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Research Article

Semisimple-direct-injective modules

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Abstract

The notion of simple-direct-injective modules which are a generalization of injective modules unifies $C2$ and $C3$-modules. In the present paper, we introduce the notion of the semisimple-direct-injective module which gives a unified viewpoint of $C2$, $C3$, SSP properties and simple-direct-injective modules. It is proved that a ring R is Artinian serial with the Jacobson radical square zero if and only if every semisimple-direct-injective right R-module has the SSP and, for any family of simple injective right R-modules $\{S_i\}_{\mathcal{I}}$, $\oplus_{\mathcal{I}} S_i$ is injective. We also show that R is a right Noetherian right V-ring if and only if every right R-module has a semisimple-direct-injective envelope if and only if every right R-module has a semisimple-direct-injective cover.

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1. Introduction

Throughout this article, unless otherwise stated, all rings have unity and all modules are unital. A right R-module M is called

a $C1$-*module* provided that every submodule of M is essential in a direct summand of M;

a $C2$-*module* (or direct-injective) provided that A is a direct summand in M whenever A is a submodule of M such that A is isomorphic to a direct summand in M and

a $C3$-*module* if A and B are direct summands in M and $A \cap B = 0$, then $A + B$ is a direct summand in M.

It is easy to see that each $C2$-module is also a $C3$-module. Conversely, for each module M, if $M \oplus M$ is a $C3$-module, then M is a $C2$-module (see also [1, Corollary 2.6]). However, $C3$ is a weaker property in general: if R is any integral domain which is not a field, then R is $C3$, but not $C2$. Recently, the classes of Ci-modules ($i = 1, 2, 3$) are studied and generalizations of them are considered ([1, 5, 6, 12, 14]).

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We recall also that a module M has *the summand sum property (SSP)* if the sum of two direct summands is a direct summand of M ([10] and [17]). Clearly, modules having (SSP) are $C3$.

Recently, Camillo, Ibrahim, Yousif and Zhou [5] obtained that every simple submodule which is isomorphic to a direct summand is itself a direct summand if and only if the sum of any two simple direct summands with zero intersection is again a direct summand [5, Proposition 2.1]. Such modules are called *simple-direct-injective* (see also [12]). In the present paper, we introduce the concept of semisimple-direct-injective modules. A module is called *semisimple-direct-injective* if every semisimple submodule isomorphic to a summand is itself a summand, or equivalently if the sum of any two semisimple summands (with zero intersection) is again a summand (see Proposition 2.1). Theorem 3.4 in [5] addressed the question of when every simple-direct-injective module is $C3$, and they proved that every simple-direct-injective right R-module is $C3$ if and only if R is an Artinian serial ring with Jacobson radical square zero. In Theorem 2.10, we prove that R is an Artinian serial ring with Jacobson radical square zero if and only if every semisimple-direct-injective right R-module has the SSP and $\oplus_{\mathfrak{J}} S_i$ is injective for any family of simple injective modules $\{S_i\}_{\mathfrak{J}}$.

Enochs [7] introduced the notation of injective cover as the dual notation of the injective envelope, and proved that a ring R is right Noetherian if and only if every right R-module has an injective cover. In Section 3, we are concerned with semisimple-direct-injective envelopes and covers, namely sdi-envelopes and sdi-covers. In Theorem 3.4, it is shown that the classes of semisimple-direct-injective modules over a ring R provide for sdi-envelopes and sdi-covers only if R is a right Noetherian V-ring.

A ring is called a *right V-ring* if every simple right R-module is injective. In Section 4, we study some natural connections between V-rings and semisimple-direct-injective modules which are similar to simple-direct-injective modules. For instance, we obtain that a ring is right Noetherian and a right V-ring if and only if every right R-module is semisimple-direct-injective if and only if every direct sum of two semisimple-direct-injective modules is semisimple-direct-injective (Theorem 2.11).

Throughout this article, a submodule N of an R-module M is called essential in M, denoted by $N \leq_e M$, if for any nonzero submodule L of M, $L \cap M \neq 0$. We write $J(R)$ and $Soc(R_R)$ for the Jacobson radical and the socle of R, respectively. We also write $N \leq_d M$ and $E(M)$ to indicate that N is a direct summand of M and the injective envelope of M, respectively. For a nonempty subset X of a ring R, the left annihilator of X in R is $l(X) = \{r \in R : rx = 0 \text{ for all } x \in X\}$. For any $a \in R$, we write $l(a)$ for $l(\{a\})$. Right annihilators are defined similarly. General background material can be found in [3], [6], [13] and [18].

