






Fractal quintic spline method for nonlinear boundary-value problems

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Abstract

In this article, numerical solutions of nonlinear boundary-value problems are obtained using fractal quintic spline. Convergence analysis of the proposed method is also established. Proposed method has fourth-order convergence. Numerical examples are provided to show practical usefulness of the method and numerical results are compared with the existing numerical methods.

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1. Introduction

In this article, we consider the nonlinear boundary-value problems (BVPs) of the form

$$\begin{cases} u_{xx}(x) + F(x, u(x)) = 0, & x \in (0, 1), \\ u(0) = \eta_0, & u(1) = \eta_1, \end{cases} \quad (1.1)$$

where η_0 and η_1 are constants. We assume that for $(x, u(x)) \in D = \{0 \leq x \leq 1, -\infty < u(x) < \infty\}$, the functions F and $\partial F/\partial u$ are continuous. It is known that problem (1.1) admits a unique solution, provided $\sup_{(x,u) \in D} \partial F/\partial u < \pi^2$ [20]. Here, we assume that $\frac{\partial F}{\partial u} \leq 0$

on D and $\frac{\partial F}{\partial u} < 0$ on $D^\circ = \{0 < x < 1, -\infty < u(x) < \infty\}$. The notation u_{xx} is used instead of u'' mainly for the sake of convenience in the future.

To obtain the numerical approximate solutions of nonlinear BVPs, various authors have used different methods. For example, Chawla [14] developed a fourth-order numerical method using cubic spline to get the numerical solutions of nonlinear BVPs. Ravikanth [23] used the cubic spline and Rashidinia [22] used non-polynomial cubic spline to obtain the numerical solutions of nonlinear BVPs. Also, one can see various numerical techniques to obtain the numerical solution of nonlinear BVPs in [1, 2, 7, 8, 11–13, 15–17, 19, 21, 24–26].

In this paper, first, we apply the quasilinearization technique [5, 19, 22] to linearize the given nonlinear BVP (1.1) and obtain a sequence of linear BVPs. Each of these linear BVPs is then solved by the numerical scheme obtained from the fractal quintic spline.

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The fractal quintic spline contains a parameter called the scaling factor and the scaling factor is suitably restricted to get the numerical solution of the linearized BVPs.

The rest of the paper is organized as follows: In Section 2, the numerical scheme using the fractal quintic spline has been developed. Convergence analysis of the numerical method is carried out in Section 3. In Section 4, numerical examples are provided to validate the theoretical results.

2. Numerical scheme

Here, at the beginning, a brief introduction about the fractal quintic spline is given. Then, we have obtained the relation from the continuity conditions of the fractal quintic spline. This relation is used to derive the numerical scheme of the BVP. Further, we have provided the concept of quasilinearization technique. Finally, we have constructed the numerical scheme to solve the BVP.

2.1. Fractal quintic spline

Let $0 = x_0 < x_1 < \dots < x_N = 1$ be the partition of the interval $I = [0, 1]$. Let $u(x)$ be the solution of the nonlinear BVP given in (1.1) and U_i be the numerical approximate solution of $u(x_i)$. Let M_i and S_i be the approximations of $u_{xx}(x_i)$ and $u_{xxxx}(x_i)$ respectively.

Barnsley [3] introduced the concept of fractal interpolation function (FIF) with the help of iterated function system (IFS). For more details about IFS and FIFs, one can see the references [3, 4, 9, 10].

Consider the IFS $\{I \times \mathbb{R}; w_i(x, y) = (L_i(x), F_i(x, y)) : i = 1, 2, \dots, N\}$, where $L_i : I \rightarrow I_i = [x_{i-1}, x_i]$ defined by $L_i(x) = hx + x_{i-1}$, $x \in I$, $F_i : I \times \mathbb{R} \rightarrow \mathbb{R}$ defined by $F_i(x, y) = \alpha y + r_i(x)$, $(x, y) \in I \times \mathbb{R}$, with $r_i(x) = \mathcal{A}_i(x - x_0)^5 + \mathcal{B}_i(x - x_0)^4 + \mathcal{C}_i(x - x_0)^3 + \mathcal{D}_i(x - x_0)^2 + \mathcal{E}_i(x - x_0) + \mathcal{F}_i$ and α is the scaling factor such that $|\alpha| < h^4$. For each fixed $i = 1, 2, \dots, N$, the function L_i satisfy the conditions $L_i(x_0) = x_{i-1}$, $L_i(x_N) = x_i$. We assume the following conditions on IFS:

$$\begin{aligned} F_i(x_0, U_0) &= U_{i-1}, \quad F_i(x_N, U_N) = U_i, \quad i = 1, 2, \dots, N, \\ F_{i,1}(x_N, U_{N,1}) &= F_{i+1,1}(x_0, U_{0,1}), \quad i = 1, 2, \dots, N-1, \\ F_{i,2}(x_0, M_0) &= M_{i-1}, \quad F_{i,2}(x_N, M_N) = M_i, \quad i = 1, 2, \dots, N, \\ F_{i,3}(x_N, U_{N,3}) &= F_{i+1,3}(x_0, U_{0,3}), \quad i = 1, 2, \dots, N-1, \\ F_{i,4}(x_0, S_0) &= S_{i-1}, \quad F_{i,4}(x_N, S_N) = S_i, \quad i = 1, 2, \dots, N, \end{aligned}$$

where

$$\begin{aligned} F_{i,1}(x, y) &= \frac{\alpha y + (r_i)_x(x)}{h}, \quad F_{i,2}(x, y) = \frac{\alpha y + (r_i)_{xx}(x)}{h^2}, \\ F_{i,3}(x, y) &= \frac{\alpha y + (r_i)_{xxx}(x)}{h^3}, \quad F_{i,4}(x, y) = \frac{\alpha y + (r_i)_{xxxx}(x)}{h^4}, \\ U_{0,1} &= \frac{(r_1)_x(x_0)}{h - \alpha}, \quad U_{N,1} = \frac{(r_N)_x(x_N)}{h - \alpha}, \\ U_{0,3} &= \frac{(r_1)_{xxx}(x_0)}{h^3 - \alpha}, \quad U_{N,3} = \frac{(r_N)_{xxx}(x_N)}{h^3 - \alpha}, \end{aligned}$$

here $(r_i)_x$ represents the derivative of r_i . Clearly, the IFS $\{I \times \mathbb{R}; w_i(x, y) = (L_i(x), F_i(x, y)) : i = 1, 2, \dots, N\}$ satisfies the conditions for \mathcal{C}^4 -differentiable FIFs [4, 9, 10].

Let $\mathcal{F} = \{\varphi \in \mathcal{C}^4(I, \mathbb{R}) \mid \varphi(x_0) = U_0, \varphi(x_N) = U_N, \varphi_{xx}(x_0) = M_0, \varphi_{xx}(x_N) = M_N, \varphi_{xxxx}(x_0) = S_0, \varphi_{xxxx}(x_N) = S_N\}$.

