wedge y)\rightarrow z=(x\rightarrow z)\vee(y\rightarrow z)$,

for all $x,y,z\in L$.

Note that the conditions $(L1)$ and $(L2)$ are equivalent to the conditions

$(L3) x\rightarrow(y\wedge z)=(x\rightarrow y)\wedge(x\rightarrow z)$, and

$(L4) x\rightarrow(y\vee z)=(x\rightarrow y)\vee(x\rightarrow z)$, respectively.

We can define a partial ordering $\leq$ on a lattice implication algebra L by $x\leq y$ if and only if $x\rightarrow y=I$. Therefore, the following statements hold:

(i) If $x\leq y$, then $x\rightarrow z\geq y\rightarrow z$ and $z\rightarrow x\leq z\rightarrow y$.

(ii) If $x\leq y$, then $y'\leq x'$.

Definition 3.2. ([24]) A subset F of L is called a filter of L if it satisfies the following conditions:

$(i) I\in F$,

$(ii) (\forall x,y\in L),(x\rightarrow y\in F, x\in F\rightarrow y\in F)$.

A filter P of L is prime if $x\vee y\in P$ implies $x\in P$ or $y\in P$.

Definition 3.3. ([24]) A nonempty subset A of a lattice implication algebra L is said to be an LI- ideal of L if

$(i) 0\in A$.

(ii) $(x\rightarrow y)'\in A$ and $y\in A$ imply $x\in A$, for any $x,y\in L$.

An LI- ideal A of L is prime if $x \wedge y \in A$ implies $x \in A$ or $y \in A$.

Definition 3.4. ([24]) Binary operations $\oplus$ and $\otimes$ as follows:

$$x \oplus y = x' \rightarrow y, x \otimes y = (x \rightarrow y')'.$$

Theorem 3.5. ([24]) The following statements hold for any $x, y, a, b \in L$:

$(i) x \otimes y = y \otimes x, x \oplus y = y \oplus x.$

$(ii) x \otimes y \leq x \leq x \oplus y, x \otimes y \leq y \leq x \oplus y.$

$(iii) 0 \otimes x = 0, I \otimes x = x, x \otimes x' = 0, 0 \oplus x = x, I \oplus x = I, x \oplus x' = I.$

(iv) If $x \leq a, y \leq b$, then $x \otimes y \leq a \otimes b, x \oplus y \leq a \oplus b$.

4. Graphs of lattice implication algebras based on filter and LI-ideal via the concepts of filter- annihilator and LI- ideal-annihilator

Definition 4.1. Let M be a nonempty subset of L, F and A be a filter, an LI- ideal of L, respectively. Then, the set of all zero-divisors of A by F and A are defined as follows:

(i) $Ann_F M = \{x \in L; x \rightarrow m \in F \ or \ m \rightarrow x \in F, \forall \ m \in M\}$.

(ii) $Ann_A M = \{x \in L; \ (x \rightarrow m)' \in A \ or (m \rightarrow x)' \in A, \forall \ m \in M\}$.

Proposition 4.2. Let M and N be nonempty subsets of L, F and A be a filter, an LI- ideal of L, respectively. Then, the following statements hold:

(i) $F \cup \{0\} \subseteq \ Ann_F M, \ A \cup \{I\} \subseteq \ Ann_A M.$

(ii) If $M \subseteq N$, then $Ann_F N \subseteq Ann_F M$ and $Ann_A N \subseteq \ Ann_A M.$

(iii) If $0 \in M$, then $Ann_F M = Ann_F(M - \{0\})$ and $Ann_A M = Ann_A(M - \{0\})$.

(iv) If $I \ \in M$, then $Ann_F M = \ Ann_F(M - \{I\})$ and $Ann_A M = Ann_A(M - \{I\})$.

(v) $Ann_F F = L$ and $Ann_A A = L$.

(vi) If $F = \{I\}, A = \{0\}$, then we have

$Ann_F M = \{y; y \ is \ comparable \ to \ any \ element \ in \ M\}$, $Ann_A M = \{y; y \ is \ comparable \ to \ any \ element \ in \ M\}$.

Proof. (i) Let $x \in F$, then by Definition 3.1 (iii), we have $m \rightarrow x \in F, \forall \ m \in M$. Also, $0 \rightarrow x = I, \forall x \in L$, So $F \cup \{0\} \subseteq Ann_F M$. Similarly, we can prove $A \cup \{I\} \subseteq Ann_A M$.

(ii) Suppose that $x \in Ann_F N$, then $x \to n \in F$ or $n \to x \in F$, $\forall\, n \in N$, but $M \subseteq N$, therefore $x \to n \in F$ or $n \to x \in F, \forall n \in M$. i.e $x \in Ann_F M$, hence $Ann_F N \subseteq Ann_F M$. Similarly, we can prove $Ann_A N \subseteq Ann_A M$.

(iii) According to Definition 4.1 (i), we have $Ann_F M = \cap_{m\in M} Ann_F m$. Also, $Ann_F\{0\} = L$. Then, $Ann_F M = Ann_F(M - \{0\})$. Similarly, we can prove $Ann_A M = Ann_A(M - \{0\})$.

(iv) According to Definition 4.1 (i), we have $Ann_F M = \cap_{m\in M} Ann_F m$. Also, $Ann_F\{I\} = L$. Then, $Ann_F M = Ann_F(M - \{I\})$. Similarly, we can prove $Ann_A M = Ann_A(M - \{I\})$.

(v) Let $x \in L$, we know by Definition 3.2, $x \to m \in F, \forall\, m \in F$, then $x \in Ann_F F$, hence $Ann_F F = L$. Similarly, we can prove $Ann_A A = L$

(vi) The proof is easy.

Definition 4.3. Let F and A be a filter, an LI- ideal of L, respectively. Then, we have:

(i) $\Phi_F(L)$ is a simple graph, with vertex set L and two distinct vertices x and y being adjacent if and only if $Ann_F\{x, y\} = F \cup \{0\}$.

(ii) $\Phi_A(L)$ is a simple graph, with vertex set L and two distinct vertices x and y being adjacent if and only if $Ann_A\{x, y\} = A \cup \{I\}$.

Example 4.4. Let $L = \{0, a, b, c, d, I\}$ and the operation $\to$ be defined by the following table:

TABLE 1. Binary operation $\to$ for Example 4.4

$\to$	0	a	b	c	d	I
0	I	I	I	I	I	I
a	c	I	b	c	b	I
b	d	a	I	b	a	I
c	a	a	I	I	a	I
d	b	I	I	b	I	I
I	0	a	b	c	d	I

Therefore, $(L, \wedge, \vee, ', \to, 0, I)$ is a lattice implication algebra. One can see that $F = \{b, c, I\}, A = \{0, c\}$ are a filter, an LI- ideal of L, respectively. Also, we have $Ann_F\{0\} = Ann_F\{a\} = Ann_F\{b\} = Ann_F\{c\} = Ann_F\{d\} = Ann_F\{I\} = L$ and $Ann_A\{0\} = Ann_A\{a\} = Ann_A\{b\} = Ann_A\{c\} = Ann_A\{d\} = Ann_A\{I\} = L$. Therefore, the graphs $\Phi_F(L)$ and $\Phi_A(L)$ are empty graphs.

Theorem 4.5. Let F and A be a filter, an LI- ideal of L, respectively. Then the following statements hold:

(i) $N_G(\{0\}) = N_G(\{I\}) = \emptyset$, where $G = \Phi_F(L)$.

$(ii) N_G(\{0\}) = N_G(\{I\}) = \emptyset$, where $G = \Phi_A(L)$.

Proof. (i) We know $Ann_F\{0\} = L$ and $Ann_F\{I\} = L$. Also, for all $x \in L, x \neq 0, I$, we have, $F \cup \{0, x\} \subseteq Ann_F\{x\}$. Then $F \cup \{0, x\} \subseteq Ann_F\{0, x\}$ and $F \cup \{0, x\} \subseteq Ann_F\{x, I\}$, for all $x \in L, x \neq 0, I$. So, by Definition 4.3 (i) of graph $\Phi_F(L)$, for all $x \in L, x \neq 0, I$, if x is connected to elements $0, I$, then $x \in F$. So by proposition 4.2 (v), $Ann_F\{x\} = L$. So, $0, I$ are not connected to x, for all $x \in L$.

(ii) We know $Ann_A\{0\} = L$ and $Ann_A\{I\} = L$. Also, for all $x \in L, x \neq 0, I$, we have, $A \cup \{0, x\} \subseteq Ann_A\{x\}$. Then $A \cup \{0, x\} \subseteq Ann_A\{0, x\}$ and $A \cup \{0, x\} \subseteq Ann_A\{x, I\}$, for all $x \in L, x \neq 0, I$. So, by Definition 4.3 (ii) of graph $\Phi_A(L)$, for all $x \in L, x \neq 0, I$, if x is connected to elements $0, I$, then $x \in A$. So by Proposition 4.2 (v), $Ann_A\{x\} = L$. So, $0, I$ are not connected to x, for all $x \in L$.