2. Semisimple-direct-injective modules

Proposition 2.1. *The followings are equivalent for a right R-module M.*

(1) *For any semisimple submodules A, B of M with $A \cong B \leq_d M$, A is a summand of M.*
(2) *For any semisimple summands A, B of M with $A \cap B = 0$, the sum $A \oplus B$ is a summand of M.*
(3) *For any semisimple summands A, B of M, $A + B \leq_d M$.*
(4) *If $M = A_1 \oplus A_2$ with A_1 semisimple and $f : A_1 \to A_2$ is a homomorphism, then* $\mathrm{Im}(f) \leq_d A_2$.

Proof. (1) ⇒ (2) Assume $M = A \oplus A'$ and let $\pi : A \oplus A' \to A'$ be the canonical projection. Then $A \oplus B = A \oplus \pi(B)$ is a direct summand of M as $\pi(B) \cong B$.

(2) ⇒ (3) Straightforward.

(3) $\Rightarrow$ (4) Let $X := \{a - f(a) : a \in A_1\}$. Clearly, $X \oplus A_2 = M$. Furthermore, $A_1 \oplus \mathrm{Im}(f) = A_1 + X$ which is a direct summand of M by the hypothesis. Now, the conclusion follows.

(4) $\Rightarrow$ (1) Let $B \oplus B' = M$ and $\theta : B \to A$ be an isomorphism. Also set $f := \pi_{|A}\theta$, where $\pi : B \oplus B' \to B'$ is the canonical projection. Then $\mathrm{Im}(f) = \pi(A) \leq_d A_2$ by the assumption, so that $B + A = B \oplus \pi(A) \leq_d M$. Since $A \leq_d A + B$, we get $A \leq_d M$ as well. □

A module M is called *semisimple-direct-injective* if M satisfies the equivalent conditions of Proposition 2.1. A ring R is called right semisimple-direct-injective if R_R is semisimple-direct-injective.

Example 2.2. Every indecomposable module is semisimple-direct-injective. In particular, $\mathbb{Z}_{\mathbb{Z}}$ is a semisimple-direct-injective module which is not direct-injective.

Example 2.3. Every semisimple-direct-injective module is simple-direct-injective. The converse is true if the module is finitely generated or it has ACC on summands by [5, Proposition 2.5] and [5, Corolllary 2.9], respectively.

Proposition 2.4. *If any semisimple summand of a right R-module M is invariant under all idempotents of* $\mathrm{End}(M)$*, then M is semisimple-direct-injective.*

Proof. Let A, B be semisimple summands of the module M with $A \cap B = 0$. Let $M = A \oplus A'$ for some submodule A' of M. Consider the projections $\pi_1 : M \to A$ and $\pi_2 : M \to A'$. Since B is invariant under all idempotents of $\mathrm{End}(M)$, we obtain

$$\begin{aligned} B &\leq \pi_1(B) \oplus \pi_2(B) \\ &\leq [\pi_1(M) \cap B] \oplus [\pi_2(M) \cap B] \\ &= (A \cap B) \oplus (A' \cap B) \\ &= A' \cap B \leq A' \end{aligned}$$

This follows that B is a direct summand of M and so $A' = B \oplus B'$ for some submodule B' of A'. Thus,

$$M = A \oplus A' = A \oplus (B \oplus B') = (A \oplus B) \oplus B'.$$

□

Recall that R is called a *right V-ring* if every simple right R-module is injective. By Theorem 2.11 below, a ring R is right Noetherian and a right V-ring if and only if every right R-module is semisimple-direct-injective. On the other hand, a ring R is a right V-ring if and only if every right R-module is simple-direct-injective by [5, Proposition 4.1].

Example 2.5. (i) Let $Q := \prod_{i=1}^{\infty} F_i$ with $F_i := \mathbb{Z}_2$ and R be the subring of Q generated by $\bigoplus_{i=1}^{\infty} F_i$ and 1_Q. Then R is a commutative, non self-injective V-ring and $\mathrm{Soc}(R)$ is essential in R. We deduce that R is not Noetherian. Thus one infers that there exists a simple-direct-injective module over R which is not semisimple-direct-injective.