Then (\mathcal{F}, d) is a complete metric space where the metric d is induced by the \mathcal{C}^4 -norm on \mathcal{F} . Define the Read-Bajraktarević operator T on (\mathcal{F}, d) as

$$T\varphi(L_i(x)) = \alpha\varphi(x) + \mathcal{A}_i(x-x_0)^5 + \mathcal{B}_i(x-x_0)^4 + \mathcal{C}_i(x-x_0)^3 + \mathcal{D}_i(x-x_0)^2 + \mathcal{E}_i(x-x_0) + \mathcal{F}_i, \quad x \in [x_0, x_N],$$

$i = 1, 2, \dots, N$. The map T is a contraction operator, hence it has a unique fixed-point Φ (say). The fixed-point Φ satisfies the following functional equation:

$$\Phi(L_i(x)) = \alpha\Phi(x) + \mathcal{A}_i(x-x_0)^5 + \mathcal{B}_i(x-x_0)^4 + \mathcal{C}_i(x-x_0)^3 + \mathcal{D}_i(x-x_0)^2 + \mathcal{E}_i(x-x_0) + \mathcal{F}_i, \quad x \in [x_0, x_N], \quad (2.1)$$

$i = 1, 2, \dots, N$. The conditions $F_i(x_0, U_0) = U_{i-1}$, $F_i(x_N, U_N) = U_i$, $F_{i,2}(x_0, M_0) = M_{i-1}$, $F_{i,2}(x_N, M_N) = M_i$, $F_{i,4}(x_0, S_0) = S_{i-1}$, $F_{i,4}(x_N, S_N) = S_i$ are equivalent to the following conditions:

$$\begin{cases} \Phi(x_{i-1}) = U_{i-1}, \Phi(x_i) = U_i, \Phi_{xx}(x_{i-1}) = M_{i-1}, \Phi_{xx}(x_i) = M_i, \\ \Phi_{xxxx}(x_{i-1}) = S_{i-1}, \Phi_{xxxx}(x_i) = S_i, \end{cases} \quad (2.2)$$

respectively. For fractal quintic spline Φ , the constants \mathcal{A}_i , \mathcal{B}_i , \mathcal{C}_i , \mathcal{D}_i , \mathcal{E}_i , and \mathcal{F}_i are evaluated by using the conditions given in (2.2) and hence we get

$$\begin{aligned} \mathcal{A}_i &= \frac{h^4}{120} \left[\left(S_i - \frac{\alpha}{h^4} S_N \right) - \left(S_{i-1} - \frac{\alpha}{h^4} S_0 \right) \right], \\ \mathcal{B}_i &= \frac{h^4}{24} \left(S_{i-1} - \frac{\alpha}{h^4} S_0 \right), \\ \mathcal{C}_i &= -\frac{h^2}{6} \left(M_{i-1} - \frac{\alpha}{h^2} M_0 \right) + \frac{h^2}{6} \left(M_i - \frac{\alpha}{h^2} M_N \right) \\ &\quad - \frac{h^4}{12} \left(S_{i-1} - \frac{\alpha}{h^4} S_0 \right) - \frac{h^4}{36} \left[\left(S_i - \frac{\alpha}{h^4} S_N \right) - \left(S_{i-1} - \frac{\alpha}{h^4} S_0 \right) \right], \\ \mathcal{D}_i &= \frac{h^2}{2} \left(M_{i-1} - \frac{\alpha}{h^2} M_0 \right), \\ \mathcal{E}_i &= \left(U_i - \alpha U_N \right) - \left(U_{i-1} - \alpha U_0 \right) + \frac{8h^4}{360} \left(S_{i-1} - \frac{\alpha}{h^4} S_0 \right) \\ &\quad + \frac{7h^4}{360} \left(S_i - \frac{\alpha}{h^4} S_N \right) - \frac{2h^2}{6} \left(M_{i-1} - \frac{\alpha}{h^2} M_0 \right) - \frac{h^2}{6} \left(M_i - \frac{\alpha}{h^2} M_N \right), \\ \mathcal{F}_i &= U_{i-1} - \alpha U_0. \end{aligned}$$

The conditions $F_{i,1}(x_N, U_{N,1}) = F_{i+1,1}(x_0, U_{0,1})$, $i = 1, 2, \dots, N-1$ can be reformulated as $\Phi_x(L_i(x_N)) = \Phi_x(L_{i+1}(x_0))$, $i = 1, 2, \dots, N-1$. For the continuity of Φ_x at the interior nodes x_i , $i = 1, 2, \dots, N-1$, we require that $\Phi_x(x_i^-) = \Phi_x(x_i^+)$, i.e., $\Phi_x(L_i(x_N)) = \Phi_x(L_{i+1}(x_0))$, $i = 1, 2, \dots, N-1$ and this leads to the following:

$$\alpha\Phi_x(x_N) + 5\mathcal{A}_i + 4\mathcal{B}_i + 3\mathcal{C}_i + 2\mathcal{D}_i + \mathcal{E}_i = \alpha\Phi_x(x_0) + \mathcal{E}_{i+1},$$

substituting the values of \mathcal{A}_i , \mathcal{B}_i , \mathcal{C}_i , \mathcal{D}_i , \mathcal{E}_i , and \mathcal{E}_{i+1} , we get

$$\begin{aligned} -\frac{15\alpha}{h^4} S_0 + 7S_{i-1} + 16S_i + 7S_{i+1} - \frac{15\alpha}{h^4} S_N &= -\frac{360\alpha}{h^4} \Phi_x(x_0) \\ &+ \frac{360\alpha}{h^4} \Phi_x(x_N) - \frac{180\alpha}{h^4} M_0 + \frac{60}{h^2} (M_{i+1} + 4M_i + M_{i-1}) - \frac{180\alpha}{h^4} M_N \\ &- \frac{360}{h^4} (U_{i+1} - 2U_i + U_{i-1}). \end{aligned} \quad (2.3)$$

The conditions $F_{i,3}(x_N, U_{N,3}) = F_{i+1,3}(x_0, U_{0,3})$, $i = 1, 2, \dots, N-1$ can be reformulated as $\Phi_{xxx}(L_i(x_N)) = \Phi_{xxx}(L_{i+1}(x_0))$, $i = 1, 2, \dots, N-1$.

For the continuity of Φ_{xxx} at the interior nodes $x_i, i = 1, 2, \dots, N - 1$, we need that $\Phi_{xxx}(x_i^-) = \Phi_{xxx}(x_i^+)$, i.e., $\Phi_{xxx}(L_i(x_N)) = \Phi_{xxx}(L_{i+1}(x_0)), i = 1, 2, \dots, N - 1$ and this leads to the following:

$$-\frac{3\alpha}{h^4}S_0 + S_{i-1} + 4S_i + S_{i+1} - \frac{3\alpha}{h^4}S_N = \frac{6\alpha}{h^4}\Phi_{xxx}(x_0) - \frac{6\alpha}{h^4}\Phi_{xxx}(x_N) + \frac{6}{h^2}(M_{i-1} - 2M_i + M_{i+1}). \tag{2.4}$$

Multiplying (2.4) by 7 and subtract the resulting equation from (2.3), we get

$$S_i = \frac{\alpha}{2h^4}(S_0 + S_N) + \frac{30\alpha}{h^4}(\Phi_x(x_0) - \Phi_x(x_N)) + \frac{7\alpha}{2h^4}(\Phi_{xxx}(x_0) - \Phi_{xxx}(x_N)) + \frac{15\alpha}{h^4}(M_0 + M_N) - \frac{3}{2h^2}(M_{i-1} + 18M_i + M_{i+1}) + \frac{30}{h^4}(U_{i+1} - 2U_i + U_{i-1}).$$

Substitute S_i in (2.4), we obtain

$$U_{i-2} + 2U_{i-1} - 6U_i + 2U_{i+1} + U_{i+2} = -\frac{\alpha}{2}[\Phi_{xxx}(x_0) - \Phi_{xxx}(x_N)] - 3\alpha(M_0 + M_N) + \frac{h^2}{20}[M_{i-2} + 26M_{i-1} + 66M_i + 26M_{i+1} + M_{i+2}] - 6\alpha[\Phi_x(x_0) - \Phi_x(x_N)]. \tag{2.5}$$

Clearly, the equation (2.5) gives the relation between U_i 's and M_i 's.