Theorem 4.6. Let $L = \{0, I\} \cup atom(L), F = \{I\}$ and $A = \{0\}$ be a filter, an LI- ideal of L, respectively. Then, $E\big(\Phi_F(L)\big) = E\big(\Phi_A(L)\big) = \{xy; x, y \in atom(L)\}$.

Proof. We know $Ann_{\{I\}}\{I\} = L$, $Ann_{\{I\}}\{0\} = L$, by Proposition 4.2 (vi), since $L = atom(L) \cup \{0, I\}$, we have, for all $x \in atom(L)$, $Ann_{\{I\}}\{x\} = \{0, x, I\}$. On the other hand we know $Ann_{\{I\}}\{x, y\} = Ann_{\{I\}}\{x\} \cap Ann_{\{I\}}\{y\}$. Then by Definition 4.3 (i) of graph $\Phi_{\{I\}}(L)$, x and y are adjacent if and only if $x, y \in atom(L)$. Similarly, we have $Ann_{\{0\}}\{I\} = L$, $Ann_{\{0\}}\{0\} = L$ and for all $x \in atom(L)$, $Ann_{\{0\}}\{x\} = \{0, x, I\}$. Then by Definition 4.3 (ii) of graph $\Phi_{\{0\}}(L)$, x and y are adjacent if and only if $x, y \in atom(L)$.

Theorem 4.7. Let $L = \{0, I\} \cup atom(L), F = \{I\}$ and $A = \{0\}$ be a filter, an LI- ideal of L, respectively. Then, $\omega\left(\Phi_{\{I\}}(L)\right) = \omega\left(\Phi_{\{0\}}(L)\right) = |atom(L)|$.

Proof. Straightforward by Theorem 4.6.

Theorem 4.8. Let $F = \{I\}$ and $A = \{0\}$ be a filter, an LI- idealof L, respectively. Then the following statements hold:

$(i) N_G(\{x\}) = \{y; y\ is\ not\ comparable\ to\ x\}$, where $G = \Phi_F(L), x \neq 0, I$.

$(ii) N_G(\{x\}) = \{y; y\ is\ not\ comparable\ to\ x\}$, where $G = \Phi_A(L), x \neq 0, I$.

Proof. (i) We have, for all $x \in L, x \neq 0, I, Ann_{\{I\}}\{x\} = \{y; y\ is\ comparable\ to\ x\}$. On the other hand, we know $Ann_{\{I\}}\{x,y\} = Ann_{\{I\}}\{x\} \cap Ann_{\{I\}}\{y\}$. Then by Definition 4.3 (i) of graph $\Phi_{\{I\}}(L)$, x and y are adjacent if and only if x and y are not comparable to each other.

(ii) We have, for all $x \in L, x \neq 0, I,\ Ann_{\{0\}}\{x\} = \{y; y\ is\ comparable\ to\ x\}$. On the other hand, we know $Ann_{\{0\}}\{x,y\} = Ann_{\{0\}}\{x\} \cap Ann_{\{0\}}\{y\}$. Then by Definition 4.3 (ii) of graph $\Phi_{\{0\}}(L)$, x and y are adjacent if and only if x and y are not comparable to each other.

Theorem 4.9. Let F and A be a filter, an LI- ideal of L, respectively. Then the following statements hold:

$(i)\alpha\big(\Phi_F(L)\big) \geq |F|$.

$(ii)\alpha\big(\Phi_A(L)\big) \geq |A|$.

Proof. (i) We suppose that $x, y \in F$. Then by Proposition 4.2 (v), we have, $Ann_F\{x\} = L$, $Ann_F\{y\} = L$. Therefore, by Definition 4.3 (i) of graph $\Phi_F(L)$, $xy \notin E\big(\Phi_F(L)\big)$. Therefore, by Definition 2.1 of independent set, we have $\alpha\big(\Phi_F(L)\big) \geq |F|$.

(ii)We suppose that $x, y \in A$. Then by Proposition 4.2 (v), we have, $Ann_A\{x\} = L$, $Ann_A\{y\} = L$. Therefore, by Definition 4.3 (ii) of graph $\Phi_A(L)$, $xy \notin E\big(\Phi_A(L)\big)$. Therefore, by Definition 2.1 of independent set, we have $\alpha\big(\Phi_A(L)\big) \geq |A|$.

Theorem 4.10. Let F be a prime filter, A be a prime LI- ideal of $L, |L - F| > 1, |L - A| > 1$. Then the following statements hold:

$(i)\Phi_F(L)$ is an empty graph.

$(ii)\Phi_A(L)$ is an empty graph.

Proof. (i) We suppose, on the contrary, that $\Phi_F(L)$ is not an empty graph. Therefore, there exist $x, y \in L$, such that $xy \in E\big(\Phi_F(L)\big)$. So, by Definition 4.3 (i) of graph $\Phi_F(L)$, we have, $Ann_F\{x,y\} = F \cup \{0\}$. On the other hand, since $|L - F| > 1$, we can choose $z \in L, z \notin F, z \neq 0$. Since F is a prime filter, then $z \to x \in F$ or $x \to z \in F$, and $z \to y \in F$ or $y \to z \in F$, hence $z \in Ann_F\{x,y\}$ that is contradiction, complete proof.

(ii) We suppose, on the contrary, that $\Phi_A(L)$ is not an empty graph. Therefore, there exist $x, y \in L$, such that $xy \in E\big(\Phi_A(L)\big)$. So, by Definition 4.3 (ii) of graph $\Phi_A(L)$, we have, $Ann_A\{x,y\} = A \cup \{I\}$. On the other hand, since $|L - A| > 1$, we can choose $z \in L, z \notin A, z \neq I$. Since A is a prime LI- ideal, then $(z \to x)' \in A$ or $(x \to z)' \in A$, and $(z \to y)' \in A$or $(y \to z)' \in A$, hence $z \in Ann_A\{x,y\}$ that is contradiction, complete proof.

5. Graphs of lattice implication algebras based on filter and LI- ideal via the concepts of right- filter- annihilator, left- filter- annihilator, right- LI- ideal- annihilator, and left- LI- ideal- annihilator

Definition 5.1. Let F and A be a filter, an LI- ideal of L, respectively. Denote $Ann_F^R\{x\} = \{y \in L; x \to y \in F\}$, $Ann_F^L\{x\} = \{y \in L; y \to x \in F\}$, $Ann_A^R\{x\} = \{y \in L;\ (x \to y)' \in A\}$, $Ann_A^L\{x\} = \{y \in L; (y \to x)' \in A\}$, which are called right- filter- annihilator, left- filter- annihilator, right- LI- ideal- annihilator, left- LI- ideal- annihilator, respectively.

Definition 5.2. Let F and A be a filter, an LI- ideal of L, respectively. Then, we have:

(i) $\Delta_F(L)$ is a simple graph, with vertex set L and two distinct vertices x and y being adjacent if and only if $Ann_F^R\{x\} \subseteq Ann_F^R\{y\}$ or $Ann_F^R\{y\} \subseteq Ann_F^R\{x\}$, there is an edge between x and y in the graph $\Sigma_F(L)$ if and only if $Ann_F^L\{x\} \subseteq Ann_F^L\{y\}$ or $Ann_F^L\{y\} \subseteq Ann_F^L\{x\}$.

(ii) $\Delta_A(L)$ is a simple graph, with vertex set L and two distinct vertices x and y being adjacent if and only if $Ann_A^R\{x\} \subseteq Ann_A^R\{y\}$ or $Ann_A^R\{y\} \subseteq Ann_A^R\{x\}$, there is an edge between x and y in the graph $\Sigma_A(L)$ if and only if $Ann_A^L\{x\} \subseteq Ann_A^L\{y\}$ or $Ann_A^L\{y\} \subseteq Ann_A^L\{x\}$.

Example 5.3. Let $L = \{0, a, b, c, I\}$. Define the partially ordered relation on L as $0 < a < b < c < I$, and define $,'' \to ''$ as follows:

TABLE 2. Binary operation → for Example 5.3

$\to$	0	a	b	c	I
0	I	I	I	I	I
a	c	I	I	I	I
b	b	c	I	I	I
c	a	b	c	I	I
I	0	a	b	c	I

Then $(L, \vee, \wedge, ', \to)$ is a lattice implication algebra. Let $F = \{I\}$ and $A = \{0\}$. Therefore, we have $Ann_F^R\{0\} = Ann_A^R\{0\} = L, Ann_F^R\{a\} = Ann_A^R\{a\} = \{a, b, c, I\}, Ann_F^R\{b\} = Ann_A^R\{b\} = \{b, c, I\}, Ann_F^R\{c\} = Ann_A^R\{c\} = \{c, I\}, Ann_F^R\{I\} = Ann_A^R\{I\} = \{I\}$.

Also, $Ann_F^L\{0\} = Ann_A^L\{0\} = \{0\}, Ann_F^L\{a\} = Ann_A^L\{a\} = \{0, a\}, Ann_F^L\{b\} = Ann_A^L\{b\} = \{0, a, b\}, Ann_F^L\{c\} = Ann_A^L\{c\} = \{0, a, b, c\}, Ann_F^L\{I\} = Ann_A^L\{I\} = L.$

Therefore graphs $\Delta_F(L), \Delta_A(L), \Sigma_F(L)$, and $\Sigma_A(L)$ are complete graphs, respectively.

