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Research Article

Higher dimensional algebras as ideal maps

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Abstract

In this work, we explain the close relationship between an ideal map structure $S \to End_R(R)$ on a homomorphism of commutative k-algebras $R \to S$ and an ideal simplicial algebra structure on the associated bar construction $Bar(S,R)$. We also explain this structure for crossed squares of algebras.

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1. Introduction

Crossed modules introduced by Whitehead, [16], are algebraic models of connected (weak homotopy) 2-types. The commutative algebra version of crossed modules has been introduced by Porter in [13]. Crossed squares defined by Loday and Guin-Walery,[11], can be regarded as 2-dimensional version of crossed modules as models for connected 3-types. Ellis, [7], gave these structures for Lie algebras and commutative algebras. These algebraic models are called "combinatorial algebra theory" and contain potentially important new ideas (see [4, 5]).

We consider the equivalence between the category of crossed modules of algebras (cf. [13]) and the category of simplicial commutative algebras with Moore complex of length 1 given in [3]. The main aim of this note is to associate an explicit ideal simplicial algebra structure on the bar construction given a crossed module of algebras and to give the same idea for crossed squares of algebras and bisimplicial algebras. We observed that a crossed module structure $(S \to End_R(R))$ or an ideal map structure on a homomorphism of algebras $\eta : R \to S$ directly yields a simplicial algebra structure on the usual bar construction namely on the simplicial k-module $Bar(S,R) = (S \times R^k)_{k \geqslant 0}$, where k is a commutative ring with 1. Thus, $Bar(S,R)$ is isomorphic, as a simplicial k-module, to a simplicial algebra, which is compatible with the action of R on the bar construction. Moreover, this process is reversible. Therefore, we can summarize the result as follows: Given an algebra homomorphism $\eta : R \to S$, a crossed module structure or an ideal map structure on the homomorphism η gives an ideal simplicial algebra structure on the simplicial k-module $Bar(S,R)$, and conversely, any ideal simplicial algebra structure on the simplicial k-module $Bar(S,R)$ determines a crossed module structure on the homomorphism η. These two explicit associations are mutual inverses. In the last section, we

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explain how to give an extension of this result to Ellis's (crossed) squares of k-algebras (cf. [6]). In section 5, considering a *crossed ideal structure* over the map $\alpha : \eta_1 \to \eta_2$ between crossed modules η_1 and η_2, we proved that a *crossed ideal map* preserves the *crossed ideals* in the category of crossed modules of commutative k-algebras.

These constructions in the category of groups can be found in [9]. In fact, the results and general methods given in this work are inspired by those proved for the corresponding case of groups using homotopy normal maps in [9]. For further work about homotopy normal maps, see [8] and [14] and for the free normal closure of a homotopy normal map, see [10].

2. Simplicial sets and simplicial algebras

Let k be a fixed commutative ring with identity. By a k-algebra, we mean a unital k-module C endowed with a k-bilinear associative multiplication $C \times C \to C$, $(c, c') \mapsto cc'$. The algebra C will as usual be called commutative if $cc' = c'c$ for all $c, c' \in C$. In this work, all algebras will be commutative and will be over the same fixed commutative ring k. We will denote the category of all algebras over the commutative ring k by Alg.

A simplicial set E consists of a family of sets E_n, for $n \geqslant 0$, together with face and degeneracy maps $d_i = d_i^n : E_n \to E_{n-1}$, $0 \leqslant i \leqslant n$, $(n \neq 0)$ and $s_i = s_i^n : E_n \to E_{n+1}$, $0 \leqslant i \leqslant n$. These maps are required to satisfy the following *simplicial identities*:

(i) $d_i d_j = d_{j-1} d_i$ for $0 \leqslant i < j \leqslant n$,

(ii) $s_i s_j = s_{j+1} s_i$ for $0 \leqslant i \leqslant j \leqslant n$,

(iii) $d_i s_j = \begin{cases} s_{j-1} d_i & (\text{if } 0 \leqslant i < j \leqslant n), \\ Id & (\text{if } i = j \text{ or } i = j+1), \\ s_j d_{i-1} & (\text{if } 0 \leqslant j < i-1 \leqslant n). \end{cases}$

For more details regarding this, see [1, 2] or [12]. In fact, a simplicial set E can be completely described as a functor $\mathsf{E} : \Delta^{op} \to \mathsf{Sets}$ where Δ is the category of finite ordinals $[n] = \{0 < 1 < \cdots < n\}$ and non-decreasing maps.

We say that the simplicial set E is a simplicial k-module (or k-algebra) if E_k is a k-module (or a k- algebra) for all k and the face and degeneracy maps are homomorphisms of k-modules (or k-algebras). Thus, a simplicial algebra can be defined as a functor from the opposite category Δ^{op} to Alg.

2.1. The simplicial k-module $Bar(X, R)$

In this section we give the usual bar construction of a simplicial k-module by using the action of a k-algebra on a k-module. First we define this action.

Let R be a k-algebra and X be a k-module. The action of R on X is defined by the function $X \times R \to X$, $r : x \mapsto x^r$ (where $r \in R, x \in X$) satisfying the following conditions:

(1) $(x)^{(r_1+r_2)} = (x^{r_1})^{r_2}$

(2) $x^{0_R} = x$

(3) $(x_1 + x_2)^{r_1+r_2} = (x_1)^{r_1} + (x_2)^{r_2}$

(4) $k(x)^r = (kx)^{kr}$

for all $r, r_1, r_2 \in R, x, x_1, x_2 \in X, k \in \mathsf{k}$.

Example 2.1. Let R be a subalgebra of a k-algebra X. Then, the function $X \times R \to X$, $r : x \mapsto x^r = x + r \in X$ (where $r \in R, x \in X$) defines the action of the algebra R on underlying k-module X of the k-algebra X.

Example 2.2. Suppose that $\eta : R \to S$ is a k-algebra homomorphism. Then, the k-algebra R acts on the underlying k-module S of the k-algebra S via η i.e. the action is

$$r : s \to s^r = s + \eta(r)$$

for all $r \in R$ and $s \in S$. Indeed, we obtain

(1) $s^{(r_1+r_2)} = s + \eta(r_1 + r_2) = (s + \eta(r_1)) + \eta(r_2) = (s^{r_1})^{r_2}$,
(2) $s^{0_R} = s + \eta(0_R) = s + 0_S = s$,
(3) $(s_1 + s_2)^{(r_1+r_2)} = (s_1 + s_2) + \eta(r_1 + r_2) = s_1 + \eta(r_1) + s_2 + \eta(r_2) = (s_1)^{r_1} + (s_2)^{r_2}$,
(4) $k(s)^r = k(s + \eta(r)) = ks + \eta(kr) = (ks)^{kr}$

for all $s, s_1, s_2 \in S$ and $r, r_1, r_2 \in R$ and $k \in \mathsf{k}$.

Let R be a k-algebra acting on the k-module X as defined above. The bar construction

$$B := Bar(X, R)$$

is the simplicial k-module consisting of the following data.

(1) for each integer $n \geqslant 0$, a k-module B_n defined by $B_0 = X$ for $n = 0$, and $B_n = X \times R^n$, for $n \geqslant 1$, where the operations in B_n are (for $x, x' \in X$ and $r_i, r'_i \in R$ and $k \in \mathsf{k}$)

$$(x, r_1, r_2, \ldots, r_n) \oplus (x', r'_1, r'_2, \ldots, r'_n) = (x + x', r_1 + r'_1, \ldots, r_n + r'_n)$$

and

$$k(x, r_1, r_2, \ldots, r_n) = (kx, kr_1, kr_2, \ldots, kr_n),$$

(2) the face k-module homomorphisms $d_i^n : d_i : B_n \to B_{n-1}$ for all $n \geqslant 1$ and $0 \leqslant i \leqslant n$ defined by:
  (i) $d_0 : (x, r_1, r_2, \ldots, r_n) \mapsto (x^{r_1}, r_2, \ldots, r_n)$
  (ii) $d_i : (x, r_1, r_2, \ldots, r_i, r_{i+1}, \ldots, r_n) \mapsto (x, r_1, r_2, \ldots, r_i + r_{i+1}, \ldots, r_n)$ for $1 \leqslant i < n$,
  (iii) $d_n : (x, r_1, r_2, \ldots, r_n) \mapsto (x, r_1, r_2, \ldots, r_{n-1})$,

(3) and together with degeneracy k-module homomorphisms; $s_j : B_n \to B_{n+1}$ defined by

$$s_j : (x, r_1, r_2, \ldots, r_n) \mapsto (x, r_1, r_2, \ldots, r_j, 0, r_{j+1}, \ldots, r_n)$$

for all $n \geqslant 0$ and $0 \leqslant j \leqslant n$.

2.2. An ideal simplicial algebra structure on $Bar(S, R)$

Lemma 2.3. *Assume that $\eta : R \to S$ is a k-algebra homomorphism and the k-algebra R acts on the underlying k-module S via η as given in Example 2.2. Then, the bar construction $Bar(S, R)$ is a simplicial k-module with the following properties:*

(i) *$B_0 = S$ and for each integer $n \geqslant 1$, $B_n = S \times R^n$ is the k-module with the operations:*

$$(s, r_1, r_2, \ldots, r_n) \oplus (s', r'_1, r'_2, \ldots, r'_n) = (s + s', r_1 + r'_1, \ldots, r_n + r'_n)$$

for $s, s' \in S$ and $r_i, r'_i \in R$ and

$$k(s, r_1, r_2, \ldots, r_n) = (ks, kr_1, kr_2, \ldots, kr_n)$$

for $k \in \mathsf{k}$.