2.2. Quasilinearization technique

We use the quasilinearization technique to convert the nonlinear BVP (1.1) into a sequence of linear BVPs. We choose reasonable initial approximation for the function $u(x)$ in $F(x, u(x))$, denote it by $u^{(0)}(x)$, and expand $F(x, u(x))$ around the function $u^{(0)}(x)$ to obtain

$$F(x, u^{(1)}(x)) = F(x, u^{(0)}(x)) + (u^{(1)}(x) - u^{(0)}(x))\left(\frac{\partial F}{\partial u}\right)_{(x, u^{(0)}(x))} + \dots,$$

and in general, we can write for $r = 0, 1, 2, \dots$ (r is the iteration index)

$$F(x, u^{(r+1)}(x)) = F(x, u^{(r)}(x)) + (u^{(r+1)}(x) - u^{(r)}(x))\left(\frac{\partial F}{\partial u}\right)_{(x, u^{(r)}(x))} + \dots$$

The nonlinear BVP (1.1) can be written as

$$\begin{cases} u_{xx}^{(r+1)}(x) + F(x, u^{(r+1)}(x)) = 0, & x \in (0, 1), \\ u^{(r+1)}(0) = \eta_0, & u^{(r+1)}(1) = \eta_1. \end{cases} \tag{2.6}$$

By substituting

$$F(x, u^{(r+1)}(x)) = F(x, u^{(r)}(x)) + (u^{(r+1)}(x) - u^{(r)}(x))\left(\frac{\partial F}{\partial u}\right)_{(x, u^{(r)}(x))}$$

in (2.6), we get

$$\begin{cases} u_{xx}^{(r+1)}(x) + q^{(r)}(x)u^{(r+1)}(x) = f^{(r)}(x), & x \in (0, 1), & r = 0, 1, \dots, \\ u^{(r+1)}(0) = \eta_0, & u^{(r+1)}(1) = \eta_1, \end{cases} \tag{2.7}$$

where

$$q^{(r)}(x) = \left(\frac{\partial F}{\partial u}\right)_{(x, u^{(r)}(x))}, f^{(r)}(x) = u^{(r)}(x)\left(\frac{\partial F}{\partial u}\right)_{(x, u^{(r)}(x))} - F(x, u^{(r)}(x)).$$

Thus, we have reduced the nonlinear BVP (1.1) into a sequence of linear BVPs (2.7). Now, our aim is to solve these linear BVPs numerically.

2.3. Numerical scheme

Let $|\alpha| < h^6$. Let $U_i^{(r)}$ be the approximation of $u^{(r)}(x_i)$ and $M_i^{(r)}$ be the approximation of $u_{xx}^{(r)}(x_i)$. Now, at $x = x_i$, the differential equation (2.7) can be discretized as

$$M_i^{(r+1)} + q_i^{(r)} U_i^{(r+1)} = f_i^{(r)}$$

where

$$q_i^{(r)} = \left(\frac{\partial F}{\partial u} \right)_{(x_i, U_i^{(r)})}, f_i^{(r)} = U_i^{(r)} \left(\frac{\partial F}{\partial u} \right)_{(x_i, U_i^{(r)})} - f(x_i, U_i^{(r)}).$$

Also, the boundary conditions can be discretized as $U_0^{(r+1)} = \eta_0$, $U_N^{(r+1)} = \eta_1$.

Equation (2.5) gives the relation between U_i and M_i and hence it can be used to derive the numerical scheme for the BVP given in (2.7). Substituting

$$\begin{aligned} \Phi_x(x_0) &= \frac{U_1^{(r+1)} - U_0^{(r+1)}}{h}, \quad \Phi_x(x_N) = \frac{U_N^{(r+1)} - U_{N-1}^{(r+1)}}{h}, \\ \Phi_{xxx}(x_0) &= \frac{-U_0^{(r+1)} + 3U_1^{(r+1)} - 3U_2^{(r+1)} + U_3^{(r+1)}}{h^3}, \\ \Phi_{xxx}(x_N) &= \frac{U_N^{(r+1)} - 3U_{N-1}^{(r+1)} + 3U_{N-2}^{(r+1)} - U_{N-3}^{(r+1)}}{h^3}, \\ M_i^{(r+1)} &= f_i^{(r)} - q_i^{(r)} U_i^{(r+1)}, \end{aligned}$$

in (2.5), we get the following system of linear algebraic equations:

$$\begin{aligned} & \left[-\frac{3\alpha}{2h^3} - \frac{6\alpha}{h} \right] U_1^{(r+1)} + \frac{3\alpha}{2h^3} U_2^{(r+1)} - \frac{\alpha}{2h^3} U_3^{(r+1)} + \left[-1 - \frac{h^2}{20} q_{i-2}^{(r)} \right] U_{i-2}^{(r+1)} \\ & + \left[-2 - \frac{13h^2}{10} q_{i-1}^{(r)} \right] U_{i-1}^{(r+1)} + \left[6 - \frac{33h^2}{10} q_i^{(r)} \right] U_i^{(r+1)} \\ & + \left[-2 - \frac{13h^2}{10} q_{i+1}^{(r)} \right] U_{i+1}^{(r+1)} + \left[-1 - \frac{h^2}{20} q_{i+2}^{(r)} \right] U_{i+2}^{(r+1)} - \frac{\alpha}{2h^3} U_{N-3}^{(r+1)} \\ & + \frac{3\alpha}{2h^3} U_{N-2}^{(r+1)} + \left[-\frac{3\alpha}{2h^3} - \frac{6\alpha}{h} \right] U_{N-1}^{(r+1)} = 3\alpha f_0^{(r)} - \frac{h^2}{20} f_{i-2}^{(r)} - \frac{13h^2}{10} f_{i-1}^{(r)} \\ & - \frac{33h^2}{10} f_i^{(r)} - \frac{13h^2}{10} f_{i+1}^{(r)} - \frac{h^2}{20} f_{i+2}^{(r)} + 3\alpha f_N^{(r)} - \left[\frac{\alpha}{2h^3} + 3\alpha q_0^{(r)} + \frac{6\alpha}{h} \right] \eta_0 \\ & - \left[\frac{\alpha}{2h^3} + 3\alpha q_N^{(r)} + \frac{6\alpha}{h} \right] \eta_1, \quad i = 2, 3, \dots, N-2. \end{aligned} \quad (2.8)$$

In the system of equations given in (2.8) we have $(N-3)$ equations with $(N-1)$ unknowns $U_1^{(r+1)}, U_2^{(r+1)}, \dots, U_{N-1}^{(r+1)}$. In order to determine a unique solution, we require two more equations. For that we use the following procedure.