Proposition 5.4. Let F and A be a filter , an LI- ideal of L, respectively. Then, the following statements hold:

$(i)\omega\big(\Delta_F(L)\big) \geq max\{|A|; A\ is\ a\ chain\ in\ L\}$.

$(ii)\omega\big(\Sigma_F(L)\big) \geq max\{|A|; A\ is\ a\ chain\ in\ L\}$.

$(iii)\omega\big(\Delta_A(L)\big) \geq max\{|A|; A\ is\ a\ chain\ in\ L\}$.

$(iv)\omega\big(\Sigma_A(L)\big) \geq max\{|A|; A\ is\ a\ chain\ in\ L\}$.

Proof. (i) According to Definition 3.1 (i), if $x \leq y$ then, $y \to z \leq x \to z$. On the other hand now we let $x \leq y, z \in Ann_F^R\{y\}$. Then, by Definition 5.1, $y \to z \in F$. So, by Definition 3.2 of filter, $x \to z \in F$. So, $z \in Ann_F^R\{x\}$. Then, $Ann_F^R\{y\} \subseteq Ann_F^R\{x\}, xy \in E\big(\Delta_F(L)\big)$, complete proof.

(ii) According to Definition 3.1 (i), if $x \leq y$ then, $z \to x \leq z \to y$. On the other hand now we let $x \leq y, z \in Ann_F^L\{x\}$. Then, by Definition 5.1, $z \to x \in F$. So, by Definition 3.2 of filter, $z \to y \in F$. So, $z \in Ann_F^L\{y\}$. Then, $Ann_F^L\{x\} \subseteq Ann_F^L\{y\}, xy \in E\big(\Sigma_F(L)\big)$, complete proof.

(iii) According to Definition 3.1 $(i),(ii)$, if $x \leq y$ then, $(x \to z)' \leq (y \to z)'$. On the other hand, now, we let $x \leq y, z \in Ann_A^R\{y\}$ then, by Definition 5.1 $(y \to z)' \in A$. So, by Definition 3.3 of LI- ideal, $(x \to z)' \in A$. So, $z \in Ann_A^R\{x\}$ then, $Ann_A^R\{y\} \subseteq Ann_A^R\{x\}, xy \in E\big(\Delta_A(L)\big)$, complete proof.

(iv) According to Definition 3.1 $(i),(ii)$, if $x \leq y$ then $(z \to y)' \leq (z \to x)'$. On the other hand, now, we let $x \leq y, z \in Ann_A^L\{x\}$ then, by Definition 5.1 $(z \to x)' \in A$. So, by Definition 3.3 of LI- ideal, $(z \to y)' \in A$. So, $z \in Ann_A^L\{y\}$ then, $Ann_A^L\{x\} \subseteq Ann_A^L\{y\}, xy \in E\big(\Sigma_A(L)\big)$, complete proof.

Theorem 5.5. Let F and A be a filter, an LI- ideal of L, respectively. Then, the following statements hold:

$(i)\Delta_F(L)$ is connected, $diam\big(\Delta_F(L)\big) \leq 2, gr\big(\Delta_F(L)\big) = 3$.

$(ii)\Sigma_F(L)$is connected, $diam\big(\Sigma_F(L)\big) \leq 2$, $gr\big(\Sigma_F(L)\big) = 3$.

$(iii)\Delta_A(L)$ is connected, $diam\big(\Delta_A(L)\big) \leq 2, gr\big(\Delta_A(L)\big) = 3$.

$(iv)\Sigma_A(L)$ is connected, $diam\big(\Sigma_A(L)\big) \leq 2, gr\big(\Sigma_A(L)\big) = 3$.

Proof. (i) For all $x \in L, 0 \leq x \leq I$, then by Proposition 5.4 (i), $0, I$ are connected to any element in L. So, $\Delta_F(L)$ is connected, $diam(\Delta_F(L)) \leq 2, gr(\Delta_F(L)) = 3$.

(ii) For all $x \in L, 0 \leq x \leq I$, then by Proposition 5.4 (ii), $0, I$ are connected to any element in L. So, $\Sigma_F(L)$ is connected, $diam(\Sigma_F(L)) \leq 2, gr(\Sigma_F(L)) = 3$.

(iii) For all $x \in L, 0 \leq x \leq I$, then by Proposition 5.4 (iii), $0, I$ are connected to any element in L. So, $\Delta_A(L)$ is connected, $diam(\Delta_A(L)) \leq 2, gr(\Delta_A(L)) = 3$.

(iv) For all $x \in L, 0 \leq x \leq I$, then by Proposition 5.4 (iv), $0, I$ are connected to any element in L. So, $\Sigma_A(L)$ is connected, $diam(\Sigma_A(L)) \leq 2, gr(\Sigma_A(L)) = 3$.

Theorem 5.6. Let F and A be a filter, an LI- ideal of L, respectively. Then, the following statements hold:

$(i)\Delta_F(L)$ is regular if and only if it is complete.

$(ii)\Sigma_F(L)$ is regular if and only if it is complete.

(iii) $\Delta_A(L)$ is regular if and only if it is complete.

(iv) $\Sigma_A(L)$ is regular if and only if it is complete.

Proof. (i) Suppose that $\Delta_F(L)$ is regular. By Theorem 5.5(i), $\deg(0) = |L| - 1$. Since $\Delta_F(L)$ is regular, for all $x \in L, \deg(x) = |L| - 1$. Hence, $\Delta_F(L)$ is complete. Conversely, a complete graph is regular.

(ii) Suppose that $\Sigma_F(L)$ is regular. By Theorem 5.5(ii), $\deg(0) = |L| - 1$. Since $\Sigma_F(L)$ is regular, for all $x \in L, \deg(x) = |L| - 1$. Hence, $\Sigma_F(L)$ is complete. Conversely, a complete graph is regular.

(iii) Suppose that $\Delta_A(L)$ is regular. By Theorem 5.5(iii), $\deg(0) = |L| - 1$. Since $\Delta_A(L)$ is regular, for all $x \in L, \deg(x) = |L| - 1$. Hence, $\Delta_A(L)$ is complete. Conversely, a complete graph is regular.

(iv) Suppose that $\Sigma_A(L)$ is regular. By Theorem 5.5(iv), $\deg(0) = |L| - 1$. Since $\Sigma_A(L)$ is regular, for all $x \in L, \deg(x) = |L| - 1$. Hence, $\Sigma_A(L)$ is complete. Conversely, a complete graph is regular.

Theorem 5.7. Let L be a chain, F and A be a filter, an LI- ideal of L, respectively. Then, the following statements hold:

$(i)\Delta_F(L), \Sigma_F(L), \Delta_A(L)$, and $\Sigma_A(L)$ are planar graphs if and only if $|L| \leq 4$.

$(ii)\Delta_F(L), \Sigma_F(L), \Delta_A(L)$, and $\Sigma_A(L)$ are outerplanar graphs if and only if $|L| \leq 3$.

$(iii)\Delta_F(L), \Sigma_F(L), \Delta_A(L)$, and $\Sigma_A(L)$ are planar graphs if and only if $|L| \leq 7$.

Proof. (i) According to Proposition 5.4, $\Delta_F(L), \Sigma_F(L), \Delta_A(L)$, and $\Sigma_A(L)$ are complete graphs, respectively, if $|L| \geq 5$, then $\Delta_F(L), \Sigma_F(L), \Delta_A(L)$, and $\Sigma_A(L)$ have a subgraph isomorphic to K_5, respectively, then by Kuratowski's theorem $\Delta_F(L), \Sigma_F(L), \Delta_A(L)$, and $\Sigma_A(L)$ are not planar, respectively. Conversely, we knowK_5 has five vertices, hence if $\Delta_F(L), \Sigma_F(L), \Delta_A(L)$, and $\Sigma_A(L)$ are not planar, respectively, then $\Delta_F(L), \Sigma_F(L), \Delta_A(L)$, and $\Sigma_A(L)$ have at least five vertices, respectively, which is contrary to $|L| \leq 4$.

(ii) According to Proposition 5.4, $\Delta_F(L), \Sigma_F(L), \Delta_A(L)$, and $\Sigma_A(L)$ are complete graphs, respectively, if $|L| \geq 4$, then $\Delta_F(L), \Sigma_F(L), \Delta_A(L)$, and $\Sigma_A(L)$ have a subgraph isomorphic to K_4, respectively, then by Definition $\Delta_F(L), \Sigma_F(L), \Delta_A(L)$, and $\Sigma_A(L)$ are not outerplanar, respectively. Conversely, we knowK_4 has four vertices, hence if $\Delta_F(L), \Sigma_F(L), \Delta_A(L)$, and $\Sigma_A(L)$ are not outerplanar, respectively, then $\Delta_F(L), \Sigma_F(L), \Delta_A(L)$, and $\Sigma_A(L)$ have at least four vertices, respectively, which is contrary to $|L| \leq 3$.