(ii) Let V be an infinite-dimensional vector space over F. Let $Q := \mathrm{End}_F(V)$, $J := \{x \in Q : dim_F(xV) < +\infty\}$ and $R := F + J$. Then R is a right V-ring (see [9, Example 6.19]) and R is not right Noetherian. Similarly (i), there is a simple-direct-injective right R-module which is not semisimple-direct-injective.

Example 2.6. If M is an indecomposable right R-module which is not simple, then $M \oplus E(M)$ is a semisimple-direct-injective module. Indeed, by [5, Lemma 3.3], $M \oplus E(M)$ has no simple summands.

Example 2.7. Given a field F and an isomorphism $F \to \overline{F} \subseteq F$ defined by $a \mapsto \overline{a}$, let R be the right F-space on basis $\{1, t\}$ with multiplication given by $t^2 = 0$ and $at = t\overline{a}$ for all $a \in F$. Assume that $1 < dim_{\overline{F}}(F) < \infty$. By Example 2.6, $R \oplus E(R)$ is a semisimple-direct-injective module which is not $C3$ (has not the SSP) by [5, Example 3.6].

Proposition 2.8. *If $M = \oplus_{i \in \mathcal{I}} E_i$ is a direct sum of indecomposable injective right R-modules E_i, then M is a semisimple-direct-injective module.*

Proof. Let A be the sum of the simples E_i and B be the sum of the non-simple ones. If S is isomorphic to a semisimple direct summand of M, then all simple summands of S are clearly injective, so that $S \cap B = 0$. Since $(B \oplus S) \cap A$ is a direct summand of A, we get the former is a direct summand of M, whence S is a direct summand of M. □

Corollary 2.9. *Let $\{S_i\}_{\mathcal{I}}$ be a family of simple injective modules and $\{E(S_j)\}_{\mathcal{K}}$ be a family of injective envelopes of simple non-injective modules S_j. Then $M = (\oplus_{i \in \mathcal{I}} S_i) \oplus (\oplus_{j \in \mathcal{K}} E(S_j))$ is a semisimple-direct-injective module.*

A module is *uniserial* if the lattice of its submodules is totally ordered under inclusion. A ring R is called right uniserial if R_R is a uniserial module. A ring R is called serial if both modules ${}_RR$ and R_R are direct sums of uniserial modules.

Now we investigate when semisimple-direct-injective modules have the SSP.

Theorem 2.10. *The followings are equivalent for a ring R:*

(1) *R is an Artinian serial ring with $J(R)^2 = 0$.*

(2) (a) *Every semisimple-direct-injective right R-module is a C3-module.*

(b) *For any family of simple injective modules $\{S_i\}_{\mathcal{I}}$, $\oplus_{\mathcal{I}} S_i$ is injective .*

(3) (a) *The right socle of R is finitely generated.*

(b) *Every semisimple-direct-injective right R-module is quasi-injective.*

Proof. (1) ⇒ (3) For any module M over an Artinian serial ring R with $J(R)^2 = 0$, we have a decomposition $M = A \oplus M$, where A is semisimple and B is a sum of injective serial modules of length 2 by [6, 13.5]. So, it is obvious that semisimple-direct-injective right R-modules are precisely those with A orthogonal to B. In this case, B is injective and A is injective relative to B. Thus, M is quasi-injective.

(3) ⇒ (2) As each quasi-injective module is a C3-module, one only needs to verify (b): If every semisimple-direct-injective right R-module is quasi-injective and every module having the zero socle is a semisimple-direct-injective module, then R is right semi-Artinian (i.e., all nonzero modules have nonzero socle). So, $E(R_R) = E(T_1) \oplus E(T_2) \oplus \cdots \oplus E(T_n)$ where each T_i is a minimal right ideal of R. Let $\{S_i\}_{\mathbb{N}}$ be a family of simple right R-modules. Let $M := (\oplus_{\mathbb{N}} E(S_i)) \oplus (\oplus_{j=1}^{n} E(T_j))$. By Lemma 2.8, M is a semisimple-direct-injective module and so, by (3-b), M is a quasi-injective module. Now one infers that $\oplus_{\mathbb{N}} E(S_i)$ is $E(R_R)$-injective and hence it is injective.