Firstly, we determine the truncation error

$$\begin{aligned} T_1^{(r)}(h) &= \left[\frac{\alpha}{2h^3} + 3\alpha q^{(r)}(x_0) + \frac{6\alpha}{h} \right] u^{(r+1)}(x_0) + \left[-\frac{3\alpha}{2h^3} - \frac{6\alpha}{h} \right] u^{(r+1)}(x_1) \\ & + \frac{3\alpha}{2h^3} u^{(r+1)}(x_2) - \frac{\alpha}{2h^3} u^{(r+1)}(x_3) - \frac{\alpha}{2h^3} u^{(r+1)}(x_{N-3}) + \frac{3\alpha}{2h^3} u^{(r+1)}(x_{N-2}) \\ & + \left[-\frac{3\alpha}{2h^3} - \frac{6\alpha}{h} \right] u^{(r+1)}(x_{N-1}) + \left[\frac{\alpha}{2h^3} + 3\alpha q^{(r)}(x_N) + \frac{6\alpha}{h} \right] u^{(r+1)}(x_N) \\ & - 3\alpha f^{(r)}(x_0) - 3\alpha f^{(r)}(x_N) + \sum_{i=0}^3 \left(c_i u^{(r+1)}(x_i) + d_i h^2 u_{xx}^{(r+1)}(x_i) \right). \end{aligned} \quad (2.9)$$

Substituting $f^{(r)}(x_i) = u_{xx}^{(r+1)}(x_i) + q^{(r)}(x_i)u^{(r+1)}(x_i)$ in (2.9), we get

$$\begin{aligned} T_1^{(r)}(h) &= -\frac{\alpha}{2} \left[\frac{-u^{(r+1)}(x_0) + 3u^{(r+1)}(x_1) - 3u^{(r+1)}(x_2) + u^{(r+1)}(x_3)}{h^3} \right] \\ &+ \frac{\alpha}{2} \left[\frac{u^{(r+1)}(x_N) - 3u^{(r+1)}(x_{N-1}) + 3u^{(r+1)}(x_{N-2}) - u^{(r+1)}(x_{N-3})}{h^3} \right] \\ &- 6\alpha \left[\frac{-u^{(r+1)}(x_0) + u^{(r+1)}(x_1)}{h} \right] + 6\alpha \left[\frac{u^{(r+1)}(x_N) - u^{(r+1)}(x_{N-1})}{h} \right] \\ &- 3\alpha u_{xx}^{(r+1)}(x_0) - 3\alpha u_{xx}^{(r+1)}(x_N) + \sum_{i=0}^3 \left(c_i u^{(r+1)}(x_i) + d_i h^2 u_{xx}^{(r+1)}(x_i) \right). \end{aligned}$$

It can be seen that

$$\begin{aligned} \frac{-u^{(r+1)}(x_0) + 3u^{(r+1)}(x_1) - 3u^{(r+1)}(x_2) + u^{(r+1)}(x_3)}{h^3} &= u_{xxx}^{(r+1)}(x_0) + \mathcal{O}(h), \\ \frac{-u^{(r+1)}(x_0) + u^{(r+1)}(x_1)}{h} &= u_x^{(r+1)}(x_0) + \mathcal{O}(h), \\ \frac{u^{(r+1)}(x_N) - 3u^{(r+1)}(x_{N-1}) + 3u^{(r+1)}(x_{N-2}) - u^{(r+1)}(x_{N-3})}{h^3} &= u_{xxx}^{(r+1)}(x_N) + \mathcal{O}(h), \\ \frac{u^{(r+1)}(x_N) - u^{(r+1)}(x_{N-1})}{h} &= u_x^{(r+1)}(x_N) + \mathcal{O}(h). \end{aligned}$$

Hence, we have

$$\begin{aligned} T_1^{(r)}(h) &= -\frac{\alpha}{2} \left[u_{xxx}^{(r+1)}(x_0) + \mathcal{O}(h) \right] + \frac{\alpha}{2} \left[u_{xxx}^{(r+1)}(x_N) + \mathcal{O}(h) \right] \\ &- 6\alpha \left[u_x^{(r+1)}(x_0) + \mathcal{O}(h) \right] + 6\alpha \left[u_x^{(r+1)}(x_N) + \mathcal{O}(h) \right] \\ &- 3\alpha u_{xx}^{(r+1)}(x_0) - 3\alpha u_{xx}^{(r+1)}(x_N) + \sum_{i=0}^3 \left(c_i u^{(r+1)}(x_i) + d_i h^2 u_{xx}^{(r+1)}(x_i) \right). \end{aligned}$$

Our aim is to get $T_1^{(r)}(h) = \mathcal{O}(h^6)$. We write

$$T_1^{(r)}(h) = T_\alpha^{(r)}(h) + T_*^{(r)}(h),$$

where

$$\begin{aligned} T_\alpha^{(r)}(h) &= -\frac{\alpha}{2} \left[u_{xxx}^{(r+1)}(x_0) + \mathcal{O}(h) \right] + \frac{\alpha}{2} \left[u_{xxx}^{(r+1)}(x_N) + \mathcal{O}(h) \right] \\ &- 6\alpha \left[u_x^{(r+1)}(x_0) + \mathcal{O}(h) \right] + 6\alpha \left[u_x^{(r+1)}(x_N) + \mathcal{O}(h) \right] \\ &- 3\alpha u_{xx}^{(r+1)}(x_0) - 3\alpha u_{xx}^{(r+1)}(x_N), \end{aligned}$$

and

$$T_*^{(r)}(h) = \sum_{i=0}^3 \left(c_i u^{(r+1)}(x_i) + d_i h^2 u_{xx}^{(r+1)}(x_i) \right).$$

Since $|\alpha| < h^6$, we have $T_\alpha^{(r)}(h) = \mathcal{O}(h^6)$. Next, our aim is to choose $c_0, c_1, c_2, c_3, d_0, d_1, d_2$, and d_3 such that $T_*^{(r)}(h) = \mathcal{O}(h^6)$. By using the Taylor series expansions for $u^{(r+1)}(x_0), u^{(r+1)}(x_2), u^{(r+1)}(x_3), u_{xx}^{(r+1)}(x_0), u_{xx}^{(r+1)}(x_2)$, and $u_{xx}^{(r+1)}(x_3)$ about the point $x = x_1$, we get

$$\begin{aligned}
T_*^{(r)}(h) &= [c_0 + c_1 + c_2 + c_3]u^{(r+1)}(x_1) + [-c_0 + c_2 + 2c_3]h u_x^{(r+1)}(x_1) \\
&+ \left[\frac{1}{2!}c_0 + \frac{1}{2!}c_2 + \frac{2^2}{2!}c_3 + d_0 + d_1 + d_2 + d_3\right]h^2 u_{xx}^{(r+1)}(x_1) \\
&+ \left[-\frac{1}{3!}c_0 + \frac{1}{3!}c_2 + \frac{2^3}{3!}c_3 - d_0 + d_2 + 2d_3\right]h^3 u_{xxx}^{(r+1)}(x_1) \\
&+ \left[\frac{1}{4!}c_0 + \frac{1}{4!}c_2 + \frac{2^4}{4!}c_3 + \frac{1}{2!}d_0 + \frac{1}{2!}d_2 + \frac{2^2}{2!}d_3\right]h^4 u_{xxxx}^{(r+1)}(x_1) \\
&+ \left[-\frac{1}{5!}c_0 + \frac{1}{5!}c_2 + \frac{2^5}{5!}c_3 - \frac{1}{3!}d_0 + \frac{1}{3!}d_2 + \frac{2^3}{3!}d_3\right]h^5 u_{xxxxx}^{(r+1)}(x_1) \\
&+ \left[\frac{1}{6!}c_0 + \frac{1}{6!}c_2 + \frac{2^6}{6!}c_3 + \frac{1}{4!}d_0 + \frac{1}{4!}d_2 + \frac{2^4}{4!}d_3\right]h^6 u_{xxxxxx}^{(r+1)}(x_1) \\
&+ \mathcal{O}(h^7).
\end{aligned}$$