(iii) According to Proposition 5.4, $\Delta_F(L), \Sigma_F(L), \Delta_A(L)$, and $\Sigma_A(L)$ are complete graphs, respectively, if $|L| \geq 8$, then $\Delta_F(L), \Sigma_F(L), \Delta_A(L)$, and $\Sigma_A(L)$ have a subgraph isomorphic to K_8, respectively, then by Theorem $\Delta_F(L), \Sigma_F(L), \Delta_A(L)$, and $\Sigma_A(L)$ are not toroidal, respectively. Conversely, we knowK_8 has eight vertices, hence if $\Delta_F(L), \Sigma_F(L), \Delta_A(L)$, and $\Sigma_A(L)$ are not toroidal, respectively, then $\Delta_F(L), \Sigma_F(L), \Delta_A(L)$, and $\Sigma_A(L)$ have at least eight vertices, respectively, which is contrary to $|L| \leq 7$.

6. Graphs of lattice implication algebras based on filter and LI-ideal via the binary operations $\oplus$ and $\otimes$.

Definition 6.1. Let F and A be a filter, an LI- ideal of L, respectively. Then we have:

$(i)\Psi_F(L)$ is a simple graph, with vertex set L and two distinct vertices x and y being adjacent if and only if $x \oplus y \in F$.

(ii) $\Psi_A(L)$ is a simple graph, with vertex set L and two distinct vertices x and y being adjacent if and only if $x \otimes y \in A$.

Example 6.2. Let $L = \{0, a, b, c, d, I\}$ be lattice implication algebra defined in Example 4.4 , $F = \{b, c, I\}, A = \{0,\ c\}$ be a filter, an LI- ideal of L, respectively. Therefore, by Definition 3.2, binary operations $\oplus$ and $\otimes$ are produced by the following tables:

TABLE 3. Binary operation $\oplus$ for Example 6.2

$\oplus$	0	a	b	c	d	I
0	0	a	b	c	d	I
a	a	a	I	I	a	I
b	b	I	I	b	I	I
c	c	I	b	c	b	I
d	d	a	I	b	a	I
I	I	I	I	I	I	I

TABLE 4. Binary operation $\otimes$ for Example 6.2

$\otimes$	0	a	b	c	d	I
0	0	0	0	0	0	0
a	0	a	d	0	d	a
b	0	d	c	c	0	b
c	0	0	c	c	0	c
d	0	d	0	0	0	d
I	0	a	b	c	d	I

Therefore,$\left(\Psi_F(L)\right) = \{0b, 0c, 0I, ab, ac, aI, bc, bd, bI, cd, cI, dI\}$,and $E\left(\Psi_A(L)\right) = \{0a, 0b, 0c, 0d, 0I, ac, bc, bd, cd, cI, 0I\}$.

Theorem 6.3. Let F and A be a filter, an LI- ideal of L, respectively. Then the following statements hold:

(i) $\deg(a) = |L| - 1$ in $\Psi_F(L)$, where $a \in F$.

(ii) $\deg(a) = |L| - 1$ in $\Psi_A(L)$, where $a \in A$.

Proof. (i) We know by Theorem 3.5 (ii) , for all $x \in L, x \oplus a \geq a$, where $a \in F$. Then by Definition 3.2 of filter $x \oplus a \in F$. So by Definition 6.1 (i) of graph $\Psi_F(L), xa \in E\left(\Psi_F(L)\right)$. Then $\deg(a) = |L| - 1$.

(ii) We know by Theorem 3.5 (ii), for all $x \in L, x \otimes a \leq a$, where $a \in A$. Then by Definition 3.3 of LI- ideal $x \otimes a \in A$. So, by Definition 6.1 (ii) of graph $\Psi_A(L), xa \in E\left(\Psi_A(L)\right)$. Then $\deg(a) = |L| - 1$.

Theorem 6.4. Let F and A be a filter, an LI-ideal of L, respectively. Then the following statements hold:

$(i)\Psi_F(L)$ is regular if and only if it is complete.

$(ii)\Psi_A(L)$ is regular if and only if it is complete.

(i) Suppose that $\Psi_F(L)$ is regular. Since by Theorem 6.3 (i), $\deg(a) = |L| - 1, a \in F,$ we have $\deg(x) = |L| - 1$, for all $x \in L$. Hence $\Psi_F(L)$ is complete. Conversely a complete graph is regular.

(ii) Suppose that $\Psi_A(L)$ is regular. Since by Theorem 6.3 (ii), $\deg(a) = |L| - 1, a \in A,$ we have $\deg(x) = |L| - 1$, for all $x \in L$. Hence $\Psi_A(L)$ is complete. Conversely a complete graph is regular.

Theorem 6.5. Let F and A be a filter, an LI- ideal of L, respectively, $x, y, a, b \in L, x \leq a, y \leq b$. Then, the following statements hold:

(i) If $xy \in E\big(\Psi_F(L)\big)$, then$ab \in E\big(\Psi_F(L)\big)$.

(ii) If $ab \in E\big(\Psi_A(L)\big)$, then $xy \in E\big(\Psi_A(L)\big)$.

Proof. (i) We know by Theorem 3.5 (iv) that if $x \leq a$ and $y \leq b$, then $x \oplus y \leq a \oplus b$. If $xy \in E\big(\Psi_F(L)\big)$ based on Definition 6.1 (i) of graph $\Psi_F(L), x \oplus y \in F$. Thus by Definition 3.2 of filter $a \oplus b \in F$. Hence, $ab \in E\big(\Psi_F(L)\big)$.

(ii) We know from Theorem 3.5 (iv) that if $x \leq a$ and $y \leq b$, then $x \otimes y \leq a \otimes b$, and if $ab \in E\big(\Psi_A(L)\big)$, then by Definition 6.1 (ii) of graph $\Psi_A(L), a \otimes b \in A$. Thus by Definition 3.3 of LI- ideal $x \otimes y \in A$. Hence, $xy \in E\big(\Psi_A(L)\big)$.

Theorem 6.6. Let F and A be a filter, an LI- ideal of L, respectively. Then, the following statements hold:

(i) $\deg(0) = |F|$ in the graph $\Psi_F(L)$.

(ii) $\deg(I) = |A|$ in the graph $\Psi_A(L)$.

Proof. (i) According to Theorem 3.5 (iii), $0 \oplus x = x \in F$, for all $x \in F$. So, by Definition 6.1 (i) of graph $\Psi_F(L)$, element 0 is connected to any element of F. So, $deg(0) = |F|$.

(ii) According to Theorem 3.5 (iii), $I \otimes x = x \in A$, for all $x \in A$. So, by Definition 6.1 (ii) of graph $\Psi_A(L)$, element I is connected to any element of A. So, $deg(I) = |A|$.

Theorem 6.7. Let F and A be a filter, an LI- ideal of L, respectively. Then, the following statements hold:

$(i) x' \in N_G(x)$, where $G = \Psi_F(L)$.

$(ii) x' \in N_G(x)$, where $G = \Psi_A(L)$.

(i) According to Theorem 3.5 (iii), $x' \oplus x = I \in F$, for all $x \in F$. So, by Definition 6.1 (i) of graph $\Psi_F(L)$, element x is connected to x'. So, $x' \in N_G(x)$.

(ii) According to Theorem 3.5 (iii), $x' \otimes x = 0 \in A$, for all $x \in A$. So, by Definition 6.1 (ii) of graph $\Psi_A(L)$, element x is connected to x'. So, $x' \in N_G(x)$.

Theorem 6.8. Let F and A be a filter, an LI- ideal of L, respectively, $x \leq y$, then the following statements hold:

$(i)\deg(x) \leq \deg(y)$ in the graph $\Psi_F(L)$.

(ii) $\deg(y) \leq \deg(x)$ in the graph $\Psi_A(L)$.

Proof. (i) Let $x \leq y, z$ be connected to x then $z \oplus x \in F$. On the other hand, $z \oplus x \leq z \oplus y$, then $z \oplus y \in F$. Therefore, by Definition 6.1 (i) of graph $\Psi_F(L)$, z is connected to y, thus $\deg(x) \leq \deg(y)$.

(ii) Let $x \leq y, z$ be connected to y then $z \otimes y \in A$. On the other hand, $z \otimes x \leq z \otimes y$, then $z \otimes x \in A$. Therefore, by Definition 6.1 (ii) of graph $\Psi_A(L), z$ is connected to x. Thus, $\deg(y) \leq \deg(x)$.

Theorem 6.9. Let F and A be a filter, an LI- ideal of L, respectively. Then, the following statements hold:

$(i) gr(\Psi_F(L)) = \{3, \infty\}$.

$(ii) gr(\Psi_A(L)) = \{3, \infty\}$.

Proof. (i) Let $|coatom(L)| \geq 2$ then we can choose $m, m' \in coatom(L)$. It is clear that $I - m - m' - I$ is a cycle of length 3. Now, suppose $|coatom(L)| = 1$, then we have $coatom(L) = \{m\}$. Now we have the following cases:

(i) If $|L| \geq 4$. Then there exist $x_i \in L, x_i \neq 0, m, I$. So, $x_i' \to m = I$. Since $x_i' \leq m$. Otherwise $x_i' > m$ that implies $x_i' = I$, then $x_i = 0$ that is contradiction. Then $I - x_i - m - I$ is a cycle of length 3. Thus, in this case we have $gr(\Psi_F(L)) = 3$.