(ii) *the face k-module homomorphisms $d_i^n : d_i : B_n \to B_{n-1}$ for all $n \geqslant 1$ and $0 \leqslant i \leqslant n$ are defined by:*

$$d_0(s, r_1, r_2, \ldots, r_n) = (s + \eta(r_1), r_2, \ldots, r_n),$$

$$d_i(s, r_1, r_2, \ldots, r_i, r_{i+1}, \ldots, r_n) = (s, r_1, r_2, \ldots, r_i + r_{i+1}, \ldots, r_n) \text{ for } 1 \leqslant i < n,$$

and

$$d_n(s, r_1, r_2, \ldots, r_n) = (s, r_1, r_2, \ldots, r_{n-1}),$$

(iii) *the degeneracy k-module homomorphisms; $s_j : B_n \to B_{n+1}$ are defined by*

$$s_j(s, r_1, r_2, \ldots, r_n) = (s, r_1, r_2, \ldots, r_j, 0, r_{j+1}, \ldots, r_n)$$

for all $n \geqslant 0$ and $0 \leqslant j \leqslant n$.

Proof. (i) $B_0 = S$ is the underlying k-module of the k-algebra S and by definition of direct product of k-modules, we obtain that $B_n = S \times R^n$ is a k-module for each integer $n \geqslant 1$.

(ii) We show that the face maps d_i for $0 \leqslant i \leqslant n$ are k-module homomorphisms from B_n to B_{n-1}. For $u = (s, r_1, r_2, \dots, r_n)$,$v = (s', r'_1, r'_2, \dots, r'_n) \in B_n$ and $k \in \mathsf{k}$, we obtain

$$\begin{aligned} d_0(u \oplus v) &= d_0(s + s', r_1 + r'_1, \dots, r_n + r'_n) \\ &= ((s + s') + \eta(r_1) + \eta(r'_1), r_2 + r'_2, \dots, r_n + r'_n) \\ &= (s + \eta(r_1) + s' + \eta(r'_1), r_2 + r'_2, \dots, r_n + r'_n) \\ &= (s + \eta(r_1), r_2, \dots, r_n) \oplus (s' + \eta(r'_1), r'_2, \dots, r'_n) \\ &= d_0(u) \oplus d_0(v), \end{aligned}$$

and

$$\begin{aligned} d_0(ku) &= d_0(ks, kr_1, \dots, kr_n) \\ &= ((ks) + \eta(kr_1), kr_2, \dots, kr_n) \\ &= (k(s + \eta(r_1)), kr_2, \dots, kr_n) \\ &= k(s + \eta(r_1), r_2, \dots, r_n) \\ &= kd_0(u). \end{aligned}$$

Similarly we have

$$\begin{aligned} d_i(u \oplus v) &= d_i(s + s', r_1 + r'_1, \dots, r_n + r'_n) \\ &= (s + s', r_1 + r'_1, r_2 + r'_2, \dots, r_i + r'_i + r_{i+1} + r'_{i+1}, \dots, r_n + r'_n) \\ &= (s + s', r_1 + r'_1, r_2 + r'_2, \dots, r_i + r_{i+1} + r'_i + r'_{i+1}, \dots, r_n + r'_n) \\ &= (s, r_1, r_2, \dots, r_i + r_{i+1}, \dots, r_n) \oplus (s', r'_1, r'_2, \dots, r'_i + r'_{i+1}, \dots, r'_n) \\ &= d_i(u) \oplus d_i(v) \end{aligned}$$

and

$$\begin{aligned} d_i(ku) &= d_i(ks, kr_1, \dots, kr_n) \\ &= (ks, kr_1, kr_2, \dots, kr_i + kr_{i+1}, \dots, kr_n) \\ &= k(s, r_1, r_2, \dots, r_i + r_{i+1}, \dots, r_n) \\ &= kd_i(u). \end{aligned}$$

We also obtain

$$\begin{aligned} d_n(u \oplus v) &= d_n(s + s', r_1 + r'_1, \dots, r_n + r'_n) \\ &= (s + s', r_1 + r'_1, \dots, r_{n-1} + r'_{n-1}) \\ &= (s, r_1, \dots, r_{n-1}) \oplus (s', r'_1, \dots, r'_{n-1}) \\ &= d_n(u) \oplus d_n(v) \end{aligned} \qquad \begin{aligned} d_n(ku) &= d_n(ks, kr_1, \dots, kr_n) \\ &= (ks, kr_1, \dots, kr_{n-1}) \\ &= k(s, r_1, \dots, r_{n-1}) \\ &= kd_n(u). \end{aligned}$$

It is easy to see s_j is a k-module homomorphism. □

Definition 2.4. Let $B := Bar(S, R)$. By an *ideal simplicial algebra structure* on B, we mean the following

(i) $B_0 = S$ is the k-algebra S,

(ii) $B_k := S \times R^k$ is endowed with a k-algebra structure for all $k \geqslant 1$ and we denote the multiplication by

$$(s, r_1, \dots, r_k) * (s', r'_1, \dots, r'_k).$$

(iii) the face map d^k_i and the degeneracy map s^k_j are k-algebra homomorphisms.

(iv) for all $s, s' \in S$

$$(s, 0, \dots, 0) * (s', 0, \dots, 0) = (ss', 0, \dots, 0),$$

where the operations take place in B_k.

Remark 2.5. By the natural action of S on $Bar(S,R)$, we mean

$$s:(s',r_1,\ldots,r_k)\mapsto s\cdot(s',r_1,\ldots,r_k)=(ss',r_1,\ldots,r_k),$$

for all $k\geqslant 0$, $(s',r_1,\ldots,r_k)\in B_k$ and $s\in S$. When we say that the multiplication in $Bar(S,R)$ is compatible with the natural action of S, we mean that condition (iv) of Definition 2.4 holds.

Notation 2.6. Let $k\geqslant 1$. We denote

(1) $S_k:=\{(s,0_R,0_R,\ldots,0_R):s\in S\}$ is a subalgebra of B_k.

(2) $R_k:=\{(0_S,r_1,r_2,\ldots,r_k):r_i\in R\}$ is an algebra ideal of B_k.

Lemma 2.7. *Suppose that $Bar(S,R)$ is endowed with an ideal simplicial algebra structure. Let $k\geqslant 1$. Then S_k is an ideal of B_k which is isomorphic to S, R_k is an ideal of B_k, $B_k=S_k+R_k$ and $S_k\cap R_k=\{0\}$.*

Proof. S_k is the image of S_{k-1} under s_{k-1}, so by induction it is a subalgebra of B_k and since s_{k-1} is injective, it is isomorphic to S. Since R_k is the kernel of $d_k\circ d_{k-1}\circ\cdots\circ d_1$, it is an ideal of B_k. Also by Definition 2.4 (iv), S_k is an ideal of B_k. Clearly $S_k\cap R_k=\{0\}$ and $B_k=S_k+R_k$. □

2.3. Crossed modules, ideal maps and ideal structures

Crossed modules of groups were initially defined by Whitehead in [16]. The algebra analogue has been studied by Porter in [13].

A crossed module of algebras consists of an algebra homomorphism $\eta:R\to S$ which here we call an ideal map (see Remark 2.8) together with a homomorphism $l:S\to End_R(R)$ which here we call an ideal structure (or a crossed module structure) on η. We denote by $s\cdot r$ the image of $r\in R$ under l_s for $s\in S$. Explicitly, the following hold (for all $k\in\mathsf{k}$, $r,r'\in R$ and $s,s'\in S$):

(1) $k(s\cdot r)=(ks)\cdot r=s\cdot(kr)$

(2) $s\cdot(r+r')=s\cdot r+s\cdot r'$

(3) $(s+s')\cdot r=s\cdot r+s'\cdot r$

(4) $s\cdot(rr')=(s\cdot r)r'=r(s\cdot r')$, $0_S\cdot r=0_R$, $s\cdot 0_R=0_R$,

(5) $(ss')\cdot r=s\cdot(s'\cdot r)$.

The maps η and l are required to satisfy the following:

$(CM1)$ $\eta(s\cdot r)=s\eta(r)$, for all $s\in S$ and $r\in R$.

$(CM2)$ $\eta(r)\cdot r'=rr'$, for all $r,r'\in R$.

Remark 2.8. Let S and R be algebras and let $\eta:R\to S$ be an algebra homomorphism. If $l:S\to End_R(R)$ is a crossed module structure on the homomorphism $\eta:R\to S$, then $\mathrm{im}\,(\eta)$ is an ideal of S. Indeed, for all $s\in S$ and $s'\in\mathrm{im}\,(\eta)$ with $s'=\eta(r);r\in R$, we obtain from $(CM1)$,

$$ss'=s\eta(r)=\eta(s\cdot r)\in\mathrm{im}\,(\eta).$$

Thus, $\mathrm{im}\,(\eta)$ is an ideal of S. Conversely, if I is an ideal of the algebra S, then the inclusion map $I\to S$ is a crossed module with the natural action of S on I. Further, $\ker\eta$ is an ideal in R and a module over S. The ideal $\mathrm{im}\,(\eta)$ of S acts trivially on $\ker\eta$, hence $\ker\eta$ inherits an action of $S/\mathrm{im}\,(\eta)$ to become an $S/\mathrm{im}\,(\eta)$-module.