Hence to get $T_*^{(r)}(h) = \mathcal{O}(h^6)$, we need

$$\begin{aligned}
c_0 + c_1 + c_2 + c_3 &= 0, & -c_0 + c_2 + 2c_3 &= 0, \\
\frac{1}{2!}c_0 + \frac{1}{2!}c_2 + \frac{2^2}{2!}c_3 + d_0 + d_1 + d_2 + d_3 &= 0, \\
-\frac{1}{3!}c_0 + \frac{1}{3!}c_2 + \frac{2^3}{3!}c_3 - d_0 + d_2 + 2d_3 &= 0, \\
\frac{1}{4!}c_0 + \frac{1}{4!}c_2 + \frac{2^4}{4!}c_3 + \frac{1}{2!}d_0 + \frac{1}{2!}d_2 + \frac{2^2}{2!}d_3 &= 0, \\
-\frac{1}{5!}c_0 + \frac{1}{5!}c_2 + \frac{2^5}{5!}c_3 - \frac{1}{3!}d_0 + \frac{1}{3!}d_2 + \frac{2^3}{3!}d_3 &= 0.
\end{aligned}$$

Here, we have 6 equations with 8 unknowns $c_0, c_1, c_2, c_3, d_0, d_1, d_2$, and d_3 . Let us fix two unknowns c_1 and c_2 as $c_1 = 7$ and $c_2 = -2$. Hence we can solve the system of equations and obtain $c_0 = -4, c_3 = -1, d_0 = \frac{1}{3}, d_1 = \frac{41}{12}, d_2 = \frac{7}{6}$, and $d_3 = \frac{1}{12}$. The constants $c_1 = 7$ and $c_2 = -2$ are carefully chosen so that the matrix $B^{(r)}$ (given below) would be a monotone matrix.

It can be seen that, by substituting $c_0 = -4, c_1 = 7, c_2 = -2, c_3 = -1, d_0 = \frac{1}{3}, d_1 = \frac{41}{12}, d_2 = \frac{7}{6}$, and $d_3 = \frac{1}{12}$, we get $T_*^{(r)}(h) = \frac{h^6}{48}u_{xxxxxx}^{(r+1)}(x_1) + \mathcal{O}(h^7)$. Hence we have $T_*^{(r)}(h) = \mathcal{O}(h^6)$. Thus, by the assumption $|\alpha| < h^6, c_0 = -4, c_1 = 7, c_2 = -2, c_3 = -1, d_0 = \frac{1}{3}, d_1 = \frac{41}{12}, d_2 = \frac{7}{6}$, and $d_3 = \frac{1}{12}$, we get

$$\begin{aligned}
T_1^{(r)}(h) &= \left[\frac{\alpha}{2h^3} + 3\alpha q^{(r)}(x_0) + \frac{6\alpha}{h}\right]u^{(r+1)}(x_0) + \left[-\frac{3\alpha}{2h^3} - \frac{6\alpha}{h}\right]u^{(r+1)}(x_1) \\
&+ \frac{3\alpha}{2h^3}u^{(r+1)}(x_2) - \frac{\alpha}{2h^3}u^{(r+1)}(x_3) - \frac{\alpha}{2h^3}u^{(r+1)}(x_{N-3}) + \frac{3\alpha}{2h^3}u^{(r+1)}(x_{N-2}) \\
&+ \left[-\frac{3\alpha}{2h^3} - \frac{6\alpha}{h}\right]u^{(r+1)}(x_{N-1}) + \left[\frac{\alpha}{2h^3} + 3\alpha q^{(r)}(x_N) + \frac{6\alpha}{h}\right]u^{(r+1)}(x_N) \\
&- 3\alpha f^{(r)}(x_0) - 3\alpha f^{(r)}(x_N) - 4u^{(r+1)}(x_0) + 7u^{(r+1)}(x_1) \\
&- 2u^{(r+1)}(x_2) - u^{(r+1)}(x_3) + \frac{1}{3}h^2u_{xx}^{(r+1)}(x_0) + \frac{41}{12}h^2u_{xx}^{(r+1)}(x_1) \\
&+ \frac{7}{6}h^2u_{xx}^{(r+1)}(x_2) + \frac{1}{12}h^2u_{xx}^{(r+1)}(x_3). \tag{2.10}
\end{aligned}$$

When $h \rightarrow 0$, we have $|T_1^{(r)}(h)| \leq K_1^{(r)}h^6$, where $K_1^{(r)}$ is a constant. Thus $T_1^{(r)}(h) = \mathcal{O}(h^6)$. Hence from (2.10), replacing the original values $u^{(r)}(x_i)$ and $u_{xx}^{(r)}(x_i)$ by the approximate

values $U_i^{(r)}$ and $M_i^{(r)}$, respectively, we get the difference equation (associated with the truncation error $T_1^{(r)}(h)$) as

$$\begin{aligned}
 & \left[7 - \frac{41h^2}{12}q_1^{(r)} - \frac{3\alpha}{2h^3} - \frac{6\alpha}{h}\right]U_1^{(r+1)} + \left[-2 - \frac{7h^2}{6}q_2^{(r)} + \frac{3\alpha}{2h^3}\right]U_2^{(r+1)} \\
 & + \left[-1 - \frac{h^2}{12}q_3^{(r)} - \frac{\alpha}{2h^3}\right]U_3^{(r+1)} - \frac{\alpha}{2h^3}U_{N-3}^{(r+1)} + \frac{3\alpha}{2h^3}U_{N-2}^{(r+1)} \\
 & + \left[-\frac{3\alpha}{2h^3} - \frac{6\alpha}{h}\right]U_{N-1}^{(r+1)} = 3\alpha f_0^{(r)} - \frac{h^2}{3}f_0^{(r)} - \frac{41h^2}{12}f_1^{(r)} - \frac{7h^2}{6}f_2^{(r)} \\
 & - \frac{h^2}{12}f_3^{(r)} + 3\alpha f_N^{(r)} - \left[-4 - \frac{h^2}{3}f_0^{(r)} + \frac{\alpha}{2h^3} + 3\alpha q_0^{(r)} + \frac{6\alpha}{h}\right]\eta_0 \\
 & - \left[\frac{\alpha}{2h^3} + 3\alpha q_N^{(r)} + \frac{6\alpha}{h}\right]\eta_1.
 \end{aligned} \tag{2.11}$$

Similarly, we can define the truncation error

$$\begin{aligned}
 T_{N-1}^{(r)}(h) &= \left[\frac{\alpha}{2h^3} + 3\alpha q^{(r)}(x_0) + \frac{6\alpha}{h}\right]u^{(r+1)}(x_0) + \left[-\frac{3\alpha}{2h^3} - \frac{6\alpha}{h}\right]u^{(r+1)}(x_1) \\
 &+ \frac{3\alpha}{2h^3}u^{(r+1)}(x_2) - \frac{\alpha}{2h^3}u^{(r+1)}(x_3) - \frac{\alpha}{2h^3}u^{(r+1)}(x_{N-3}) \\
 &+ \frac{3\alpha}{2h^3}u^{(r+1)}(x_{N-2}) + \left[-\frac{3\alpha}{2h^3} - \frac{6\alpha}{h}\right]u^{(r+1)}(x_{N-1}) + \left[\frac{\alpha}{2h^3} \right. \\
 &+ 3\alpha q^{(r)}(x_N) + \left. \frac{6\alpha}{h}\right]u^{(r+1)}(x_N) - 3\alpha f^{(r)}(x_0) - 3\alpha f^{(r)}(x_N) \\
 &+ \sum_{i=N-3}^N \left(c_i u^{(r+1)}(x_i) + d_i h^2 u_{xx}^{(r+1)}(x_i)\right).
 \end{aligned}$$

By the assumption $|\alpha| < h^6$, $c_N = -4$, $c_{N-1} = 7$, $c_{N-2} = -2$, $c_{N-3} = -1$, $d_N = \frac{1}{3}$, $d_{N-1} = \frac{41}{12}$, $d_{N-2} = \frac{7}{6}$, and $d_{N-3} = \frac{1}{12}$, we get $T_{N-1}^{(r)}(h) = \mathcal{O}(h^6)$, that is, when $h \rightarrow 0$, we have $|T_{N-1}^{(r)}(h)| \leq K_{N-1}^{(r)}h^6$, where $K_{N-1}^{(r)}$ is a constant.