(ii) If $|L| = 3, m \in F$. Then $L = \{0, m, I\}$. Also, we have $0 \oplus m = m \in F$. Then $0 - m - I - 0$ is a cycle of length 3. Thus $gr(\Psi_F(L)) = 3$.

(iii) If $|L| = 3, F = \{I\}$, we have $0 \oplus m = m \notin F$. Thus 0 is not connected to m, so $\Psi_F(L)$ is star graph. Then $gr(\Psi_F(L)) = \infty$.

(ii) Let $|atom(L)| \geq 2$ then we can choose $a, a' \in atom(L)$. It is clear that $0 - a - a' - 0$ is a cycle of length 3. Now, suppose $|atom(L)| = 1$, then we have $atom(L) = \{a\}$. Now we have the following cases:

(i) If $|L| \geq 4$. Then there exist $y_i \in L, y_i \neq 0, a, I$. So, $(a \to y_i')' = 0$. Since $a \leq y_i'$. Otherwise $a > y_i'$ that implies $y_i' = 0$, then $y_i = I$ that is contradiction. Then $0 - a - y_i - 0$ is a cycle of length 3. Thus, in this case we have $gr(\Psi_A(L)) = 3$.

(ii) If $|L| = 3, a \in A$. Then $L = \{0, a, I\}$. Also, we have $I \otimes a = a \in A$. Then $I - a - 0 - I$ is a cycle of length 3. Thus $gr(\Psi_A(L)) = 3$.

(iii) If $|L| = 3, A = \{0\}$, we have $I \otimes a = a \notin A$. Thus I is not connected to a, so $\Psi_A(L)$ is star graph. Then $gr(\Psi_A(L)) = \infty$.

Theorem 6.10. Let F and A be a filter, an LI- ideal of L, respectively. Then, the following statements hold:

$(i)\omega(\Psi_F(L)) \geq \max\{|F|, |coatom(L)| + 1\}$.

$(ii)\omega(\Psi_A(L)) \geq \max\{|A|, |atom(L)| + 1\}$.

Proof. (i) We have $m \vee n = I$, for all $m, n \in coatom(L)$, since $m \vee n \leq m' \to n = m \oplus n$. Then $m \oplus n = I$. So, $mn \in E(\Psi_F(L))$. Also, for all $x, y \in F, x \oplus y = x' \to y \in F$. Thus $xy \in E(\Psi_F(L))$, this implies that $\omega(\Psi_F(L)) \geq \max\{|F|, |coatom(L)| + 1\}$.

(ii) We have $a \wedge b = 0$, for all $a, b \in atom(L)$, since $a \otimes b = (a \to b')' \leq a \wedge b$. Then $a \otimes b = 0$. So, $ab \in E(\Psi_A(L))$. Also, for all $x, y \in A, x \otimes y = (x \to y')' \in A$. Thus $xy \in E(\Psi_A(L))$, this implies that $\omega(\Psi_A(L)) \geq \max\{|A|, |atom(L)| + 1\}$.

Theorem 6.11. Let F and A be a filter, an LI- ideal of L, respectively. Then, the following statements hold:

$(i)\Psi_F(L)$ is an Euler graph if and only if $|L|$ is odd.

$(ii)\Psi_A(L)$ is an Euler graph if and only if $|L|$ is odd.

Proof. (i) Theorem 6.3 (i) says that $\Psi_F(L)$ is connected. So, by Euler's theorem, $\Psi_F(L)$ is an Euler graph if and only if the degree of any vertex is even. Therefore, if $\Psi_F(L)$ is an Euler graph, then $deg(I)$ is even. On the other hand, by Theorem 6.3 (i), $\deg(I) = |L| - 1$ in the graph $\Psi_F(L)$. Therefore, if $\Psi_F(L)$ is an Euler graph, then $|L|$ is odd. Hence, this is proved.

(ii) Theorem 6.3 (ii) says that $\Psi_A(L)$ is connected. So, by Euler's theorem, $\Psi_A(L)$ is an Euler graph if and only if the degree of any vertex is even. Therefore, if $\Psi_A(L)$ is an Euler graph, then $deg(0)$ is even. On the other hand, by Theorem 6.3 (ii), $\deg(0) = |L| - 1$ in the graph $\Psi_A(L)$. Therefore, if $\Psi_A(L)$ is an Euler graph, then $|L|$ is odd. Hence, this is proved.

Theorem 6.12. Let F and A be a filter, an LI- ideal of L, respectively. Then, the following statements hold:

(i) If $|coatom(L)| \geq 4$, then $\Psi_F(L)$ is not planar.

(ii) If $|coatom(L)| \geq 3$, then $\Psi_F(L)$ is not outerplanar.

(iii) If $|coatom(L)| \geq 7$, then $\Psi_F(L)$ is not toroidal.

(iv) If $|atom(L)| \geq 4$, then $\Psi_A(L)$ is not planar.

(v) If $|atom(L)| \geq 3$, then $\Psi_A(L)$ is not outerplanar.

(vi) If $|atom(L)| \geq 7$, then $\Psi_A(L)$ is not toroidal.

Proof. (i) We know vertex I is connected to any element in L. Also, for all $x, y \in coatom(L)$, we have $x \oplus y = I \in F$. Since $x \vee y \leq x' \to y, x \vee y = I$. Then, for all $x, y \in coatom(L), xy \in E(\Psi_F(L))$. So, by assumption, $\Psi_F(L)$ has a subgraph isomorphic to K_5. Then, by Kuratowski's theorem $\Psi_F(L)$ is not planar.

(ii) We know vertex I is connected to any element in L. Also, for all $x, y \in coatom(L)$, we have $x \oplus y = I \in F$. Since $x \vee y \leq x' \to y, x \vee y = I$. Then, for all $x, y \in coatom(L), xy \in E(\Psi_F(L))$. So, by assumption, $\Psi_F(L)$ has a subgraph isomorphic to K_4. Then, by Definition 2.5 $\Psi_F(L)$ is not outerplanar.

(iii) We know vertex I is connected to any element in L. Also, for all $x, y \in coatom(L)$, we have $x \oplus y = I \in F$. Since $x \vee y \leq x' \to y, x \vee y = I$. Then, for all $x, y \in coatom(L), xy \in E(\Psi_F(L))$. So, by assumption, $\Psi_F(L)$ has a subgraph isomorphic to K_8. Then, by Theorem 2.7 $\Psi_F(L)$ is not toroidal.

(iv) We know vertex 0 is connected to any element in L. Also, for all $x, y \in atom(L)$, we have $x \otimes y = 0 \in A$. Since $x \wedge y \geq (x' \to y)', x \wedge y = 0$. Then, for all $x, y \in atom(L), xy \in E(\Psi_A(L))$. So, $\Psi_A(L)$ has a subgraph isomorphic to K_5. Then, by Kuratowski's theorem $\Psi_A(L)$ is not planar.

(v) We know vertex 0 is connected to any element in L. Also, for all $x, y \in atom(L)$, we have $x \otimes y = 0 \in A$. Since $x \wedge y \geq (x' \to y)', x \wedge y = 0$. Then, for all $x, y \in atom(L), xy \in$

$E(\Psi_A(L))$. So, $\Psi_A(L)$ has a subgraph isomorphic to K_4. Then, by Definition 2.5 $\Psi_A(L)$ is not outerplanar.

(vi) We know vertex 0 is connected to any element in L. Also, for all $x, y \in atom(L)$, we have $x \otimes y = 0 \in A$. Since $x \wedge y \leq (x' \to y)', x \wedge y = 0$. Then, for all $x, y \in atom(L), xy \in E(\Psi_A(L))$. So, $\Psi_A(L)$ has a subgraph isomorphic to K_8. Then, by Theorem 2.7 $\Psi_A(L)$ is not toroidal.

7. Graphs of lattice implication algebras based on filter and LI-ideal via the concepts of annihilator, binary operations $\oplus$ and $\otimes$.

Definition 7.1. Let F and A be a filter, an LI- ideal of L, respectively. Then the set of all zero-divisors of x by F and A are defined as follows:

$(i) Ann_{\oplus}\{x\} = \{y \in L; x \oplus y \in F\}$.

$(ii) Ann_{\otimes}\{x\} = \{y \in L; x \otimes y \in A\}$.

Definition 7.2. Let F and A be a filter, an LI- ideal of L, respectively. Then we have:

$(i) \Omega_F(L)$ is a simple graph with vertex set L and two distinct vertices x and y being adjacent if and only if $Ann_{\oplus}\{x, y\} = F$.

$(ii) \Omega_A(L)$ is a simple graph with vertex set L and two distinct vertices x and y being adjacent if and only if $Ann_{\otimes}\{x, y\} = A$.