(2) ⇒ (1) We first prove R is right Noetherian. Let $\{S_i\}_{\mathbb{N}}$ be a family of simple right R-modules. We claim that $\oplus_{\mathbb{N}} E(S_i)$ is an injective module. By [4, Theorem 1.3], one infers that there exists an infinite subset $\mathcal{J}$ of $\mathbb{N}$ such that $\oplus_{\mathcal{J}} E(S_i)$ is injective. Write $\mathbb{N} = \mathcal{J}_1 \cup \mathcal{J}_2$ such that S_i is injective if $i \in \mathcal{J}_1$ and S_j is not injective if $j \in \mathcal{J}_2$. By the assumption, $\oplus_{\mathcal{J}_1} S_i$ is injective. Now we can assume that $|\mathcal{J}_2|$ is infinite. Note that $M = (\oplus_{\mathcal{J}_2} E(S_j)) \oplus E(\oplus_{\mathcal{J}_2} E(S_j))$ has no simple summands. Hence M is a semisimple-direct-injective module, and so it is a C3-module. So, $\oplus_{\mathcal{J}_2} E(S_j)$ is an injective module. Thus R is right Noetherian. Now, by the same proof of (1) ⇒ (3) of Theorem 3.4 in [5], one infers that R is an Artinian serial ring with $J(R)^2 = 0$. □

The following observations give some connections between (right Noetherian) right V-rings and semisimple-direct-injective modules.

Theorem 2.11. *The following conditions are equivalent for a ring R:*

(1) *R is a right Noetherian and right V-ring.*
(2) *Every right R-module is semisimple-direct-injective.*
(3) *Direct sum of two semisimple-direct-injective right R-modules is semisimple-direct-injective.*

Proof. Recall that R is a right Noetherian and right V-ring if and only if every semisimple module is injective.

$(1) \Rightarrow (2), (3)$ are obvious.

$(2) \Rightarrow (1)$ If A is a semisimple right R-module, then, by the assumption, $M = A \oplus E(A)$ is a semisimple-direct-injective module. By Proposition 2.1, A is a direct summand of $E(A)$ and hence A is injective. Thus R is a right Noetherian right V-ring.

$(3) \Rightarrow (1)$ is similar to $(2) \Rightarrow (1)$. □

Corollary 2.12. *R is semisimple Artinian if and only if every semisimple-direct-injective right R-module is injective.*

Proof. Assume that every semisimple-direct-injective right R-module is injective. We deduce that every semisimple right R-module is injective. So, R is a right Noetherian right V-ring.

If R is not right semi-Artinian, there exists a non-zero right R-module M with $\mathrm{Soc}(M) = 0$. Clearly, M and its submodules are injective, a contradiction. □

We recall Example 2.3 before the following corollary.

Corollary 2.13. *Let R be a right V-ring. Then R is right Noetherian if and only if every simple-direct-injective right R-module is semisimple-direct-injective.*

In [5, Theorem 4.4], authors give a new answer to Fisher's question [8]: When are regular rings right V-rings?. They proved that a regular ring R is a right V-ring if and only if every cyclic right R-module is simple-direct-injective. Recall that a ring R is called *(von Neumann) regular* if for every $a \in R$, there exists some $b \in R$ such that $a = aba$.

Theorem 2.14. *Let R be a regular ring. The following conditions are equivalent:*

(1) *R is a right V-ring.*
(2) *Every cyclic right R-module is semisimple-direct-injective.*
(3) *Every cyclic right R-module is simple-direct-injective.*

Proof. This follows from [5, Theorem 4.4] and Example 2.3. □

A right R-module M is called *strongly soc-injective* if for any right R-module N and any semisimple submodule K of N, every R-homomorphism $f : K \to M$ extends to N [2]. By [2, Proposition 16], a right R-module M is strongly soc-injective if and only if $M = E \oplus T$, where E is injective and $\mathrm{Soc}(T) = 0$. It is easy to see that every strongly soc-injective module is semisimple-direct-injective.

Proposition 2.15. *The following conditions are equivalent for a ring R:*

(1) *R is a right Noetherian right V-ring.*
(2) *Every semisimple-direct-injective module is strongly soc-injective.*

Proof. $(1) \Rightarrow (2)$. Let M be a semisimple-direct-injective module. Assume that $\mathrm{Soc}(M)$ is non-zero. Hence, M has a decomposition $M = \mathrm{Soc}(M) \oplus T$ such that $\mathrm{Soc}(M)$ is injective and $\mathrm{Soc}(T) = 0$. Thus, M is a strongly soc-injective module.