Hence, we get the difference equation (associated with $T_{N-1}^{(r)}(h)$) as follows:

$$\begin{aligned}
 & \left[-\frac{3\alpha}{2h^3} - \frac{6\alpha}{h}\right]U_1^{(r+1)} + \frac{3\alpha}{2h^3}U_2^{(r+1)} - \frac{\alpha}{2h^3}U_3^{(r+1)} + \left[-1 - \frac{h^2}{12}q_{N-3}^{(r)} \right. \\
 & - \left. \frac{\alpha}{2h^3}\right]U_{N-3}^{(r+1)} + \left[-2 - \frac{7h^2}{6}q_{N-2}^{(r)} + \frac{3\alpha}{2h^3}\right]U_{N-2}^{(r+1)} + \left[7 - \frac{41h^2}{12}q_{N-1}^{(r)} \right. \\
 & - \left. \frac{3\alpha}{2h^3} - \frac{6\alpha}{h}\right]U_{N-1}^{(r+1)} = 3\alpha f_0^{(r)} - \frac{h^2}{3}f_N^{(r)} - \frac{41h^2}{12}f_{N-1}^{(r)} - \frac{7h^2}{6}f_{N-2}^{(r)} \\
 & - \frac{h^2}{12}f_{N-3}^{(r)} + 3\alpha f_N^{(r)} - \left[\frac{\alpha}{2h^3} + 3\alpha q_0^{(r)} + \frac{6\alpha}{h}\right]\eta_0 \\
 & - \left[-4 - \frac{h^2}{3}f_N^{(r)} + \frac{\alpha}{2h^3} + 3\alpha q_N^{(r)} + \frac{6\alpha}{h}\right]\eta_1.
 \end{aligned} \tag{2.12}$$

The system given in (2.11), (2.8), and (2.12) gives the approximations $U_1^{(r+1)}, U_2^{(r+1)}, \dots, U_{N-1}^{(r+1)}$.

Remark 2.1. The system corresponding to the quintic spline can be obtained by taking $\alpha = 0$ in (2.11), (2.8), and (2.12).

3. Convergence analysis

In this section, first, we derive the truncation error corresponding to the system of equations given in (2.11), (2.8), and (2.12). Then, we obtain the error bound of the proposed numerical method.

3.1. Truncation error

We define the truncation error $T_i^{(r)}(h)$, $i = 2, 3, \dots, N-2$, associated with the equations given in (2.8) as

$$\begin{aligned}
 T_i^{(r)}(h) = & \left[\frac{\alpha}{2h^3} + 3\alpha q^{(r)}(x_0) + \frac{6\alpha}{h} \right] u^{(r+1)}(x_0) + \left[-\frac{3\alpha}{2h^3} - \frac{6\alpha}{h} \right] u^{(r+1)}(x_1) \\
 & + \frac{3\alpha}{2h^3} u^{(r+1)}(x_2) - \frac{\alpha}{2h^3} u^{(r+1)}(x_3) + \left[-1 - \frac{h^2}{20} q^{(r)}(x_{i-2}) \right] u^{(r+1)}(x_{i-2}) \\
 & + \left[-2 - \frac{13h^2}{10} q^{(r)}(x_{i-1}) \right] u^{(r+1)}(x_{i-1}) + \left[6 - \frac{33h^2}{10} q^{(r)}(x_i) \right] u^{(r+1)}(x_i) \\
 & + \left[-2 - \frac{13h^2}{10} q^{(r)}(x_{i+1}) \right] u^{(r+1)}(x_{i+1}) + \left[-\frac{h^2}{20} q^{(r)}(x_{i+2}) \right. \\
 & \left. - 1 \right] u^{(r+1)}(x_{i+2}) - \frac{\alpha}{2h^3} u^{(r+1)}(x_{N-3}) + \frac{3\alpha}{2h^3} u^{(r+1)}(x_{N-2}) \\
 & + \left[-\frac{3\alpha}{2h^3} - \frac{6\alpha}{h} \right] u^{(r+1)}(x_{N-1}) + \left[\frac{\alpha}{2h^3} + 3\alpha q^{(r)}(x_N) + \frac{6\alpha}{h} \right] u^{(r+1)}(x_N) \\
 & - 3\alpha f^{(r)}(x_0) + \frac{h^2}{20} f^{(r)}(x_{i-2}) + \frac{13h^2}{10} f^{(r)}(x_{i-1}) + \frac{33h^2}{10} f^{(r)}(x_i) \\
 & + \frac{13h^2}{10} f^{(r)}(x_{i+1}) + \frac{h^2}{20} f^{(r)}(x_{i+2}) - 3\alpha f^{(r)}(x_N). \tag{3.1}
 \end{aligned}$$

Substituting $f^{(r)}(x_i) = u_{xx}^{(r+1)}(x_i) + q^{(r)}(x_i)u^{(r+1)}(x_i)$ in (3.1), we get

$$\begin{aligned}
 T_i^{(r)}(h) = & -\frac{\alpha}{2} \left[\frac{-u^{(r+1)}(x_0) + 3u^{(r+1)}(x_1) - 3u^{(r+1)}(x_2) + u^{(r+1)}(x_3)}{h^3} \right] \\
 & + \frac{\alpha}{2} \left[\frac{u^{(r+1)}(x_N) - 3u^{(r+1)}(x_{N-1}) + 3u^{(r+1)}(x_{N-2}) - u^{(r+1)}(x_{N-3})}{h^3} \right] \\
 & - 6\alpha \left[\frac{-u^{(r+1)}(x_0) + u^{(r+1)}(x_1)}{h} \right] + 6\alpha \left[\frac{u^{(r+1)}(x_N) - u^{(r+1)}(x_{N-1})}{h} \right] \\
 & - 3\alpha u_{xx}^{(r+1)}(x_0) - 3\alpha u_{xx}^{(r+1)}(x_N) - u^{(r+1)}(x_{i-2}) - 2u^{(r+1)}(x_{i-1}) \\
 & + 6u^{(r+1)}(x_i) - 2u^{(r+1)}(x_{i+1}) - u^{(r+1)}(x_{i+2}) + \frac{h^2}{20} u_{xx}^{(r+1)}(x_{i-2}) \\
 & + \frac{13h^2}{10} u_{xx}^{(r+1)}(x_{i-1}) + \frac{33h^2}{10} u_{xx}^{(r+1)}(x_i) + \frac{13h^2}{10} u_{xx}^{(r+1)}(x_{i+1}) \\
 & + \frac{h^2}{20} u_{xx}^{(r+1)}(x_{i+2}).
 \end{aligned}$$