Now let S be an algebra and R be subalgebra of S. Let $\eta:R\to S$ be the inclusion map and let S/R be the set of cosets of R in S. Then, there is a natural action of S on the set S/R via left multiplication and it is easy to verify that the following statements are equivalent.

(i) R is an ideal of S.
(ii) There exists a crossed module structure on the inclusion map $\eta : R \to S$.
(iii) There exists an algebra structure on S/R with the action of S on S/R given by

$$s \cdot (s' + R) = ss' + R$$

for all $s, s' \in S$.

3. From an ideal simplicial algebra structure on $Bar(S, R)$ to an ideal structure on the map $\eta : R \to S$

In this section we assume that R and S are k-algebras and $\eta : R \to S$ is a k-algebra homomorphism together with a homomorphism $l : S \to End_R(R)$ satisfying the conditions $(1)-(5)$ given above. The purpose of this section is to prove that we can recover the crossed module structure (or an ideal structure) on the homomorphism $\eta : R \to S$ from an ideal simplicial algebra structure on the associated bar construction $Bar(S, R)$. Thus, we will show that the homomorphisms η and l satisfy the conditions $(CM1)$ and $(CM2)$.

Proposition 3.1. *Suppose that $Bar(S, R)$ is endowed with an ideal simplicial algebra structure. Then*

(1) $(0, r) \oplus (0, r') = (0, r + r')$ *and* $(0, r) * (0, r') = (0, rr')$ *for all* $r, r' \in R$ *where the operations take place in* R_1, *(see Notation 2.6).*
(2) *The map* $l : S \to End_R(R)$ *defined by*

$$l_s : r \mapsto s \cdot r$$

gives an ideal structure (or a crossed module structure) on η, *where*

$$(0, s \cdot r) = (s, 0) * (0, r).$$

We will give the proof of this proposition using the following Lemmas. Note that Proposition 3.1 together with Lemma 2.7 imply that the map l above is a well defined action of S on R. To prove this proposition, we assume that $Bar(S, R)$ is endowed with an ideal simplicial algebra structure as defined in subsection 2.2.

Lemma 3.2. *Let* $k \geqslant 0$ *and* $r, r' \in R$. *Then*

(i) *The zero element of* B_k *is* $(0_S, 0_R, \dots, 0_R)$,
(ii) $(0, -r, r) \oplus (0, 0, r') = (0, -r, r + r')$,
(iii) $(-\eta(r), r) \oplus (0, r') = (-\eta(r), r + r')$,
(iv) $(0, r) * (0, r') = (0, rr')$.

Proof. (i) By definition, the zero element of $B_0 = S$ is the zero element 0_S of S. Then by induction since $s_0 : B_k \to B_{k+1}$ is an algebra homomorphism, for all $k \geqslant 0$, part (i) follows.

(ii) Applying d_2^2 and using (i), we find that

$$(0_S, -r, r) \oplus (0_S, 0_R, r') = (0_S, -r, x).$$

Applying d_1^2 and using (i) again, we find that

$$(0_S, r') = (0_S, -r + x)$$

so; $x = r + r'$ and (ii) holds.

(iii) This part follows from (ii) by applying d_0^2.

(iv) Since $Bar(S, R)$ is endowed with an ideal simplicial algebra structure, by definition of B_1, we have $d_0(s, r) = s + \eta(r)$, $d_1(s, r) = s$ and $s_0(s) = (s, 0)$ for all $s \in S, r \in R$. If $(s, r) \in \ker d_1$, we get $d_1(s, r) = s = 0_S$ and then $(0, r) \in \ker d_1$. Therefore, we have

$$\ker d_1 = \{(0, r) : r \in R\} = \{0\} \times R$$

and the restriction of d_0 to $\ker d_1$ is given by $d_0(0, r) = \eta(r)$ for $r \in R$. Since d_1 is a homomorphism of algebras from B_1 to B_0, we have $\ker d_1 = \{0\} \times R$ is an ideal of B_1 with respect to the operations $\oplus$ and $*$. We can also say that there is always a *natural injection* isomorphism $\theta : R \to \{0\} \times R$ defined by $\theta(r) = (0, r)$. This satisfies $\theta(rr') = \theta(r) * \theta(r') = (0, r) * (0, r')$, for $r, r' \in R$. Thus, for all $r, r' \in R$, we get

$$(0, rr') = \theta(rr') = (0, r) * (0, r').$$

Therefore, we get the equality $(0, r) * (0, r') = (0, rr')$. □

Lemma 3.3. *Assume that $\eta : R \to S$ is a k-algebra homomorphism together with the homomorphism $l : S \to End_R(R)$ satisfying the conditions* (1)-(5)*. Then,*

$$B_1 = S \ltimes R = \{(s, r) : s \in S, r \in R\}$$

is the semi-direct product algebra of R by S with the following operations:

$$(s, r) \oplus (s', r') = (s + s', r + r'), \quad k(s, r) = (ks, kr)$$

and

$$(s, r) * (s', r') = (ss', s \cdot r' + s' \cdot r + rr')$$

for all $s, s' \in S$ and $r, r' \in R$, $k \in \mathsf{k}$ where $s \cdot r' = l_s(r')$ and $s' \cdot r = l_{s'}(r)$.

Proof. It is clear that the set $S \ltimes R$ is a k-module with the operations

$$(s, r) \oplus (s', r') = (s + s', r + r')$$

and

$$k(s, r) = (ks, kr)$$

for all $k \in \mathsf{k}, (s, r), (s', r') \in S \ltimes R$. On the other hand, we obtain

$$\begin{aligned}
(s_1, r_1) * ((s, r) \oplus (s', r')) &= (s_1, r_1) * (s + s', r + r') \\
&= (s_1(s + s'), s_1 \cdot (r + r') + (s + s') \cdot r_1 + r_1(r + r')) \\
&= (s_1 s + s_1 s', s_1 \cdot r + s_1 \cdot r' + s \cdot r_1 + s' \cdot r_1 + r_1 r + r_1 r') \\
&= (s_1 s, s_1 \cdot r + s \cdot r_1 + r_1 r) \oplus (s_1 s', s_1 \cdot r' + s' \cdot r_1 + r_1 r') \\
&= ((s_1, r_1) * (s, r)) \oplus ((s_1, r_1) * (s', r'))
\end{aligned}$$

and

$$\begin{aligned}
(s_1, r_1) * ((s, r) * (s', r')) &= (s_1, r_1) * (ss', s \cdot r' + s' \cdot r + rr') \\
&= (s_1(ss'), s_1 \cdot (s \cdot r' + s' \cdot r + rr') + (ss') \cdot r_1 \\
&\quad + r_1(s \cdot r' + s' \cdot r + rr')) \\
&= ((s_1 s)s', (s_1 s) \cdot r' + (s_1 s') \cdot r' + s_1 \cdot (rr') \\
&\quad + (ss') \cdot r_1 + s \cdot (r_1 r') + s' \cdot (r_1 r) + (r_1 r)r') \\
&= ((s_1 s)s', (s_1 s) \cdot r' + s' \cdot (s_1 \cdot r + s \cdot r_1 + r_1 r) \\
&\quad + r'(s_1 \cdot r + s \cdot r_1 + r_1 r)) \\
&= (s_1 s, s_1 \cdot r + s \cdot r_1 + r_1 r) * (s', r') \\
&= ((s_1, r_1) * (s, r)) * (s', r')
\end{aligned}$$

and

$$\begin{aligned} k\left((s,r)*(s',r')\right) &= k(ss', s\cdot r' + s'\cdot r + rr') \\ &= (k(ss'), k(s\cdot r') + k(s'\cdot r) + k(rr')) \\ &= ((ks)s', (ks)\cdot r' + s'\cdot (kr) + (kr)r') \\ &= (ks, kr)*(s',r') \\ &= (k(s,r))*(s',r') \\ &= (s,r)*(k(s',r')) \end{aligned}$$

for $k \in \mathsf{k}$, $(s_1, r_1), (s,r), (s',r') \in S \ltimes R$. Since S and R are commutative k-algebras, we get

$$\begin{aligned} (s,r)*(s',r') &= (ss', s\cdot r' + s'\cdot r + rr') \\ &= (s's, s'\cdot r + s\cdot r' + r'r) \\ &= (s',r')*(s,r). \end{aligned}$$

Therefore, $S \ltimes R$ is a commutative k-algebra. □

Remark 3.4. We assume that $\eta : R \to S$ is a k-algebra homomorphism together with the homomorphism $l : S \to End_R(R)$. Using the semi-direct product algebra of R by S, we get

$$(0_S, r)*(0_S, r') = (0_S 0_S, 0_S\cdot r' + 0_S\cdot r + rr') = (0_S, rr')$$

where the action of S on R is given by l. For zero element of S, we get zero homomorphism $l_{0_S} : R \to R$ in $End_R(R)$ which is defined by $l_{0_S} : r \mapsto 0_R$.