Example 7.3. Consider lattice implication algebra is defined in Example 4.4, $F = \{b, c, I\}$, and $A = \{0, c\}$. Then, we have $Ann_{\oplus}\{b\} = Ann_{\oplus}\{c\} = Ann_{\oplus}\{I\} = L, Ann_{\oplus}\{0\} = Ann_{\oplus}\{a\} = Ann_{\oplus}\{d\} = F, Ann_{\otimes}\{0\} = Ann_{\otimes}\{c\} = L, Ann_{\otimes}\{a\} = Ann_{\otimes}\{I\} = A, Ann_{\otimes}\{b\} = Ann_{\otimes}\{d\} = \{0, b, c, d\}$, Then $E(\Omega_F(L)) = \{0a, 0b, 0c, 0d, 0I, ab, ac, ad, aI, bd, cd, dI\}$ and $E(\Omega_A(L)) = \{0a, ab, ac, ad, aI, 0I, bI, cI, dI\}$.

Theorem 7.4. Let F and A be a filter, an LI- ideal of L, respectively. Then the following statements hold:

(i) $\deg(0) = |L| - 1, \deg(I) = |D_{\oplus}(L)|$, in the graph $G = \Omega_F(L)$, where $D_{\oplus}(L) = \{x \in L;\ Ann_{\oplus}\{x\} = F\}$.

(ii) $\deg(I) = |L| - 1, \deg(0) = |D_{\otimes}(L)|$, in the graph $G = \Omega_A(L)$, where $D_{\otimes}(L) = \{x \in L;\ Ann_{\otimes}\{x\} = A\}$.

Proof. (i) We know $N_G(\{0\}) = \{x \in L;\ Ann_{\oplus}\{0, x\} = F\}, Ann_{\oplus}\{0, x\} = Ann_{\oplus}\{0\} \cap Ann_{\oplus}\{x\} = F \cap Ann_{\oplus}\{x\} = F$, then $\deg(0) = |L| - 1, N_G(\{I\}) = \{x \in L;\ Ann_{\oplus}\{x, I\} = F\} = D_{\oplus}(L)$. Thus, $\deg(I) = |D_{\oplus}(L)|$.

(ii) We know $N_G(\{I\}) = \{x \in L;\ Ann_{\otimes}\{x, I\} = A\}, Ann_{\otimes}\{x, I\} = Ann_{\otimes}\{x\} \cap Ann_{\otimes}\{I\} = Ann_{\otimes}\{x\} \cap A = A$, then $\deg(I) = |L| - 1, N_G(0) = \{x \in L;\ Ann_{\otimes}\{0, x\} = A\} = D_{\otimes}(L)$. Thus, $\deg(0) = |D_{\otimes}(L)|$.

Theorem 7.5. Let F and A be a filter, an LI- ideal of L, respectively. Then the following statements hold:

$(i) diam\left(\Omega_F(L)\right) \leq 2$.

$(ii) diam\left(\Omega_A(L)\right) \leq 2$.

Proof. (i) We know by Theorem 7.4 that the vertex 0 is connected to every element in L. Now, if there exist $x, y \in L, x, y \neq 0$ and $xy \in E\left(\Omega_F(L)\right)$, then $diam\left(\Omega_F(L)\right) = 1$; otherwise, $diam\left(\Omega_F(L)\right) = 2$.

(ii) We know by Theorem 7.4 that vertex I is connected to every element in L. Now, if there exist $x, y \in L, x, y \neq I, xy \in E\left(\Omega_A(L)\right)$, then $diam\left(\Omega_A(L)\right) = 1$; otherwise, $diam\left(\Omega_A(L)\right) = 2$.

Theorem 7.6. Let F and A be a filter, an LI- ideal of L, respectively. Then the following statements hold:

(i) Graph $\Omega_F(L)$ is regular if and only if it is complete.

(ii) Graph $\Omega_A(L)$ is regular if and only if it is complete.

Proof. (i) Suppose that $\Omega_F(L)$ is regular. We have $\deg(0) = |L| - 1$. Since $\Omega_F(L)$ is regular, $\deg(x) = |L| - 1$, for all $x \in L$. Hence, $\Omega_F(L)$ is complete. Conversely, a complete graph is regular.

(ii) Suppose that $\Omega_A(L)$ is regular. We have $\deg(I) = |L| - 1$. Since $\Omega_A(L)$ is regular, $\deg(x) = |L| - 1$, for all $x \in L$. Hence, $\Omega_A(L)$ is complete. Conversely, a complete graph is regular.

Theorem 7.7. Let F and A be a proper filter, a proper LI- ideal of L, respectively. Then the following statements hold:

$(i)\alpha(\Omega_F(L)) \geq |F|$, where $G = \Omega_F(L)$.

$(ii)\alpha(\Omega_A(L)) \geq |A|$, where $G = \Omega_A(L)$.

Proof. (i) We know for all $x, y \in F, Ann_{\oplus}\{x\} = L$ and $Ann_{\oplus}\{y\} = L$. Then, $Ann_{\oplus}\{x, y\} = L$. Now, if $xy \in E(\Omega_F(L))$, then $L = F$ which is contradiction. Then $\alpha(\Omega_F(L)) \geq |F|$.

(ii) We know for all $x, y \in A, Ann_{\otimes}\{x\} = L$ and $Ann_{\otimes}\{y\} = L$. Then, $Ann_{\otimes}\{x, y\} = L$. Now, if $xy \in E(\Omega_A(L))$, then $L = A$ which is contradiction. Then $\alpha(\Omega_A(L)) \geq |A|$.

Theorem 7.8. Let F be a filter of L. Then $\Omega_F(L)$ is a star graph if satisfies the two following conditions:

$(i)|D_{\oplus}(L)| = 1, D_{\oplus}(L) = \{x \in L;\ Ann_{\oplus}\{x, y\} = F\}$.

$(ii)|atom(L)| = 1$.

Proof. We know the vertex 0 is connected to every element of L. Now, suppose there exist $x, y \neq 0$ in such away that $xy \in E(\Omega_F(L))$, thus $Ann_{\oplus}\{x, y\} = F$. On the other hand $|atom(L)| = 1$. Let $a \in atom(L)$, thus $a \leq x$ and $a \leq y$, if there exists t where $t \oplus a \in F$, then $t \oplus x \in F$ and $t \oplus y \in F$, since $Ann_{\oplus}\{x, y\} = F$, we have $t \in F$, which implies $a \in D_{\oplus}(L)$, this is contrary to $|D_{\oplus}(L)| = 1$ as $0 \in D_{\oplus}(L), a \neq 0$.

Theorem 7.9. Let A be an LI- ideal of L. Then $\Omega_A(L)$ is a star graph if satisfies the two following conditions:

$(i)|D_{\otimes}(L)| = 1, D_{\otimes}(L) = \{x \in L;\ Ann_{\otimes}\{x, y\} = A\}$.

$(ii)|coatom(L)| = 1$.

Proof. We know the vertex I is connected to every element of L. Now, suppose there exist $x, y \neq I$ in such away that $xy \in E(\Omega_A(L))$, thus $Ann_{\otimes}\{x, y\} = A$. On the other hand $|coatom(L)| = 1$. Let $m \in coatom(L)$, thus $x \leq m$ and $y \leq m$, if there exists s where $s \otimes m \in A$, then $s \otimes x \in A$ and $s \otimes y \in A$, since $Ann_{\otimes}\{x, y\} = A$, we have $s \in A$, which implies $m \in D_{\otimes}(L)$, this is contrary to $|D_{\otimes}(L)| = 1$ as $I \in D_{\otimes}(L), m \neq I$.

Proposition 7.10. Suppose that $|D_{\oplus}(L)| = n, |D_{\otimes}(L)| = n$, then the following statements hold:

$(i)\omega(\Omega_F(L)) \geq n + 1$.

$(ii)\omega\big(\Omega_A(L)\big) \geq n+1$.

Proof. (i) Let $\big|D_{\oplus}(L)\big| = n$, then there exist $x_1, x_2, \dots, x_n \in D_{\oplus}(L)$. So, for all $i = 1, 2, \dots, n, t \oplus x_i \in F$ implies $t \in F$, then $x_i x_j \in E\big(\Omega_F(L)\big)$ for all $i, j = 1, 2, \dots, n$. Also, the vertex 0 is connected to every element in L. Hence, $\Omega_F(L)$ contains a clique of length $n+1$. So, by Definition 2.1 of clique number $\omega(\Omega_F(L)) \geq n+1$.

(ii) Let $\big|D_{\otimes}(L)\big| = n$, then there exist $x_1, x_2, \dots, x_n \in D_{\otimes}(L)$. So, for all $i = 1, 2, \dots, n, t \otimes x_i \in A$ suggests $t \in A$. Then $x_i x_j \in E\big(\Omega_A(L)\big)$ for all $i, j = 1, 2, \dots, n$. Also, the vertex I is connected to every element in L. Hence, $\Omega_A(L)$ contains a clique of length $n+1$. So, by Definition 2.1 of clique number $\omega\big(\Omega_A(L)\big) \geq n+1$.

Theorem 7.11. Let F and A be a filter, an LI- ideal of L, respectively. Then the following statements hold:

$(i)\Omega_F(L)$ is an Euler graph if and only if $|L|$ is odd.