$(2) \Rightarrow (1)$ Let M be a semisimple module. Then, M is a strongly soc-injective module, write $M = E \oplus T$, where E is injective and $\mathrm{Soc}(T) = 0$. Furthermore, we have $T = \mathrm{Soc}(T)$ and so $M = E$ is injective. □

Recall that a right R-module M is called *mininjective* if, for every simple right ideal K of R, each R-homomorphism $f : K \to M$ extends to $g : R \to M$; that is, $f = m\cdot$ is multiplication by some $m \in M$ ([14]).

Lemma 2.16 ([14, Theorem 2.36])**.** *The following conditions are equivalent for a ring R:*

(1) *Every right R-module is mininjective.*
(2) *Every cyclic right R-module is mininjective.*
(3) *$K^2 \neq 0$ for every simple right ideal K of R.*
(4) $\mathrm{Soc}(R_R) \cap J(R) = 0$.
(5) *R is right mininjective and* $\mathrm{Soc}(R_R)$ *is projective as a right R-module.*

A ring R is called right *universally mininjective* if it satisfies the conditions in Lemma 2.16.

Lemma 2.17. *The following conditions are equivalent for a ring R:*

(1) *R is right universally mininjective.*
(2) *R is right semisimple-direct-injective and every minimal right ideal of R is projective as a right R-module.*

Proof. (1) $\Rightarrow$ (2). Assume that R is right universally mininjective. Then, every minimal right ideal of R is a direct summand of R_R by Lemma 2.16. It follows that R is a right simple-direct-injective ring, and so it is semisimple-direct-injective.

(2) $\Rightarrow$ (1). We show that R is right mininjective. Indeed, let K be a minimal right ideal of R. Then, K is a projective module, and so K is isomorphic to a direct summand of R_R. We have that R is right semisimple-direct-injective and obtain that K is a direct summand of R_R. We deduce that R is right mininjective. Thus, R is right universally mininjective by Lemma 2.16. □

Theorem 2.18. *The following conditions are equivalent for a ring R:*

(1) *R is semisimple Artinian.*
(2) *R satisfies the following conditions:*
 (a) *R is right semisimple-direct-injective with* $\mathrm{Soc}(R_R) \leq_e R_R$ *and projective as a right R-module.*
 (b) *Every ascending chain*

$$r(a_1) \subseteq r(a_2a_1) \subseteq \cdots$$

 terminates for every infinite sequence $a_1, a_2, \ldots$ of elements in R.

Proof. (1) $\Rightarrow$ (2) This is obvious.

(2) $\Rightarrow$ (1) By (2-a), R is a right universally mininjective ring and $\mathrm{Soc}(R_R) \leq \mathrm{Soc}({}_RR)$ by Lemma 2.17. Hence $\mathrm{Soc}({}_RR)$ is essential in R_R. From [15, Theorem 2.2] we infer that R is a right perfect ring. Furthermore, $\mathrm{Soc}(R_R) \cap J(R) = 0$ and $\mathrm{Soc}(R_R) \leq_e R_R$, which implies that $J(R) = 0$. Thus R is a semisimple Artinian ring. □

We denote the nil radical $N(R)$ of R by $N(R) = \sum\{I |\ I$ is nil right ideal of $R\}$.

Corollary 2.19. *If $N(R) = 0$,* $\mathrm{Soc}(R_R) \leq_e R_R$ *and every ascending chain*

$$r(a_1) \subseteq r(a_2a_1) \subseteq \cdots$$

terminates for every infinite sequence $a_1, a_2, \ldots$ of elements in a ring R, then R is a semisimple Artinian ring.

Proof. Let I be an arbitrary minimal right ideal of R. From the hypothesis $N(R) = 0$ it immediately follows that $I^2 \neq 0$. Therefore, I is a direct summand of R_R. It follows that R is right semisimple-direct-injective and every minimal right ideal of R is projective as a right R-module. Thus R is a semisimple Artinian ring. □

Corollary 2.20 ([18, 4.3]). *A right Artinian ring R with $N(R) = 0$ is semisimple Artinian.*

We finish this section with the study of the following question:

" Does there exist a right semisimple-direct-injective ring that is not left semisimple-direct-injective?"

Rings of formal triangular matrices also serve as a source of examples of rings with non-symmetrical properties. Below we give an example of a formal triangular matrices ring that answers positively the previous question.