Lemma 3.5. *The map $\Phi : S \ltimes R \to S$ defined by $\Phi(s,r) = s + \eta(r)$ is a homomorphism of algebras if and only if η satisfies $(CM1)$ above.*

Proof. First we suppose that η satisfies condition $(CM1)$. Then, we obtain for all $(s,r), (s',r') \in S \ltimes R$,

$$\begin{aligned} \Phi((s,r)\oplus(s',r')) =& \Phi((s+s', r+r')) \\ =& s + s' + \eta(r+r') \\ =& s + \eta(r) + s' + \eta(r') \\ =& \Phi(s,r) + \Phi(s',r') \end{aligned}$$

and

$$\begin{aligned} \Phi((s,r)*(s',r')) =& \Phi(ss', s\cdot r' + s'\cdot r + rr') \\ =& ss' + \eta(s\cdot r' + s'\cdot r + rr') \\ =& ss' + s\eta(r') + s'\eta(r) + \eta(r)\eta(r') \quad \text{since } (CM1) \\ =& s(s' + \eta(r')) + \eta(r)(s' + \eta(r')) \\ =& (s + \eta(r))(s' + \eta(r')) \\ =& \Phi((s,r))\Phi((s',r')). \end{aligned}$$

Conversely, we suppose now that Φ is a homomorphism of algebras. We get $\Phi((s,0)*(0,r)) = \Phi(0, s\cdot r) = \eta(s\cdot r)$. On the other hand, we have $\Phi(s,0)\Phi(0,r) = (s+\eta(0))(0+\eta(r)) = s\eta(r)$. That is, we obtain $\eta(s\cdot r) = s\eta(r)$ and this is $(CM1)$. □

Lemma 3.6. *Consider the action of R on itself via multiplication and form the semi-direct product $R \ltimes R$ with respect to this action. Thus*

$$(a,b) \oplus (c,d) = (a+c, b+d)$$

and

$$(a,b) * (c,d) = (ac, ad+bc+bd), a,b,c,d \in R.$$

Then, the map $\Phi : R \ltimes R \to S \ltimes R$ defined by $(a,b) \mapsto (\eta(a), b)$ is a homomorphism if and only if η satisfies $(CM2)$.

Proof. Suppose that η satisfies condition $(CM2)$. Then, for all $(a,b),(c,d) \in R \ltimes R$, we obtain

$$\begin{aligned}\Phi((a,b) \oplus (c,d)) =& \Phi((a+c, b+d)) \\ =& (\eta(a+c), b+d) \\ =& (\eta(a), b) + (\eta(c), d) \\ =& \Phi(a,b) + \Phi(c,d)\end{aligned}$$

and

$$\begin{aligned}\Phi(a,b) * \Phi(c,d) =& (\eta(a), b) * (\eta(c), d) \\ =& (\eta(a)\eta(c), \eta(a) \cdot d + \eta(c) \cdot b + bd) \\ =& (\eta(ac), ad+bc+bd) \quad \text{since } (CM2) \\ =& \Phi(ac, ad+bc+bd) \\ =& \Phi((a,b) * (c,d)).\end{aligned}$$

Now suppose that Φ is a homomorphism. Then we have $\Phi((a,0) * (0,r)) = \Phi(0, ar) = (\eta(0), ar) = (0, ar)$. On the other hand, we have $\Phi(a,0) * \Phi(0,r) = (\eta(a), 0) * (0,r) = (0, \eta(a) \cdot r)$. Therefore we obtain $\eta(a) \cdot r = ar$ and this is $(CM2)$. □

Lemma 3.7. *Let $a_i, b_i \in R$. Then*

(i)

$$(0_S, a_1, \ldots, a_k) * (0_S, b_1, \ldots, b_k) = (0_S, a_1b_1, a_1b_2 + a_2(b_1+b_2), \ldots, (\sum_{i=1}^{k-1} a_i)b_k + a_k \sum_{i=1}^{k} b_i).$$

(ii) *Let $s \in S$ and $(0_S, a_1, a_2, \ldots, a_k) \in R_k$. Then*

$$(0_S, a_1, \ldots, a_k) * (s, 0_R, \ldots, 0_R) = (0_S, s \cdot a_1, s \cdot a_2, \ldots, s \cdot a_k).$$

Proof. We prove (i) by induction on k. For $k=1$, from Lemma 3.2 (iv), it is easy to see that

$$(0_S, a_1) * (0_S, b_1) = (0_S, a_1b_1).$$

Then, by applying d_k and induction, we see that

$$(0_S, a_1, \ldots, a_k) * (0_S, b_1, \ldots, b_k) = (0_S, a_1b_1, \ldots, (\sum_{i=1}^{k-2} a_i)b_{k-1} + a_{k-1} \sum_{i=1}^{k-1} b_i, x).$$

Applying d_{k-1} and induction once more we get that

$$\begin{aligned}(0_S, a_1b_1, \ldots, (\sum_{i=1}^{k-2} a_i)b_{k-1} + a_{k-1}\sum_{i=1}^{k-1} b_i + x)\\ =&(0_S, a_1, \ldots, a_{k-1} + a_k) * (0_S, b_1, \ldots, b_{k-1} + b_k)\\ =&(0_S, a_1b_1, \ldots, \sum_{i=1}^{k-2} a_i(b_{k-1} + b_k) + (a_{k-1} + a_k)\sum_{i=1}^{k} b_i)\\ =&(0_S, a_1b_1, \ldots, \sum_{i=1}^{k-2} a_i(b_{k-1}) + \sum_{i=1}^{k-2} a_i(b_k) + a_{k-1}\sum_{i=1}^{k} b_i + a_k\sum_{i=1}^{k} b_i)\\ =&(0_S, a_1b_1, \ldots, \sum_{i=1}^{k-2} a_i(b_{k-1}) + a_{k-1}\sum_{i=1}^{k-1} b_i + a_{k-1}b_k + \sum_{i=1}^{k-2} a_i(b_k) + (a_k)\sum_{i=1}^{k} b_i).\end{aligned}$$

It follows that

$$x = (\sum_{i=1}^{k-1} a_i)b_k + a_k\sum_{i=1}^{k} b_i.$$

(*ii*) By induction on k similarly, we prove Part (*ii*). For $k = 1$, we have

$$(0_S, a_1) * (s, 0_R) = (0_S, s \cdot a_1).$$

Applying d_k using induction we see that for $k - 1$

$$(0_S, a_1, \ldots, a_k) * (s, 0_R, \ldots, 0_R) = (0_S, s \cdot a_1, s \cdot a_2, \ldots, s \cdot a_{k-1}, x).$$

Then applying d_{k-1} using induction, we get that

$$\begin{aligned}(0_S, s \cdot a_1, \ldots, s \cdot a_{k-1} + x) =&(0_S, a_1, \ldots, a_{k-1} + a_k) * (s, 0_R, \ldots, 0_R)\\ =&(0_S, s \cdot a_1, \ldots, s \cdot (a_{k-1} + a_k))\end{aligned}$$

and so, $x = s \cdot a_k$. □

Proposition 3.8. *The homomorphism $l : S \to End_R(R)$ is an ideal structure (or a crossed module structure) on the map $\eta : R \to S$.*

Proof. Since $B_1 = S \ltimes R$, and since the homomorphism

$$d_0 : S \ltimes R = B_1 \to B_0 = S$$

is defined by $d_0(s, r) = s^r = s + \eta(r)$, Lemma 3.5 implies that $(CM1)$ holds for the map $\eta : R \to S$. Notice that by Lemma 3.7 the subalgebra R_2 is isomorphic to $R \ltimes R$. Further, the map d_0 restricted to R_2 is given by $d_0(0_S, a, b) = (\eta(a), b)$ and it is a homomorphism from $R \ltimes R$ to $S \ltimes R$ given by $(a, b) \mapsto (\eta(a), b)$. Hence by Lemma 3.6, $(CM2)$ holds for the map η. □

Let $(s, a_1, \ldots, a_k), (s', b_1, \ldots, b_k) \in B_k$. Then from the above results we get

$$\begin{aligned}(s, a_1, \ldots, a_k) * (s', b_1, \ldots, b_k) =&(ss', s \cdot b_1 + s' \cdot a_1 + a_1b_1, s \cdot b_2 + s' \cdot a_2 + a_1b_2 + a_2(b_1 + b_2),\\ &\ldots, s \cdot b_k + s' \cdot a_k + \sum_{i=1}^{k-1} a_ib_k + a_k\sum_{i=1}^{k} b_i)\end{aligned}$$

and

$$(s, a_1, \ldots, a_k) \oplus (s', b_1, \ldots, b_k) = (s + s', a_1 + b_1, \ldots, a_k + b_k).$$

4. From an ideal structure on $\eta : R \to S$ to an ideal simplicial algebra structure on $Bar(S,R)$.

In this section S and R are algebras and $\eta : R \to S$, $l : S \to End_R(R)$ are algebra homomorphisms. Recall that we denote

$$l_s : r \mapsto l_s(r) = s \cdot r$$

for $s \in S$ and $r \in R$. We assume that l is an ideal structure or a crossed module structure on η. We let $Bar(S,R)$ denote the bar construction using the action of the k-algebra R on the underlying k-module S of the algebra S via $s \mapsto s + \eta(r)$ for all $s \in S$ and $r \in R$. Our aim in this section is to show that the crossed module structure l leads to an ideal simplicial algebra structure on $Bar(S,R)$.