After further simplification, we obtain that

$$\begin{aligned}
 T_i^{(r)}(h) = & -\frac{\alpha}{2} \left[u_{xxx}^{(r+1)}(x_0) + \mathcal{O}(h) \right] + \frac{\alpha}{2} \left[u_{xxx}^{(r+1)}(x_N) + \mathcal{O}(h) \right] \\
 & - 6\alpha \left[u_x^{(r+1)}(x_0) + \mathcal{O}(h) \right] + 6\alpha \left[u_x^{(r+1)}(x_N) + \mathcal{O}(h) \right] \\
 & - 3\alpha u_{xx}^{(r+1)}(x_0) - 3\alpha u_{xx}^{(r+1)}(x_N) - \frac{h^6}{120} u_{xxxxxx}^{(r+1)}(x_i) + \mathcal{O}(h^7),
 \end{aligned}$$

$i = 2, 3, \dots, N - 2$. When $h \rightarrow 0$, we have $|T_i^{(r)}(h)| \leq K_i^{(r)}h^6$ where $K_i^{(r)}$ is a constant. It can be seen that $T_i^{(r)}(h) = \mathcal{O}(h^6)$, $i = 2, 3, \dots, N - 2$.

Remark 3.1. The system given in (2.11), (2.8), and (2.12) can be written in the form $A^{(r)}U^{(r+1)} = d^{(r)}$, where

$$A^{(r)} = \begin{pmatrix} A_{1,1}^{(r)} & A_{1,2}^{(r)} & A_{1,3}^{(r)} & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & A_{1,N-3}^{(r)} & A_{1,N-2}^{(r)} & A_{1,N-1}^{(r)} \\ A_{2,1}^{(r)} & A_{2,2}^{(r)} & A_{2,3}^{(r)} & A_{2,4}^{(r)} & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & A_{2,N-3}^{(r)} & A_{2,N-2}^{(r)} & A_{2,N-1}^{(r)} \\ A_{3,1}^{(r)} & A_{3,2}^{(r)} & A_{3,3}^{(r)} & A_{3,4}^{(r)} & A_{3,5}^{(r)} & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & A_{3,N-3}^{(r)} & A_{3,N-2}^{(r)} & A_{3,N-1}^{(r)} \\ A_{4,1}^{(r)} & A_{4,2}^{(r)} & A_{4,3}^{(r)} & A_{4,4}^{(r)} & A_{4,5}^{(r)} & A_{4,6}^{(r)} & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & A_{4,N-3}^{(r)} & A_{4,N-2}^{(r)} & A_{4,N-1}^{(r)} \\ A_{5,1}^{(r)} & A_{5,2}^{(r)} & A_{5,3}^{(r)} & A_{5,4}^{(r)} & A_{5,5}^{(r)} & A_{5,6}^{(r)} & A_{5,7}^{(r)} & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & A_{5,N-3}^{(r)} & A_{5,N-2}^{(r)} & A_{5,N-1}^{(r)} \\ A_{6,1}^{(r)} & A_{6,2}^{(r)} & A_{6,3}^{(r)} & A_{6,4}^{(r)} & A_{6,5}^{(r)} & A_{6,6}^{(r)} & A_{6,7}^{(r)} & A_{6,8}^{(r)} & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & A_{6,N-3}^{(r)} & A_{6,N-2}^{(r)} & A_{6,N-1}^{(r)} \\ A_{7,1}^{(r)} & A_{7,2}^{(r)} & A_{7,3}^{(r)} & 0 & A_{7,5}^{(r)} & A_{7,6}^{(r)} & A_{7,7}^{(r)} & A_{7,8}^{(r)} & A_{7,9}^{(r)} & 0 & \dots & 0 & 0 & 0 & 0 & 0 & A_{7,N-3}^{(r)} & A_{7,N-2}^{(r)} & A_{7,N-1}^{(r)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{N-7,1}^{(r)} & A_{N-7,2}^{(r)} & A_{N-7,3}^{(r)} & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & A_{N-7,N-9}^{(r)} & A_{N-7,N-8}^{(r)} & A_{N-7,N-7}^{(r)} & A_{N-7,N-6}^{(r)} & A_{N-7,N-5}^{(r)} & 0 & A_{N-7,N-3}^{(r)} & A_{N-7,N-2}^{(r)} & A_{N-7,N-1}^{(r)} \\ A_{N-6,1}^{(r)} & A_{N-6,2}^{(r)} & A_{N-6,3}^{(r)} & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & A_{N-6,N-8}^{(r)} & A_{N-6,N-7}^{(r)} & A_{N-6,N-6}^{(r)} & A_{N-6,N-5}^{(r)} & A_{N-6,N-4}^{(r)} & A_{N-6,N-3}^{(r)} & A_{N-6,N-2}^{(r)} & A_{N-6,N-1}^{(r)} \\ A_{N-5,1}^{(r)} & A_{N-5,2}^{(r)} & A_{N-5,3}^{(r)} & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & A_{N-5,N-7}^{(r)} & A_{N-5,N-6}^{(r)} & A_{N-5,N-5}^{(r)} & A_{N-5,N-4}^{(r)} & A_{N-5,N-3}^{(r)} & A_{N-5,N-2}^{(r)} & A_{N-5,N-1}^{(r)} \\ A_{N-4,1}^{(r)} & A_{N-4,2}^{(r)} & A_{N-4,3}^{(r)} & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & A_{N-4,N-6}^{(r)} & A_{N-4,N-5}^{(r)} & A_{N-4,N-4}^{(r)} & A_{N-4,N-3}^{(r)} & A_{N-4,N-2}^{(r)} & A_{N-4,N-1}^{(r)} \\ A_{N-3,1}^{(r)} & A_{N-3,2}^{(r)} & A_{N-3,3}^{(r)} & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & A_{N-3,N-5}^{(r)} & A_{N-3,N-4}^{(r)} & A_{N-3,N-3}^{(r)} & A_{N-3,N-2}^{(r)} & A_{N-3,N-1}^{(r)} \\ A_{N-2,1}^{(r)} & A_{N-2,2}^{(r)} & A_{N-2,3}^{(r)} & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & A_{N-2,N-4}^{(r)} & A_{N-2,N-3}^{(r)} & A_{N-2,N-2}^{(r)} & A_{N-2,N-1}^{(r)} \\ A_{N-1,1}^{(r)} & A_{N-1,2}^{(r)} & A_{N-1,3}^{(r)} & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & A_{N-1,N-3}^{(r)} & A_{N-1,N-2}^{(r)} & A_{N-1,N-1}^{(r)} \end{pmatrix}$$

$A_{i,j}^{(r)}$ is the coefficient of $U_j^{(r+1)}$, $U^{(r+1)} = (U_1^{(r+1)}, U_2^{(r+1)}, \dots, U_{N-1}^{(r+1)})^T$, $d^{(r)} = (d_1^{(r)}, d_2^{(r)}, \dots, d_{N-1}^{(r)})^T$, $d_i^{(r)}$ is the right-hand side of the system.

Let $B^{(r)}$ be the matrix corresponding to the quintic spline system given in Remark 2.1.