$(ii)\Omega_A(L)$ is an Euler graph if and only if $|L|$ is odd.

Proof. (i) According to Theorem 7.4 (i), we know $\Omega_F(L)$ is a connected graph. So, based on Euler's theorem, which states that a connected graph is an Euler graph if and only if the degree of every vertex is even, hence $\Omega_F(L)$ is an Euler graph, then $\deg(0)$ is even. Meanwhile, according to Theorem 7.4 (i), we have $\deg(0) = |L| - 1$, therefore, if $\Omega_F(L)$ is an Euler graph, then $|L|$ is odd. Hence, this is proved completely.

(ii) According to Theorem 7.4 (ii), we know $\Omega_A(L)$ is a connected graph. So, based on Euler's theorem , which states that a connected graph is an Euler graph if and only if the degree of every vertex is even, thus, if $\Omega_A(L)$ is an Euler graph, then $\deg(I)$ is even. On the other hand, with Theorem 7.4 (ii), we have $\deg(I) = |L| - 1$, so, if $\Omega_A(L)$ is an Euler graph, then $|L|$ is odd. Hence, this is proved completely.

Theorem 7.12. Let F and A be a filter, an LI- ideal of L, respectively. Also, $D_{\oplus}(L) = \{m, I\}, D_{\otimes}(L) = \{0, a\}$, where $m \in coatom(L), a \in atom(L), A = \{x \in L; m\ covers\ x\}$, and $B = \{x \in L; x\ covers\ a\}$. Then the following statements hold:

(i) If $|A| \geq 3$, then $\Omega_F(L)$ is not planar.

(ii) If $|A| \geq 2$, then $\Omega_F(L)$ is not outerplanar.

(iii) If $|A| \geq 6$, then $\Omega_F(L)$ is not toroidal.

(iv) If $|B| \geq 3$, then $\Omega_A(L)$ is not planar.

(v) If $|B| \geq 2$, then $\Omega_A(L)$ is not outerplanar.

(vi) If $|B| \geq 6$, then $\Omega_A(L)$ is not toroidal.

Proof. (i) Let $|A| \geq 3$, then there exist $x_1, x_2, x_3 \in A$. We have $t \oplus x_i \leq t \oplus m$ for all $i = 1, 2, 3$. If there exists t where $t \oplus x_i \in F, i = 1, 2, 3$, then $t \oplus m \in F$. Since $D_{\oplus}(L) = \{m, I\}$, then $t \in F$. So, $mx_i, x_ix_j \in E\big(\Omega_F(L)\big)$ for all $i, j = 1, 2, 3$. Also, the vertex 0 is connected to every element in L. So, the induced subgraph of $\Omega_F(L)$ on $\{0, x_1, x_2, x_3, m\}$ is isomorphic to K_5. Thus, based on Kuratowski's theorem, $\Omega_F(L)$ is not planar.

(ii) Let $|A| \geq 2$, then there exist $x_1, x_2 \in A$. We have $t \oplus x_i \leq t \oplus m$ for all $i = 1, 2$. If there exists t such that $t \oplus x_i \in F, i = 1, 2$, then $t \oplus m \in F$. Since $D_{\oplus}(L) = \{m, I\}$, then $t \in F$. So, $x_ix_j, x_im \in E\big(\Omega_F(L)\big)$ for all $i, j = 1, 2$. Further, the vertex 0 is connected to every element in L. So, the induced subgraph of $\Omega_F(L)$ on $\{0, x_1, x_2, m\}$ is isomorphic to K_4. Hence, based on Definition 2.5, $\Omega_F(L)$ is not outerplanar.

(iii) Let $|A| \geq 6$, then there exist $x_1, \dots, x_6 \in A$. We have $t \oplus x_i \leq t \oplus m$ for all $i = 1, \dots, 6$. If there exists t where $t \oplus x_i \in F, i = 1, \dots, 6$, then $t \oplus m \in F$. Since $D_{\oplus}(L) = \{m, I\}$, then $t \in F$. So, $x_ix_j, x_im \in E\big(\Omega_F(L)\big)$ for all $i, j = 1, \dots, 6$. Also, the vertex 0 is connected to every element in L. So, the induced subgraph of $\Omega_F(L)$ on $\{0, x_1, \dots, x_6, m\}$ is isomorphic to K_8. Then, according to Theorem 2.7, $\Omega_F(L)$ is not toroidal.

(iv) Let $|B| \geq 3$, then there exist $x_1, x_2, x_3 \in B$. We have $s \otimes a \leq s \otimes x_i$ for all $i = 1, 2, 3$. If there exists s such that $s \otimes x_i \in A, i = 1, 2, 3$, then $s \otimes a \in A$. Therefore, $s \in A$. So, $ax_i, x_ix_j \in E\big(\Omega_A(L)\big)$ for all $i, j = 1, 2, 3$. In addition, the vertex I is connected to every element in L. Hence, the induced subgraph of $\Omega_A(L)$ on $\{a, x_1, x_2, x_3, I\}$ is isomorphic to K_5. Then, with Kuratowski's theorem, $\Omega_A(L)$ is not planar.

(v) Let $|B| \geq 2$, then there exist $x_1, x_2 \in B$. We have $s \otimes a \leq s \otimes x_i$ for all $i = 1, 2$, then $s \otimes a \in A$. Thus, $s \in A$. So, $ax_i, x_ix_j \in E\big(\Omega_A(L)\big)$ for all $i, j = 1, 2$. Additionally, the vertex I is connected to every element in L. So, the induced subgraph of $\Omega_A(L)$ on $\{a, x_1, x_2, I\}$ is isomorphic to K_4. Hence, based on Definition 2.5, $\Omega_A(L)$ is not outerplanar.

(vi) Let $|B| \geq 6$, then there exist $x_1, \dots, x_6 \in B$. We have $s \otimes a \leq s \otimes x_i$ for all $i = 1, \dots, 6$. If there exists s such that $s \otimes x_i \in A, i = 1, \dots, 6$. Also, the vertex I is connected to every element in L. So, the induced subgraph of $\Omega_A(L)$ on $\{a, x_1, \dots, x_6, I\}$ is isomorphic to K_8. Hence, by Theorem 2.7, $\Omega_A(L)$ is not toroidal.

8. Graphs of lattice implication algebras based on filter and LI-ideal via the binary operations ∨ and ∧.

Definition 8.1. Let F and A be a filter, an LI- ideal of L, respectively. Then, we have:

$(i)\Upsilon_F(L)$ is a simple graph, with vertex set L and two distinct vertices x and y are adjacent if and only if $x \vee y \in F$.

$(ii)\Upsilon_A(L)$ is a simple graph, with vertex set L and two distinct vertices x and y are adjacent if and only if $x \wedge y \in A$.

Example 8.2. Let $L = \{0, a, b, I\}$ and operators of L be defined in the following tables:

TABLE 5. Binary operation ∨ for Example 8.2

$\vee$	0	a	b	I
0	0	a	b	I
a	a	a	I	I
b	b	I	b	I
I	I	I	I	I

TABLE 6. Binary operation ∧ for Example 8.2

$\wedge$	0	a	b	I
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
I	0	a	b	I

TABLE 7. Binary operation → for Example 8.2

$\rightarrow$	0	a	b	I
0	I	I	I	I
a	b	I	b	I
b	a	a	I	I
I	0	a	b	I

TABLE 8. Unary operation ´ for Example 8.2

′	0	a	b	I
	I	b	a	0

Then $(L, \vee, \wedge, ', \rightarrow)$ is a lattice implication algebra. We suppose $F = \{I\}$ and $A = \{0\}$ be a filter, an LI- ideal of L, respectively. Then $E(\Upsilon_F(L)) = \{0I, ab, aI, bI\}$, and $E(\Upsilon_A(L)) = \{0a, 0b, 0I, ab\}$.

Lemma 8.3. Let F and A be a filter, an LI- ideal of L, respectively. Then, the following statements hold:

(i) $\deg(x) = |L| - 1$, in the graph $\Upsilon_F(L)$, where $x \in F$.

(ii) $\deg(x) = |L| - 1$, in the graph $\Upsilon_A(L)$, where $x \in A$.

Proof. (i) Let $x \in F, y$ be an arbitrary element in L, then $x \vee y \in F$. Since $x \leq x \vee y, F$ is a filter of L. So, $xy \in E(\Upsilon_F(L))$, complete proof.

(ii) Let $x \in A, y$ be an arbitrary element in L, then $x \wedge y \in A$. Since $x \wedge y \leq x, A$ is an LI- ideal of L. So, $xy \in E(\Upsilon_A(L))$, complete proof.

Theorem 8.4. Let F and A be a filter, an LI- ideal of L, respectively. Then, the following statements hold:

$(i)\Upsilon_F(L)$ is regular if and only if it is complete.

$(ii)\Upsilon_A(L)$ is regular if and only if it is complete.

Proof. (i) Let $\Upsilon_F(L)$ be a regular graph. By Lemma 8.3 (i), we have $\deg(I) = |L| - 1$. Now, since $\Upsilon_F(L)$ is regular, then for any $x \in L, \deg(x) = |L| - 1$. This means that $\Upsilon_F(L)$ is a complete graph. Conversely, a complete graph is regular.