Given the R-S-bimodule M we denote

$$l(M) = \{r \in R \,|\, rM = 0\},\ r(M) = \{s \in S \,|\, Ms = 0\}$$

Theorem 2.21. *The following conditions are equivalent for a formal triangular matrices ring* $K = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$

(1) *K is a right semisimple-direct-injective ring;*

(2) (a) *For any semisimple submodules A, B of $l(M)$ with $A \cong B \leq_d R_R$, A is a summand of R_R.*

(b) *For any semisimple submodules A, B of S_S with $A \cong B \leq_d S_S$, A is a summand of S_S and $A \leq r(M)$.*

Proof. (1) $\Rightarrow$ (2) (a) Let A be a semisimple submodule of R_R, $A \cong B \leq_d R_R$ and $A, B \leq l(M)$. Then, there exists a submodule B' of R_R such that $R_R = B \oplus B'$. It follows that there is a decomposition $K_K = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} B' & M \\ 0 & S \end{pmatrix}$. We have that an K-isomorphism $\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \cong \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}$ of K-modules and obtain that there exists a submodule L of K_K such that we have a decomposition $K_K = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \oplus L$. Let $A' = \{a \in R \mid \exists m \in M, \exists s \in S : \begin{pmatrix} a & m \\ 0 & s \end{pmatrix} \in L\}$. One can check that $R_R = A \oplus A'$.

(b) Let A be a semisimple submodule of S_S, $A \cong B \leq_d S_S$. Using arguments similar to those in the proof of (a), we can show that $A \leq_d S_S$. Assume that $MA \neq 0$. Then, there exists a simple submodule A_0 of A such that $MA_0 \neq 0$. One can check that there is an isomorphism of K-modules $\begin{pmatrix} 0 & 0 \\ 0 & A_0 \end{pmatrix} \cong \begin{pmatrix} 0 & MA_0 \\ 0 & 0 \end{pmatrix}$. Since $\begin{pmatrix} 0 & 0 \\ 0 & A_0 \end{pmatrix} \leq_d K_K$, then we get a contradiction with the condition of (1). It follows that $MA = 0$ or $A \leq r(M)$.

(2) $\Rightarrow$ (1) Firstly, let A be a simple submodule of K_K, $A \cong A' \leq_d K_K$. It follows, from the condition of (2), that either $A' = \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix} K = \begin{pmatrix} eR & 0 \\ 0 & 0 \end{pmatrix}$, or $A' = \begin{pmatrix} 0 & 0 \\ 0 & e' \end{pmatrix} K = \begin{pmatrix} 0 & 0 \\ 0 & e'S \end{pmatrix}$ for some $e^2 = e \in R$ and $e'^2 = e' \in S$.

Assume that $A' = \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix} K$ and $f : A' \to A$ is an isomorphism of K-modules. Since $A' \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 0$, then $S = \begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix}$, where A_0 is a simple submodule of R_R.

Assume that $A' = \begin{pmatrix} 0 & 0 \\ 0 & e' \end{pmatrix} K$ and $f : A' \to A$ is an isomorphism of K-modules. Since $A' \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 0$ then $f(\begin{pmatrix} 0 & 0 \\ 0 & e' \end{pmatrix}) = \begin{pmatrix} 0 & m \\ 0 & s \end{pmatrix}$ with $m \in M, s \in S$. We have $f(\begin{pmatrix} 0 & 0 \\ 0 & e' \end{pmatrix}) \begin{pmatrix} 0 & 0 \\ 0 & e' \end{pmatrix} = f(\begin{pmatrix} 0 & 0 \\ 0 & e' \end{pmatrix})$ and get $\begin{pmatrix} 0 & m \\ 0 & s \end{pmatrix} = \begin{pmatrix} 0 & me' \\ 0 & se' \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & se' \end{pmatrix}$. Thus $A = \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}$, where B is a simple submodule of S_S.

Now, we assume that A is a semisimple submodule of K_K and $A \cong B \leq_d K_K$. It follows, from the above reasoning, that there are submodules C, C' of R_R and D, D' of S_S such that $A = \begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix}$ and $B = \begin{pmatrix} C' & 0 \\ 0 & D' \end{pmatrix}$. Since $A \cong B$, it is easy to verify that $C_R \cong C'_R$ and $D_R \cong D'_R$. We have that $B \leq_d K_K$ and obtain that $C' \leq_d R_R$, and $D' \leq_d S_S$. Then, it follows, from the conditions of (2), that there are submodules $E \leq R_R, F \leq S_S$ such that we have a decomposition $C \oplus E = R_R, D \oplus F = S_S$. Thus, we have a decomposition $K_K = \begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix} \oplus \begin{pmatrix} E & M \\ 0 & F \end{pmatrix} = A \oplus \begin{pmatrix} E & M \\ 0 & F \end{pmatrix}$. □