We start by defining a multiplication on B_k for all $k \geqslant 0$. For $k = 0$, $B_0 = S$ and the operations are as in S. Obviously, from simplicial structure $Bar(S,R)$, for $k \geqslant 1$, we can denote the addition by

$$(s, a_1, \dots, a_k) \oplus (s', b_1, \dots, b_k) = (s + s', a_1 + b_1, \dots, a_k + b_k).$$

We can define the multiplication by

$$\begin{aligned}(s, a_1, \dots, a_k) * (s', b_1, \dots, b_k) = &(ss', s \cdot b_1 + s' \cdot a_1 + a_1 b_1, s \cdot b_2 + s' \cdot a_2 + a_1 b_2 + a_2(b_1 + b_2), \\ &\dots, s \cdot b_k + s' \cdot a_k + \sum_{i=1}^{k-1} a_i b_k + a_k \sum_{i=1}^{k} b_i)\end{aligned}$$

as illustrated above.

Theorem 4.1. *Let $k \geqslant 0$. Then*

(i) *B_k is an algebra,*

(ii) *the k-module homomorphism*

$$d_0 : (s, a_1, \dots, a_k) \mapsto (s + \eta(a_1), a_2, \dots, a_k)$$

is a k-algebra homomorphism from B_k to B_{k-1},

(iii) *the k-module homomorphisms*

$$d_i : (s, a_1, \dots, a_k) \mapsto (s, a_1, \dots, a_{i-1} + a_i, \dots, a_k)$$

are k-algebra homomorphisms from B_k to B_{k-1} for all $1 \leqslant i \leqslant k-1$,

(iv) *the k-module homomorphism*

$$d_k : (s, a_1, \dots, a_k) \mapsto (s, a_1, \dots, a_{k-1})$$

is a k-algebra homomorphism from B_k to B_{k-1},

(v) *the k-module homomorphisms*

$$s_i : (s, a_1, \dots, a_k) \mapsto (s, a_1, \dots, a_i, 0, a_{i+1}, \dots, a_k)$$

are k-algebra homomorphisms for all $0 \leqslant i \leqslant k$.

Proof. (i) For each $k \geqslant 1$ define

$$\eta_k : (s, a_1, \dots, a_k) \mapsto s + \eta(a_1 + \dots + a_k)$$

from B_k to S. We prove that B_k is an algebra and that η_k is an algebra homomorphism. For $k = 1$, this is Lemma 3.5. Suppose that this holds for $k-1$. Then B_{k-1} acts on R via

$$(s, a_1, \dots, a_{k-1}) : a \mapsto a \cdot (s + \eta(a_1 + \dots + a_{k-1}))$$

for $(s, a_1, \dots, a_{k-1}) \in B_{k-1}$ and $a \in R$. Notice that B_k is just the semi-direct product algebra $B_{k-1} \ltimes R$ with respect to this action, so B_k is an algebra. To show that η_k is an algebra homomorphism, we obtain

$$\begin{aligned}\eta_k(&(s,a_1,\dots,a_k)*(s',b_1,\dots,b_k))\\ =&\eta_k(ss', s\cdot b_1+s'\cdot a_1+a_1b_1, s\cdot b_2+s'\cdot a_2+a_1b_2+a_2(b_1+b_2),\\ &\dots, s\cdot b_k+s'\cdot a_k+\sum_{i=1}^{k-1}a_ib_k+a_k\sum_{i=1}^{k}b_i)\\ =&ss'+\eta(s\cdot b_1+s'\cdot a_1+a_1b_1+s\cdot b_2+s'\cdot a_2+a_1b_2+a_2(b_1+b_2)+\\ &\dots+s\cdot b_k+s'\cdot a_k+\sum_{i=1}^{k-1}a_ib_k+a_k\sum_{i=1}^{k}b_i)\\ =&ss'+s\eta(b_1)+s'\eta(a_1)+\eta(a_1)\eta(b_1)+s\eta(b_2)+s'\eta(a_2)+\eta(a_1b_2+a_2(b_1+b_2))\\ &\dots+s\eta(b_k)+s'\eta(a_k)+\sum_{i=1}^{k-1}\eta(a_i)\eta(b_k)+\eta(a_k)\sum_{i=1}^{k}\eta(b_i)\\ =&s(s'+\sum_{i=1}^{k}\eta(b_i))+(\sum_{i=1}^{k}\eta(a_i))(s'+\sum_{i=1}^{k}\eta(b_i))\\ =&(s+\eta(a_1+\dots+a_k))(s'+\eta(b_1+\dots+b_k))\\ =&\eta_k(s,a_1,\dots,a_k)\eta_k(s',b_1,\dots,b_k).\end{aligned}$$

(ii) Let

$$u=(s,a_1,\dots,a_k), v=(s',b_1,\dots,b_k)\in B_k.$$

Then we obtain

$$\begin{aligned}d_0(u*v)=&d_0(ss', s\cdot b_1+s'\cdot a_1+a_1b_1, s\cdot b_2+s'\cdot a_2+a_1b_2+a_2(b_1+b_2),\\ &\dots, s\cdot b_k+s'\cdot a_k+\sum_{i=1}^{k-1}a_ib_k+a_k\sum_{i=1}^{k}b_i)\\ =&(ss'+s\eta(b_1)+s'\eta(a_1)+\eta(a_1)\eta(b_1), s\cdot b_2+s'\cdot a_2+a_1b_2+a_2(b_1+b_2),\\ &\dots, s\cdot b_k+s'\cdot a_k+\sum_{i=1}^{k-1}a_ib_k+a_k\sum_{i=1}^{k}b_i)\\ =&((s+\eta(a_1)(s'+\eta(b_1)), s\cdot b_2+s'\cdot a_2+a_1b_2+a_2(b_1+b_2),\\ &\dots, s\cdot b_k+s'\cdot a_k+\sum_{i=1}^{k-1}a_ib_k+a_k\sum_{i=1}^{k}b_i)\\ =&(s+\eta a_1, a_2,\dots,a_k)(s'+\eta b_1, b_2,\dots,b_k)\\ =&d_0(u)*d_0(v).\end{aligned}$$

(iii) Let

$$u=(s,a_1,\dots,a_k), v=(s',b_1,\dots,b_k)\in B_k.$$

We shall show that the k-module homomorphisms d_i are k-algebra homomorphisms for $0\leqslant i\leqslant k-1$. We calculate

$$\begin{aligned}d_i(u*v)=&d_i(ss', s\cdot b_1+s'\cdot a_1+a_1b_1, s\cdot b_2+s'\cdot a_2+a_1b_2+a_2(b_1+b_2),\\ &\dots, s\cdot b_k+s'\cdot a_k+\sum_{i=1}^{k-1}a_ib_k+a_k\sum_{i=1}^{k}b_i)\\ =&(ss', s\cdot b_1+s'\cdot a_1+a_1b_1, s\cdot b_2+s'\cdot a_2+a_1b_2+a_2(b_1+b_2),\end{aligned}$$

$$\begin{aligned}
&\ldots, s\cdot b_{i-1}+s'\cdot a_{i-1}+b_{i-1}\sum_{j=1}^{i-2}a_j+a_{i-1}\sum_{j=1}^{i-1}b_j\\
&+s\cdot b_i+s'\cdot a_i+b_i\sum_{j=1}^{i-1}a_j+a_i\sum_{j=1}^{i}b_j,\\
&\ldots, s\cdot b_k+s'\cdot a_k+\sum_{i=1}^{k-1}a_ib_k+a_k\sum_{i=1}^{k}b_i)\\
=&(ss', s\cdot b_1+s'\cdot a_1+a_1b_1,\ldots, s'\cdot(a_{i-1}+a_i)+s\cdot(b_{i-1}+b_i)\\
&+(b_{i-1}+b_i)\sum_{j=1}^{i-2}a_j+(a_{i-1}+a_i)\sum_{j=1}^{i-1}b_j+(a_{i-1}+a_i)b_i,\\
&\ldots, s\cdot b_k+s'\cdot a_k+\sum_{i=1}^{k-1}a_ib_k+a_k\sum_{i=1}^{k}b_i)\\
=&(s,a_1,\ldots,a_{i-1}+a_i,\ldots,a_k)(s',b_1,\ldots,b_{i-1}+b_i,\ldots,b_k)\\
=&d_i(u)*d_i(v)
\end{aligned}$$

for $0\leqslant i\leqslant k-1$, so Part (iii) holds.

(iv) In any semi-direct product, since the projection on to the first coordinate is a homomorphism, the map

$$d_k:(s,a_1,\ldots,a_k)\mapsto(s,a_1,\ldots,a_{k-1})$$

is a homomorphism from B_k to B_{k-1} for $k\geqslant 1$.

(v) We leave it to the reader. □

5. The mutual inverse relation between above associations

Let $\eta:R\to S$ be an algebra homomorphism together with $l:S\to End_R(R)$. We showed how to start with an ideal simplicial algebra structure on $Bar(S,R)$ and obtain a crossed module structure on η and we showed how to start with a crossed module structure on η and obtain an ideal simplicial algebra structure on the associated simplicial k-module $Bar(S,R)$. Our aim in this section is to make the observation that these two associations are mutual inverses.