(ii) Let $\Upsilon_A(L)$ be a regular graph. By Lemma 8.3 (ii), we have $\deg(0) = |L| - 1$. Now, since $\Upsilon_A(L)$ is regular, then for any $x \in L, \deg(x) = |L| - 1$. This means that $\Upsilon_A(L)$ is a complete graph. Conversely, a complete graph is regular.

Proposition 8.5. Let F and A be a filter, an LI- ideal of L, respectively. Then, the following statements hold:

$(i)\omega(\Upsilon_F(L)) \geq |F|$.

$(ii)\omega(\Upsilon_A(L)) \geq |A|$.

Proof. (i) Straightforward by Lemma 8.3 (i).

(ii) Straightforward by Lemma 8.3 (ii).

Theorem 8.6. Let F and A be a filter, an LI- ideal of L, respectively. Then, the following statements hold:

(i) $\Upsilon_F(L)$ is connected, $diam(\Upsilon_F(L)) \leq 2$.

(ii) $\Upsilon_A(L)$ is connected, $diam(\Upsilon_A(L)) \leq 2$.

Proof. (i) Straightforward by Lemma 8.3 (i).

(ii) Straightforward by Lemma 8.3 (ii).

Theorem 8.7. Let $F \neq \{I\}$ and $A \neq \{0\}$ be a filter, an LI- ideal of L, respectively. Then, the following statements hold:

$(i)gr(\Upsilon_F(L)) = 3$.

$(ii)gr(\Upsilon_A(L)) = 3$.

Proof. (i) Let $a \neq I$ be an element in F, x be an arbitrary element in L, then $I - a - x - I$ is a cycle of length 3 in $\Upsilon_F(L)$, complete proof.

(ii) Let $a \neq 0$ be an element in A, x be an arbitrary element in L, then $0 - a - x - 0$ is a cycle of length 3 in $\Upsilon_A(L)$, complete proof.

Proposition 8.8. Let F and A be a filter, an LI- ideal of L, respectively. Then, the following statements hold:

(i) If $\Upsilon_F(L)$ is planar, then $|F| \leq 4$.

(ii) If $\Upsilon_F(L)$ is outerplana, then $|F| \leq 3$.

(iii) If $\Upsilon_F(L)$ is toroidal, then $|F| \leq 7$.

(iv) If $\Upsilon_A(L)$ is planar, then $|A| \leq 4$.

(v) If $\Upsilon_A(L)$ is outerplanar, then $|A| \leq 3$.

(vi) If $\Upsilon_A(L)$ is toroidal, then $|A| \leq 7$.

Proof. (i) According to Lemma 8.3 (i), $\Upsilon_F(L)$ is a complete graph on F, if $|F| \geq 5$ then $\Upsilon_F(L)$ has a subgraph isomorphic to K_5 which by Kuratowski's theorem, $\Upsilon_F(L)$ is not planar.

(ii) According to Lemma 8.3 (i), $\Upsilon_F(L)$ is a complete graph on F, if $|F| \geq 4$ then $\Upsilon_F(L)$ has a subgraph isomorphic to K_4 which by Definition 2.5, $\Upsilon_F(L)$ is not outerplanar.

(iii) According to Lemma 8.3 (i), $\Upsilon_F(L)$ is a complete graph on F, if $|F| \geq 7$ then $\Upsilon_F(L)$ has a subgraph isomorphic to K_8 which by Theorem 2.7, $\Upsilon_F(L)$ is not toroidal.

(iv) According to Lemma 8.3 (ii), $\Upsilon_A(L)$ is a complete graph on A, if $|A| \geq 5$ then $\Upsilon_A(L)$ has a subgraph isomorphic to K_5 which by Kuratowski's theorem, $\Upsilon_A(L)$ is not planar.

(v) According to Lemma 8.3 (ii), $\Upsilon_A(L)$ is a complete graph on A, if $|A| \geq 4$ then $\Upsilon_A(L)$ has a subgraph isomorphic to K_4 which by Definition 2.5, $\Upsilon_A(L)$ is not outerplanar.

(vi) According to Lemma 8.3 (ii), $\Upsilon_A(L)$ is a complete graph on A, if $|A| \geq 7$ then $\Upsilon_A(L)$ has a subgraph isomorphic to K_8 which by Theorem 2.7, $\Upsilon_A(L)$ is not toroidal.

Theorem 8.9. Let F and A be a filter, an LI- ideal of L, respectively. Then, the following statements hold:

(i) If $\Upsilon_F(L)$ is an Euler graph then $|L|$ is odd.

(ii) If $\Upsilon_A(L)$ is an Euler graph then $|L|$ is odd.

Proof. (i) According to Lemma 8.3 (i), for all $x \in F$, $\deg(x) = |L| - 1$. Now, if $\Upsilon_F(L)$ is an Euler graph then degree of every vertex in F is even. So, $|L|$ is odd, complete proof.

(ii) According to Lemma 8.3 (ii), for all $x \in A$, $\deg(x) = |L| - 1$. Now, if $\Upsilon_A(L)$ is an Euler graph then degree of every vertex in A is even. So, $|L|$ is odd, complete proof.

Theorem 8.10. Let F and A be a filter, an LI- ideal of L, respectively. Then, the following statements hold:

(i) If $F = \cap_{1\le i\le n} P_i$ and, for each $1 \le j \le n, F \neq \cap_{1\le i\le n, i\neq j} P_i$, where P_i are prime filters of L. Then $\omega(\Upsilon_F(L)) = n = \chi(\Upsilon_F(L))$.

(ii) If $A = \cap_{1\le i\le n} P_i$ and, for each $1 \le j \le n,\ A \neq \cap_{1\le i\le n, i\neq j} P_i$, where P_i are prime LI- ideals of L. Then $\omega(\Upsilon_A(L)) = n = \chi(\Upsilon_A(L))$.

Proof. (i) For each j with $1 \le j \le n$, consider an element x_j in $(\cap_{1\le i\le n, i\neq j} P_i) - P_j$. We have $A = \{x_1, \dots, x_n\}$ is a clique in $\Upsilon_F(L)$. Hence $\omega(\Upsilon_F(L)) \ge n$. Now, we prove that $\chi(\Upsilon_F(L)) \le n$. Define a coloring f by putting $f(x) = \min\{i; x \notin P_i\}$. Let $f(x) = k$, x and y be adjacent vertices. So, $x \notin P_k$ and $x \vee y \in F$. Since P_k is prime, $y \in P_k$, and so $f(y) \neq k$. Now, since $\omega(\Upsilon_F(L)) \le \chi(\Upsilon_F(L))$, the result hold.

(ii) For each j with $1 \le j \le n$, consider an element x_j in $(\cap_{1\le i\le n, i\neq j} P_i) - P_j$. We have $A = \{x_1, \dots, x_n\}$ is a clique in $\Upsilon_A(L)$. Hence $\omega(\Upsilon_A(L)) \ge n$. Now, we prove that $\chi(\Upsilon_A(L)) \le n$. Define a coloring f by putting $f(x) = \min\{i; x \notin P_i\}$. Let $f(x) = k$, x and y be adjacent vertices. So, $x \notin P_k$ and $x \wedge y \in A$. Since P_k is prime, $y \in P_k$, and so $f(y) \neq k$. Now, since $\omega(\Upsilon_A(L)) \le \chi(\Upsilon_A(L))$, the result hold.

Theorem 8.11. Let F and A be a filter, an LI- ideal of L, respectively. Then, the following statements hold:

(i) If $F = \cap_{j\in J} P_j$, where P_j are prime filters of L, J is an infinite set and, for each $i \in J$, $F \neq \cap_{j\neq i} P_j$. Then $\omega(\Upsilon_F(L)) = \infty = \chi(\Upsilon_F(L))$.

(ii) If $A = \cap_{j\in J} P_j^{\vee}$where $P_j^{\vee}$ are prime LI- ideals of L, J is an infinite set and, for each $i \in J$, $A \neq \cap_{j\neq i} P_j^{\vee}$. Then $\omega(\Upsilon_A(L)) = \infty = \chi(\Upsilon_A(L))$.

Proof. (i) For each $i \in J$, there exists $x_i \in (\cap_{j\neq i} P_j - P_i)$. Now, one can easily see that the set of x_i forms an infinite clique in $\Upsilon_F(L)$. Since $\omega(\Upsilon_F(L)) \le \chi(\Upsilon_F(L))$, the assertion holds.

(ii) For each $i \in J$, there exists $x_i \in (\cap_{j\neq i} P_j - P_i)$. Now, one can easily see that the set of x_i forms an infinite clique in $\Upsilon_A(L)$. Since $\omega(\Upsilon_A(L)) \le \chi(\Upsilon_A(L))$, the assertion holds.

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Atena Tahmasbpour Meikola: Conceptualization, Methodology, Validation, Investigation, Writing- Original Draft, Writing- Review and Editing, Visualization, Project administration, Funding acquisition.

Declaration of Competing Interest

The author declares that there is no competing financial interests or personal relationships that influence the work in this paper.

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