Example 2.22. Let $Q := \prod_{i=1}^{\infty} F_i$ with $F_i := \mathbb{Z}_2$ and R be the subring of Q generated by $\bigoplus_{i=1}^{\infty} F_i$ and 1_Q. Consider the right action R on $T_2(\mathbb{Z}_2) = \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{pmatrix}$ which are defined by the relations

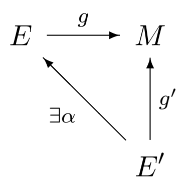
$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} (\alpha 1_Q + \beta) = \begin{pmatrix} a\alpha & b \\ 0 & c\alpha \end{pmatrix},$$

where $\alpha \in \mathbb{Z}_2, \beta \in \bigoplus_{i=1}^{\infty} F_i$. Then $T_2(\mathbb{Z}_2)$ is $T_2(\mathbb{Z}_2)$-R-bimodule. Consider the formal triangular matrices ring $K = \begin{pmatrix} T_2(\mathbb{Z}_2) & {}_{T_2(\mathbb{Z}_2)}T_2(\mathbb{Z}_2)_R \\ 0 & R \end{pmatrix}$. Since the ring $T_2(\mathbb{Z}_2)$ is not left (and right) semisimple-direct-injective, it follows, from the left-sided analogue of Theorem 2.21, that the ring K is not left semisimple-direct-injective. Since $l({}_{T_2(\mathbb{Z}_2)}T_2(\mathbb{Z}_2)_R) = 0$ and $r({}_{T_2(\mathbb{Z}_2)}T_2(\mathbb{Z}_2)_R) = \mathrm{Soc}(R)$, then conditions (2)(a) and (2)(b) of Theorem 2.21 hold. Thus, the ring K is right semisimple-direct-injective.

3. Semisimple-direct-injective envelopes and covers

An R-homomorphism $g : E \to M$ is called a *semisimple-direct-injective cover* (a $C3$-cover [1], respectively) for short an sdi-cover, of a right R-module M if E is a semisimple-direct-injective module (a $C3$ module, respectively) such that:

(i) Any diagram



with E a semisimple-direct-injective module (a $C3$ module, respectively), can be commutatively completed.

(ii) If any endomorphism $\alpha : E \to E$ satisfies $g\alpha = g$, then α must be an automorphism of E.

Dually, the notion of the semisimple-direct-injective envelope can be defined.

Lemma 3.1. *Assume that N is a non-injective semisimple module. Then the module $M = N \oplus E(N)$ does not have an sdi-envelope and an sdi-cover.*

Proof. Consider the inclusion map (note that, it is the semisimple-direct-injective envelope monomorphism)

$$\iota : N \oplus E(N) \to E,$$

where E is a semisimple-direct-injective module. Since the modules N and $E(N)$ are semisimple-direct-injective, there exist $f_1 : E \to N$ and $f_2 : E \to E(N)$ such that $f_i\iota = \pi_i$, where $\pi_1 : M \to N$ and $\pi_2 : M \to E(N)$ are the projections. Now there exists $f : E \to$

$N \oplus E(N)$ such that $\pi_i f = f_i$, which implies that $(\iota f)\iota = \iota$. Since E is semisimple-direct-injective envelope of M, we have ιf is an isomorphism. It follows that $E \cong N \oplus E(N)$ is a semisimple-direct-injective module. Thus $N = E(N)$ is injective, a contradiction.

The rest is similar. □

Lemma 3.2. *If A is a $C3$-module and $A \oplus E(A)$ has a $C3$-cover, then A is injective.*

Proof. This similar to Lemma 3.1. □

Theorem 3.3. *The followings are equivalent for a ring R:*

(1) *R is an Artinian serial ring with $J(R)^2 = 0$.*
(2) *Every simple-direct-injective right R-module has a $C3$-cover.*
(3) (a) *Every semisimple-direct-injective right R-module has a C3-cover.*
   (b) *The module $\oplus_{\mathfrak{J}} S_i$ is injective for any family of simple injective modules $\{S_i\}_{\mathfrak{J}}$.*

Proof. (1) ⇒ (2) This is clear.