First assume that the simplicial k-module $Bar(S,R)$ is endowed with an ideal simplicial algebra structure, and denote the multiplication in B_k as

$$(s,a_1,\ldots,a_k)*(s',b_1,\ldots,b_k).$$

We showed that the action $l:S\to End_R(R)$ given by $l_s:r\mapsto s\cdot r$ gives an crossed module structure on η. Further, given this crossed module structure on η, the equation

$$\begin{aligned}
(s,a_1,\ldots,a_k)*(s',b_1,\ldots,b_k)=&(ss', s\cdot b_1+s'\cdot a_1+a_1b_1, s\cdot b_2+s'\cdot a_2+a_1b_2+a_2(b_1+b_2),\\
&\ldots, s\cdot b_k+s'\cdot a_k+\sum_{i=1}^{k-1}a_ib_k+a_k\sum_{i=1}^{k}b_i).
\end{aligned}$$

tells us how to define an ideal simplicial algebra structure on B_k with the multiplication '$*$'.

Conversely let $l:S\to End_R(R)$ be an ideal structure (or a crossed module structure) on η. Let '$*$' be the multiplication in B_k as given above. Let $l':S\to End_R(R)$ be the crossed module structure on η. That is for all $s\in S$, $l'_s:r\mapsto s'$ where $(0,s')=(s,0)*(0,r)$. Now by definition of the operation $*$, we obtain

$$(s,0)*(0,r)=(0s,s\cdot r+0.0+0\cdot r)=(0,s\cdot r).$$

We thus see that $s' = s \cdot r$ for all $r \in R$, $s \in S$, that is $l'_s = l_s$ for all $s \in S$. This completes the observation that the two associations are mutual inverses.

6. Crossed ideal maps between ideal maps

We explored above that a homomorphism of algebras $\eta : R \to S$ together with an ideal structure (or crossed module structure) preserves the ideals of R. This ideal approach to crossed modules shades some light on ideals of Loday's crossed square (cf. [11]). That is, we consider the same thing for crossed ideals of crossed modules. In this section, we will provide an extension of this result for higher dimensional crossed modules of algebras. We see that if there is a (crossed) ideal structure over a morphism between crossed modules, then this map preserves the (crossed) ideals.

First we recall the definition of 'crossed ideal' of a crossed module of algebras from [15].

Definition 6.1. A homomorphism of algebras $\eta' : R' \to S'$ will be called a *crossed ideal* of the crossed module $\eta : R \to S$ in the category of crossed modules over k-algebras if:

$\mathfrak{CI}1$: $\eta' : R' \to S'$ is a subcrossed module of $\eta : R \to S$, that is, the following conditions are satisfied:

(i) R' is a subalgebra of R and S' is a subalgebra of S.
(ii) the action of S' on R' induced by the action of S on R.
(iii) $\eta' : R' \to S'$ is a crossed module.
(iv) the following diagram of morphisms of crossed modules

$$\begin{array}{ccc} R' & \xrightarrow{\mu} & R \\ {\scriptstyle \eta'}\downarrow & & \downarrow{\scriptstyle \eta} \\ S' & \xrightarrow[\nu]{} & S \end{array}$$

commutes, where μ and ν are the inclusions,

$\mathfrak{CI}2$: $R'R = RR' \subseteq R'$ and $SS' = S'S \subseteq S'$,
$\mathfrak{CI}3$: $R \cdot S' = S' \cdot R \subseteq R'$,
$\mathfrak{CI}4$: R' is closed under the action of S, i.e. $S \cdot R' = R' \cdot S \subseteq R'$.

6.1. Crossed ideal structure over maps between crossed modules

Assume that $\eta_1 : R_1 \to S_1$ and $\eta_2 : R_2 \to S_2$ are crossed modules. Let $\alpha : (\alpha_1, \alpha_2)$ be a morphism from η_1 to η_2 in the category of crossed modules of k-algebras, where $\alpha_1 : R_1 \to R_2$ and $\alpha_2 : S_1 \to S_2$ are homomorphisms of k-algebras. In this case, the morphism $\alpha := (\alpha_1, \alpha_2)$ satisfies the following conditions:

(i) the diagram

$$\begin{array}{ccc} R_1 & \xrightarrow{\alpha_1} & R_2 \\ {\scriptstyle \eta_1}\downarrow & & \downarrow{\scriptstyle \eta_2} \\ S_1 & \xrightarrow[\alpha_2]{} & S_2 \end{array}$$

commutes, i.e. $\alpha_2\eta_1 = \eta_2\alpha_1$,

(ii) for all $s_1 \in S_1$ and $r_1 \in R_1$,

$$\alpha_1(l_{s_1}(r_1)) = l_{\alpha_2(s_1)}(\alpha_1(r_1)) \text{ or } \alpha_1(s_1 \cdot r_1) = \alpha_2(s_1) \cdot (\alpha_1(r_1)).$$

Definition 6.2. A morphism $\alpha := (\alpha_1, \alpha_2)$ between crossed modules η_1 and η_2 is called a *crossed ideal map* if

(i) there are ideal map structures over the homomorphisms α_1, α_2 and $\eta_2\alpha_1 = \alpha_2\eta_1$, and
(ii) there is an S_2-bilinear map $h : R_2 \times S_1 \to R_1$ satisfying the conditions:

(a) $\alpha_1(h(r_2, s_1)) = \alpha_2(s_1) \cdot r_2$,
(b) $\eta_1(h(r_2, s_1)) = \eta_2(r_2) \cdot s_1$,
(c) $h(\alpha_1(r_1), s_1) = s_1 \cdot r_1$,
(d) $h(r_2, \eta_1(r_1)) = r_2 \cdot r_1$
for all $r_2 \in R_2, s_1 \in S_1$.

Remark 6.3. A crossed ideal structure over the map α between crossed modules η_1 and η_2 gives a crossed square structure of algebras on the square

$$\begin{array}{ccc} R_1 & \xrightarrow{\alpha_1} & R_2 \\ {\scriptstyle\eta_1}\downarrow & & \downarrow{\scriptstyle\eta_2} \\ S_1 & \xrightarrow[\alpha_2]{} & S_2 \end{array}$$

defined by Ellis in [6].

Thus, we get the following result.

Proposition 6.4. *If the morphism $\alpha : (\alpha_1, \alpha_2)$ is a crossed ideal map from $(\eta_1 : R_1 \to S_1)$ to $(\eta_2 : R_2 \to S_2)$ in the category of crossed modules of* k*-algebras, then $\alpha(\eta_1) : \alpha_1(R_1) \to \alpha_2(S_1)$ is a crossed ideal of the crossed module $\eta_2 : R_2 \to S_2$.*

Proof. First we consider the following square

$$\begin{array}{ccc} \alpha_1(R_1) = R_1' & \xrightarrow{\mu} & R_2 \\ {\scriptstyle\overline{\eta_2}}\downarrow & & \downarrow{\scriptstyle\eta_2} \\ \alpha_2(S_1) = S_1' & \xrightarrow[\nu]{} & S_2 \end{array}$$

where μ and ν are the inclusions. The map $\overline{\eta_2} : R_1' \to S_1'$ is defined by the restriction of the map η_2 to the subalgebra $\alpha_1(R_1)$ of R_2. We will show that the restricted homomorphism $\overline{\eta_2}$ is a crossed ideal of η_2.

$\mathfrak{CI}1$. We will show that $\overline{\eta_2}$ is a subcrossed module of η_2.

(i) It is clear that R_1' is a subalgebra of R_2 and similarly $\alpha_2(S_1) = S_1'$ is a subalgebra of S_2.

(ii) Since the map $\alpha := (\alpha_1, \alpha_2)$ is a crossed module morphism, it satisfies the condition $\alpha_2(s_1) \cdot (\alpha_1 r_1) = \alpha_1(s_1 \cdot r_1)$ for all $r_1 \in R_1$ and $s_1 \in S_1$. Then the algebra action of $\alpha_2(s_1) \in S_1'$ on $\alpha_1(r_1) \in R_1'$ can be given by $\alpha_2(s_1) \cdot \alpha_1(r_1) = \alpha_1(s_1 \cdot r_1) \in R_1'$.

(iii) We will show that $\overline{\eta_2} : R_1' \to S_1'$ is a crossed module. For all $\alpha_2(s_1) \in S_1'$ and $\alpha_1(r_1), \alpha_1(r_1') \in R_1'$, we obtain

$$\begin{aligned} \overline{\eta_2}(\alpha_2(s_1) \cdot (\alpha_1(r_1))) &= \eta_2\alpha_1(s_1 \cdot r_1) \\ &= \alpha_2\eta_1(s_1 \cdot r_1) \\ &= \alpha_2(s_1\eta_1(r_1)) \quad (\text{ since } \eta_1 \text{ is a crossed module }) \\ &= \alpha_2(s_1)\alpha_2\eta_1(r_1) \\ &= \alpha_2(s_1)\eta_2\alpha_1(r_1) \\ &= \alpha_2(s_1)\overline{\eta_2}(\alpha_1(r_1)), \end{aligned}$$

and

$$\begin{aligned} \overline{\eta_2}(\alpha_1(r_1)) \cdot \alpha_1(r_1') &= \alpha_2(\eta_1(r_1)) \cdot \alpha_1(r_1') \\ &= \alpha_1(\eta_1(r_1) \cdot (r_1')) \\ &= \alpha_1(r_1 r_1') \quad (\text{ since } \eta_1 \text{ is a crossed module }) \\ &= \alpha_1(r_1)\alpha_1(r_1'). \end{aligned}$$

(iv) the square

$$\begin{array}{ccc} R_1' & \xrightarrow{\mu} & R_2 \\ {\scriptstyle \overline{\eta_2}}\downarrow & & \downarrow{\scriptstyle \eta_2} \\ S_1' & \xrightarrow[\nu]{} & S_2 \end{array}$$

is commutative, because μ and ν are the inclusions and $\overline{\eta_2}$ is given by the restriction of η_2. Thus $\overline{\eta_2}$ is a subcrossed module of η_2.

$\mathfrak{CI}2$. Since there are ideal structures over the maps $\alpha_1 : R_1 \to R_2$ and $\alpha_2 : S_1 \to S_2$, we obtain that $\alpha_1(R_1) = R_1'$ and $\alpha_2(S_1) = S_1'$ are ideals of R_2 and S_2 respectively. Therefore, we obtain

$$R_1'R_2 = R_2R_1' \subseteq R_1' \text{ and } S_1'S_2 = S_2S_1' \subseteq S_1'.$$

$\mathfrak{CI}3$. We have to show that $R_2 \cdot S_1' = S_1' \cdot R_2 \subseteq R_1'$. We use the h-map to prove it. For all $\alpha_2(s_1) \in S_1'$ and $r_2 \in R_2$ we have $r_2 \cdot \alpha_2(s_1) = \alpha_2(s_1) \cdot r_2 = l_{\alpha_2(s_1)}(r_2) = \alpha_1(h(r_2, s_1))$, where $h(r_2, s_1) \in R_1$, then we obtain $r_2 \cdot \alpha_2(s_1) = \alpha_2(s_1) \cdot r_2 \in \alpha_1(R_1) = R_1'$ so that $R_2 \cdot S_1' = S_1' \cdot R_2 \subseteq R_1'$.

$\mathfrak{CI}4$. We have to show that $S_2 \cdot R_1' = R_1' \cdot S_2 \subseteq R_1'$. For all $s_2 \in S_2$ and $\alpha_1(r_1) \in R_1'$, we can define the action by $s_2 \cdot \alpha_1(r_1) = l_{s_2}(\alpha_1(r_1)) = \alpha_1(s_2 \cdot (r_1)) \in R_1'$. Thus R_1' is closed under the action of S_2 and this completes the proof. □

Conversely, we can easily state that given a crossed ideal $\overline{\eta_2} : R_1' \to S_1'$ of the crossed module $\eta_2 : R_2 \to S_2$, then inclusion morphism from $\overline{\eta_2}$ to η_2 is a crossed ideal map in the category of crossed modules of k-algebras.

Indeed, if $\overline{\eta_2}$ is a crossed ideal of η_2 in the following diagram,

$$\begin{array}{ccc} R_1' & \xrightarrow{\mu} & R_2 \\ {\scriptstyle \overline{\eta_2}}\downarrow & & \downarrow{\scriptstyle \eta_2} \\ S_1' & \xrightarrow[\nu]{} & S_2 \end{array}$$

the inclusion morphisms μ and ν are crossed modules with the natural actions of R_2 and S_2 on their ideals R_1' and S_1' given by the multiplication, respectively. Further, the h-map $h : R_2 \times S_1' \to R_1'$ is defined by $h(r_2, s_1') = (l|_{S_1'})_{s_1'}(r_2)$, where $l|_{S_1'}$ is the restriction of $l : S_2 \to End(R_2)$ to S_1'.

7. From the morphism $\alpha : \eta_1 \to \eta_2$ to the usual bar construction

In [9], Farjoun and Segev proved that a crossed module map $l : G \to Aut(N)$, which they call a normal structure on the map $N \to G$ is inversely associated with a group structure on the homotopy quotient $G//N := hocomlim_N G$ by taking $G//N$ to be the usual bar construction. They also stated in section 6 of their work, for a morphism from a normal map $X \to G$ to a normal map $Y \to H$ in the category of normal maps, one can form a simplicial group morphism $X//G \to Y//H$ and the homotopy quotient $(Y//H)//(X//G)$. In fact, if there is a normal map structure over the simplicial group morphism $X//G \to Y//H$, then $(Y//H)//(X//G)$ is a *bisimplicial* group. In this section, we make some remarks concerning these ideas over k-algebras.

Recall that a functor $\mathbf{E}_{\cdot,\cdot} : (\Delta \times \Delta)^{op} \to \mathbf{Alg}$ is called a *bisimplicial algebra*, where Δ is the category of finite ordinals and $\mathbf{Alg}$ is the category of (commutative) k-algebras.

Hence $\mathbf{E}_{\cdot,\cdot}$ is equivalent to giving for each (p, q) an algebra $E_{p,q}$ and morphisms:

$$\begin{aligned} d_i^{h(pq)} &: E_{p,q} \to E_{p-1,q} \\ s_i^{h(pq)} &: E_{p,q} \to E_{p+1,q} \quad 0 \leq i \leq p \\ d_j^{v(pq)} &: E_{p,q} \to E_{p,q-1} \\ s_j^{v(pq)} &: E_{p,q} \to E_{p,q+1} \quad 0 \leq j \leq q \end{aligned}$$

such that the maps $d_i^{h(pq)}, s_i^{h(pq)}$ commute with $d_j^{v(pq)}, s_j^{v(pq)}$ and that $d_i^{h(pq)}, s_i^{h(pq)}$ (resp. $d_j^{v(pq)}, s_j^{v(pq)}$) satisfy the usual simplicial identities.

Now suppose that $\alpha : (\alpha_1, \alpha_2)$ is a morphism from $\eta_1 : R_1 \to S_1$ to $\eta_2 : R_2 \to S_2$ in the category of crossed modules of k-algebras. Using the usual bar construction, we can form the simplicial algebras $S_1//R_1$ and $S_2//R_2$ from η_1 and η_2 respectively as above. Analogously to [9], we obtain a simplicial algebra morphism

$$\Phi : S_1//R_1 \to S_2//R_2$$

and we can define this map on each step by

$$\Phi_n : (S_1 \ltimes (R_1)^{\ltimes^n}) \to (S_2 \ltimes (R_2)^{\ltimes^n})$$

with

$$\Phi_n : (s_1, r_1, r_2, \ldots, r_n) = (\alpha_2(s_1), \alpha_1(r_1), \alpha_1(r_2), \ldots, \alpha_1(r_n))$$

for all $s_1 \in S_1$ and $r_i \in R_1$ and where the maps Φ_n are homomorphisms of algebras.

An action of the algebra $(S_1 \ltimes (R_1)^{\ltimes^n})$ on the underlying k-module of the algebra $(S_2 \ltimes (R_2)^{\ltimes^n})$ can be given by this map, namely,

$$(s_1, \ltimes_{i=1}^n (r_i)) : (s_2, \ltimes_{i=1}^n (r_i')) = (s_2 + \alpha_1(s_1), \ltimes_{i=1}^n (r_i' + \alpha_1(r_i)))$$

where $s_1 \in S_1, s_2 \in S_1$ and $r_i \in R_1$, $r_i' \in R_2$ for $i = 1, 2, \ldots, n$.

Using this action on each step, and considering the usual bar construction, we can form a *bisimplicial* k-module,

$$\mathfrak{Bar}^{(2)} : (S_2//R_2)//(S_1//R_1)$$

and, on each directions, this can be defined by the k-modules

$$\mathfrak{Bar}_{n,m}^{(2)} := (S_2 \ltimes (R_2)^{\ltimes^n}) \times (S_1 \ltimes (R_1)^{\ltimes^n})^{\times^m}.$$

The horizontal homomorphisms between these k-modules can be defined as follows:

1. For all

$$(s_2, r_{21}, \cdots, r_{2n}) \in S_2 \ltimes (R_2)^{\ltimes^n}$$

and

$$((s_1^1, r_{11}^1, \cdots, r_{1n}^1), \cdots, (s_1^m, r_{11}^m, \cdots, r_{1n}^m)) \in (S_1 \ltimes (R_1)^{\ltimes^n})^{\times^m},$$

where, for $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$, $r_{1i}^j \in R_1$, $r_{2i} \in R_2$, $s_2 \in S_2$, $s_1^j \in S_1$, the $d_0^h : \mathfrak{Bar}_{n,m}^{(2)} \to \mathfrak{Bar}_{n,m-1}^{(2)}$ is defined by

$$\begin{aligned} &d_0^h((s_2, r_{21}, \cdots, r_{2n}), (s_1^1, r_{11}^1, \cdots, r_{1n}^1), \cdots, (s_1^m, r_{11}^m, \cdots, r_{1n}^m)) \\ &= \Big((s_2, r_{21}, \cdots, r_{2n}) + \Phi_n(s_1^1, r_{11}^1, \cdots, r_{1n}^1), (s_1^2, r_{11}^2, \cdots, r_{1n}^2), \cdots, (s_1^m, r_{11}^m, \cdots, r_{1n}^m)\Big). \end{aligned}$$