(2) ⇒ (1) Consider the family $\{E_i\}_{i\in I}$ of injective right R-modules E_i, $i \in I$. By the assumption, $M = E \oplus (\oplus_{i\in I} E_i)$ with $E = E(\oplus_{i\in I} E_i)$ has a $C3$-cover, say $\alpha : C \to M$. Let $E_{i_0} := E$ and $\iota_i : E_i \to M$ be the inclusion maps for all $i \in I \cup \{i_0\}$. Since E_i is injective (hence simple-direct-injective), there exists a linear map $\beta_i : E_i \to C$ such that $\alpha\beta_i = \iota_i$. Hence $id = \oplus\iota_i = \alpha(\oplus\beta_i)$ which implies that M is a direct summand of C. So M is a $C3$-module. By [5, Lemma 3.2], $\oplus_{i\in I} E_i$ is injective. Thus R is right Noetherian.

We next prove that R is right semi-Artinian. Without loss of generality, we can assume that M is a non-zero indecomposable right R-module with $\mathrm{Soc}(M) = 0$ (since R is right Noetherian). Then M is a $C3$-module. Since $\mathrm{Soc}(M \oplus E(M)) = 0$, we get $M \oplus E(M)$ is a simple-direct-injective module. By the assumption, $M \oplus E(M)$ has a $C3$-cover. By Lemma 3.2, M is injective. Hence M is uniform and every submodule of M is $C3$. Let N be a non-zero arbitrary submodule of M. By the same argument, we have N is injective. So, N is a direct summand of M. This shows that M is a semisimple module, a contradiction. Thus, every non-zero indecomposable right R-module has non-zero socle. It follows that R is right semi-Artinian and hence R is right Artinian.

By the same technique of [5, Theorem 3.4 (1) ⇒ (3)], we can obtain that every right R-module is a direct sum of a semisimple module and a family of injective uniserial modules of length 2. Thus R is an Artinian serial ring with $J(R)^2 = 0$.

(1) ⇔ (3) This is similar to (1) ⇔ (2). □

Now, we can prove that the classes of semisimple-direct-injective modules over a ring R provide for sdi-envelopes and sdi-covers only if R is a right Noetherian right V-ring:

Theorem 3.4. *The following conditions are equivalent:*

(1) *R is a right Noetherian right V-ring.*
(2) *Every right R-module has an sdi-cover.*
(3) *Direct sums of semisimple-direct-injective modules have sdi-covers.*
(4) *Every right R-module has an sdi-envelope.*
(5) *Direct sums of semisimple-direct-injective modules has an sdi-envelope.*

Proof. (1) ⇒ (2), (3) are obvious.

(2) ⇒ (1) For any semisimple right R-module S, then by the assumption, $M = S \oplus E(S)$ has an sdi-cover, say $\alpha : C \to M$. Let $\iota_1 : S \to M$ and $\iota_2 : E(S) \to M$ be the inclusion maps for all $i = 1, 2$. Note that S and $E(S)$ are semisimple-direct-injective modules, and there are homomorphisms $\beta_1 : S \to C, \beta_2 : E(S) \to C$ such that $\alpha\beta_i = \iota_i$. Clearly, $id_M = \iota_1 \oplus \iota_2 = \alpha(\iota_1 \oplus \iota_2)$. This shows that M is isomorphic to a direct summand of C, which implies that M is a semisimple-direct-injective module. Hence S is injective.

(3) ⇒ (1) is similar to (2) ⇒ (1).

(4) ⇒ (1) Let N be an arbitrary semisimple module. Assume that $\iota : M = N \oplus E(N) \to E$ is the sdi-envelope, where E is a simple-direct-injective module. Since N and $E(N)$

are semisimple-direct-injective modules, there exist $f_1 : E \to N$, $f_2 : E \to E(N)$ such that $f_i\iota = \pi_i$, where $\pi_1 : M \to N_i$ and $\pi_2 : M \to E(N)$ are the projections. There exists $\phi : E \to M$ such that $\pi_i\phi = f_i$ for all $i = 1, 2$. It follows that $\phi\iota = id_M$, and so the monomorphism ι splits. Thus $N \oplus E(N)$ is isomorphic to a direct summand of E. It follows that $N \oplus E(N)$ is also a semisimple-direct-injective module. Hence N is injective.

(5) $\Rightarrow$ (1) is similar to (4) $\Rightarrow$ (1). □

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