2. For $0 < i < m$, the $d_i^h : \mathfrak{Bar}_{n,m}^{(2)} \to \mathfrak{Bar}_{n,m-1}^{(2)}$ is defined by

$$\begin{aligned} &d_i^h((s_2, r_{21}, \cdots, r_{2n}), (s_1^1, r_{11}^1, \cdots, r_{1n}^1), \cdots, (s_1^m, r_{11}^m, \cdots, r_{1n}^m)) \\ &\qquad = ((s_2, r_{21}, \cdots, r_{2n}), (s_1^1, r_{11}^1, \cdots, r_{1n}^1), \cdots, \\ &\qquad (s_1^i, r_{11}^i, \cdots, r_{1n}^i) + (s_1^{i+1}, r_{11}^{i+1}, \cdots, r_{1n}^{i+1}), \cdots, (s_1^m, r_{11}^m, \cdots, r_{1n}^m)). \end{aligned}$$

3. $d^h_m : \mathfrak{Bar}^{(2)}_{n,m} \to \mathfrak{Bar}^{(2)}_{n,m-1}$ is defined by

$$d^h_m((s_2, r_{21}, \cdots, r_{2n}), (s^1_1, r^1_{11}, \cdots, r^1_{1n}), \cdots, (s^m_1, r^m_{11}, \cdots, r^m_{1n})) \\ = ((s_2, r_{21}, \cdots, r_{2n}), (s^1_1, r^1_{11}, \cdots, r^1_{1n}), \cdots, (s^{m-1}_1, r^{m-1}_{11}, \cdots, r^{m-1}_{1n})).$$

4. For all $0 \leqslant i \leqslant m$, the $s^h_i : \mathfrak{Bar}^{(2)}_{n,m} \to \mathfrak{Bar}^{(2)}_{n,m+1}$ is defined by

$$s^h_i((s_2, r_{21}, \cdots, r_{2n}), (s^1_1, r^1_{11}, \cdots, r^1_{1n}), \cdots, (s^m_1, r^m_{11}, \cdots, r^m_{1n})) \\ = ((s_2, r_{21}, \cdots, r_{2n}), (s^1_1, r^1_{11}, \cdots, r^1_{1n}), \cdots, \\ (s^i_1, r^i_{11}, \cdots, r^i_{1n}), (0, 0, \cdots, 0), (s^{i+1}_1, r^{i+1}_{11}, \cdots, r^{i+1}_{1n}), \cdots, (s^m_1, r^m_{11}, \cdots, r^m_{1n})).$$

Similarly, the vertical homomorphisms can be defined as follows:

1. the $d^v_0 : \mathfrak{Bar}^{(2)}_{n,m} \to \mathfrak{Bar}^{(2)}_{n-1,m}$ is defined by

$$d^v_0((s_2, r_{21}, \cdots, r_{2n}), (s^1_1, r^1_{11}, \cdots, r^1_{1n}), \cdots, (s^m_1, r^m_{11}, \cdots, r^m_{1n})) \\ = \Big((s_2 + \eta_2(r_{21}), r_{22} \cdots, r_{2n}), (s^1_1 + \eta_1(r^1_{11}), r^2_{12} \cdots, r^2_{1n}), \cdots, (s^m_1 + \eta_1(r^m_{11}), r^m_{12} \cdots, r^m_{1n})\Big).$$

2. For $0 < i < n$, the $d^v_i : \mathfrak{Bar}^{(2)}_{n,m} \to \mathfrak{Bar}^{(2)}_{n-1,m}$ is defined by

$$d^v_i((s_2, r_{21}, \cdots, r_{2n}), (s^1_1, r^1_{11}, \cdots, r^1_{1n}), \cdots, (s^m_1, r^m_{11}, \cdots, r^m_{1n})) \\ = ((s_2, r_{21}, \cdots, r_{2i} + r_{2(i+1)}, \cdots, r_{2n}), (s^1_1, r^1_{11}, \cdots, r^1_{1i} + r^1_{1(i+1)}, \cdots, r^1_{1n}), \cdots, \\ (s^m_1, r^m_{11}, \cdots, r^m_{1i} + r^m_{1(i+1)} \cdots, r^m_{1n})).$$

3. $d^v_n : \mathfrak{Bar}^{(2)}_{n,m} \to \mathfrak{Bar}^{(2)}_{n-1,m}$ is defined by

$$d^v_n((s_2, r_{21}, \cdots, r_{2n}), (s^1_1, r^1_{11}, \cdots, r^1_{1n}), \cdots, (s^m_1, r^m_{11}, \cdots, r^m_{1n})) \\ = ((s_2, r_{21}, \cdots, r_{(2n-1)}), (s^1_1, r^1_{11}, \cdots, r^1_{1(n-1)}), \cdots, (s^{m-1}_1, r^m_{11}, \cdots, r^m_{1(n-1)})).$$

4.For all $0 \leqslant i \leqslant n$, the $s^v_i : \mathfrak{Bar}^{(2)}_{n,m} \to \mathfrak{Bar}^{(2)}_{n+1,m}$ is defined by

$$s^v_i((s_2, r_{21}, \cdots, r_{2n}), (s^1_1, r^1_{11}, \cdots, r^1_{1n}), \cdots, (s^m_1, r^m_{11}, \cdots, r^m_{1n})) \\ = ((s_2, r_{21}, \cdots, r_{2i}, 0, r_{2(i+1)}, \cdots, r_{2n}), (s^1_1, r^1_{11}, \cdots, r^1_{1i}, 0, r^1_{1(i+1)}, \cdots, r^1_{1n}), \cdots, \\ (s^m_1, r^m_{11}, \cdots, r^m_{1i}, 0, r^m_{1(i+1)}, \cdots r^m_{1n})).$$

In low dimensions, we can illustrate this bisimplicial k-module by the diagram:

$$\begin{array}{ccccc}
\cdots & \Rrightarrow & (S_2 \ltimes (R_2)^2) \times (S_1 \ltimes (R_1)^2) & \rightrightarrows & (S_2 \ltimes R_2^2) \\
\downdownarrows\downarrow & & \downdownarrows\downarrow & & \downdownarrows\downarrow \\
(S_2 \ltimes R_2) \times (S_1 \ltimes R_1)^2 & \Rrightarrow & (S_2 \ltimes R_2) \times (S_1 \ltimes R_1) & \rightrightarrows & S_2 \ltimes R_2 \\
\downdownarrows & & \downdownarrows & & \downdownarrows \\
\ldots S_2 \times (S_1)^2 & \Rrightarrow & (S_2 \times S_1) & \rightrightarrows & S_2
\end{array}$$

For instance, in this diagram, the homomorphisms in the first square are given by:

$$d^v_0(s_2, r_2) = s_2 + \eta_2(r_2), \quad d^h_0(s_2, s_1) = s_2 + \alpha_2(s_1) \\ d^v_1(s_2, r_2) = s_2, \quad d^h_1(s_2, s_1) = s_2 \\ s^v_0(s_2) = (s_2, 0), \quad s^h_0(s_2) = (s_2, 0).$$

and

$$d_0^v((s_2,r_2),(s_1,r_1)) = (s_2+\eta_2(r_2), s_1+\eta_1(r_1)), \quad d_0^h((s_2,r_2),(s_1,r_1)) = (s_2+\alpha_2(s_1), r_2+\alpha_1(r_1))$$
$$d_1^v((s_2,r_2),(s_1,r_1)) = (s_2,s_1), \quad d_1^h((s_2,r_2),(s_1,r_1)) = (s_2,r_2)$$
$$s_0^v(s_2,s_1) = ((s_2,0),(s_1,0)), \quad s_0^h(s_2,r_2) = ((s_2,r_2),(0,0)).$$

Therefore, we obtained a bisimplicial k-module, from the morphism α in the category of crossed modules of k-algebras. Thus we expect to give the following result.

Theorem 7.1. *Given a morphism $\alpha : \eta_1 \to \eta_2$ in the category of crossed modules of k-algebras, a crossed ideal map structure on the morphism α gives an ideal bisimplicial algebra structure on the associated bisimplicial k-module $\mathfrak{Bar}^{(2)} : (S_2//R_2)//(S_1//R_1)$, and conversely, any ideal bisimplicial algebra structure on the bisimplicial k-module $\mathfrak{Bar}^{(2)} : (S_2//R_2)//(S_1//R_1)$ determines a crossed ideal map structure on the morphism $\alpha : \eta_1 \to \eta_2$.*

Remark 7.2. In order to prove this theorem, we would need to introduce the notion of '*ideal bisimplicial algebra structure*' over the associated bisimplicial k-module $\mathfrak{Bar}^{(2)}$ explicitly. The proof will be analysed in a separate paper. Of course, this result can be iterated to the crossed n-cube structure defined by Ellis in [6]. In this case, we would need to give a detailed definition of a *crossed n-ideal* of a crossed n-cube and a *crossed n-ideal structure* over the morphism between crossed $(n-1)$ cubes. Then it would give a multi-simplicial algebra in dimension n, or an n-simplicial algebra together with this structure.

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