Note that, $\|A^{(r)} - B^{(r)}\|_\infty = \max_i \sum_{j=1}^{N-1} |A_{i,j}^{(r)} - B_{i,j}^{(r)}|$. Thus, we get

$$\|A^{(r)} - B^{(r)}\|_\infty = 2 \left| -\frac{3\alpha}{2h^3} - \frac{6\alpha}{h} \right| + 2 \left| \frac{3\alpha}{2h^3} \right| + 2 \left| -\frac{\alpha}{2h^3} \right|.$$

Theorem 3.2 ([6]). *Let Q_1 and Q_2 be any two $n \times n$ matrices, and let $\|\cdot\|$ be any matrix norm. Then the eigenvalues of Q_1 and Q_2 can be enumerated as $\lambda_1, \lambda_2, \dots, \lambda_n$ and $\mu_1, \mu_2, \dots, \mu_n$ in such a way that*

$$\max_i |\lambda_i - \mu_i| \leq 2^{\frac{2n-1}{n}} n^{\frac{1}{n}} (2P)^{\frac{n-1}{n}} \|Q_1 - Q_2\|^{\frac{1}{n}}, \tag{3.2}$$

where $P = \max\{\|Q_1\|, \|Q_2\|\}$.

In our case, we take the matrices $A^{(r)} = Q_1$, $B^{(r)} = Q_2$, $n = N - 1$, $\|\cdot\|_\infty$ norm in the previous theorem, then from (3.2), we get

$$\max_i |\lambda_i^{(r)} - \mu_i^{(r)}| \leq 2^{\frac{2N-3}{N-1}} (N - 1)^{\frac{1}{N-1}} (2P^{(r)})^{\frac{N-2}{N-1}} \|A^{(r)} - B^{(r)}\|_\infty^{\frac{1}{N-1}}, \tag{3.3}$$

where $P^{(r)} = \max\{\|A^{(r)}\|_\infty, \|B^{(r)}\|_\infty\}$, $\lambda_i^{(r)}$ and $\mu_i^{(r)}$, $i = 1, 2, \dots, N - 1$ are the eigenvalues of $A^{(r)}$ and $B^{(r)}$ respectively.

Note that, when h is sufficiently small, $B^{(r)}$ becomes irreducible, $B_{i,i}^{(r)} > 0$, $B_{i,j}^{(r)} \leq 0$, $i \neq j$, $i, j = 1, 2, \dots, N - 1$, and the row sums give

$$\begin{aligned} \mathcal{R}_1^{(r)} &= 4 - \frac{41h^2}{12}q_1^{(r)} - \frac{7h^2}{6}q_2^{(r)} - \frac{h^2}{12}q_3^{(r)} > 0, \\ \mathcal{R}_2^{(r)} &= 1 - \frac{13h^2}{10}q_1^{(r)} - \frac{33h^2}{10}q_2^{(r)} - \frac{13h^2}{10}q_3^{(r)} - \frac{h^2}{20}q_4^{(r)} > 0, \\ \mathcal{R}_i^{(r)} &= -\frac{h^2}{20}q_{i-2}^{(r)} - \frac{13h^2}{10}q_{i-1}^{(r)} - \frac{33h^2}{10}q_i^{(r)} - \frac{13h^2}{10}q_{i+1}^{(r)} - \frac{h^2}{20}q_{i+2}^{(r)} > 0, \\ & i = 3, 4, \dots, N - 3, \\ \mathcal{R}_{N-2}^{(r)} &= 1 - \frac{13h^2}{10}q_{N-1}^{(r)} - \frac{33h^2}{10}q_{N-2}^{(r)} - \frac{13h^2}{10}q_{N-3}^{(r)} - \frac{h^2}{20}q_{N-4}^{(r)} > 0, \\ \mathcal{R}_{N-1}^{(r)} &= 4 - \frac{41h^2}{12}q_{N-1}^{(r)} - \frac{7h^2}{6}q_{N-2}^{(r)} - \frac{h^2}{12}q_{N-3}^{(r)} > 0. \end{aligned}$$

Hence $B^{(r)}$ is a monotone matrix [18]. Thus, $B^{(r)-1}$ exist and the eigenvalues $\mu_i^{(r)}$, $i = 1, 2, \dots, N - 1$ of $B^{(r)}$ are non-zero when h is sufficiently small. Thus, once we take sufficiently small h (which makes $B^{(r)}$ is a monotone matrix), then α can vary in the region $(-h^6, h^6)$. Now we can choose α to satisfy the following conditions:

- $A^{(r)}$ is invertible because $\|A^{(r)} - B^{(r)}\|_\infty = 2 \left| -\frac{3\alpha}{2h^3} - \frac{6\alpha}{h} \right| + 2 \left| \frac{3\alpha}{2h^3} \right| + 2 \left| -\frac{\alpha}{2h^3} \right|$ and from (3.3), it can be seen that, the eigenvalues of $A^{(r)}$ are non-zero when α is sufficiently small.
- Since $\mathcal{R}_i^{(r)} > 0$, $i = 1, 2, \dots, N - 1$, the row sums of $A^{(r)}$,
 $\mathcal{S}_i^{(r)} = \mathcal{R}_i^{(r)} - \frac{12\alpha}{h} - \frac{\alpha}{h^3} > 0$, $i = 1, 2, \dots, N - 1$,
 when α is sufficiently small.

For sufficiently small h (which makes $B^{(r)}$ is monotone) and for sufficiently small $\alpha \in (-h^6, h^6)$ (which makes $A^{(r)}$ is invertible and row sums of $A^{(r)}$ is positive), we obtain the following error bound.

3.2. Error bound

The system (2.11), (2.8), and (2.12) with exact solutions can be written as

$$A^{(r)}\bar{u}^{(r+1)} = d^{(r)} + T^{(r)}(h)$$

where

$$\bar{u}^{(r+1)} = (u^{(r+1)}(x_1), u^{(r+1)}(x_2), \dots, u^{(r+1)}(x_{N-1}))^T$$

and

$$T^{(r)}(h) = (T_1^{(r)}(h), T_2^{(r)}(h), \dots, T_{N-1}^{(r)}(h))^T.$$

Since

$$A^{(r)}U^{(r+1)} = d^{(r)},$$

we get

$$A^{(r)}(\bar{u}^{(r+1)} - U^{(r+1)}) = T^{(r)}(h),$$

that is,

$$A^{(r)}E^{(r+1)} = T^{(r)}(h),$$

where $E^{(r+1)} = (E_1^{(r+1)}, E_2^{(r+1)}, \dots, E_{N-1}^{(r+1)})$, $E_i^{(r+1)} = u^{(r+1)}(x_i) - U_i^{(r+1)}$. Consequently, we obtain that

$$E^{(r+1)} = A^{(r)-1}T^{(r)}(h).$$

From the definition of multiplication of a matrix by its inverse, we have

$$\sum_{i=1}^{N-1} A^{(r)-1}_{j,i} \mathcal{S}_i^{(r)} = 1, \quad j = 1, 2, \dots, N - 1,$$

where $A^{(r)-1}_{j,i}$ is the (j, i) -th element of the matrix $A^{(r)-1}$. Hence, we have

$$\sum_{i=1}^{N-1} A^{(r)-1}_{j,i} \leq \frac{1}{\min_{1 \leq i \leq N-1} \mathcal{S}_i^{(r)}} = \frac{1}{C_{i_0}^{(r)} h^2},$$

where $C_{i_0}^{(r)}$ is a constant. We can write

$$E_j^{(r+1)} = \sum_{i=1}^{N-1} A^{(r)-1}_{j,i} T_i^{(r)}(h), \quad j = 1, 2, \dots, N - 1,$$

and hence $|E_j^{(r+1)}| \leq \frac{K^{(r)}h^6}{C_{i_0}^{(r)}h^2}$, where $K^{(r)}$ is a constant. Therefore, we have

$$\|E^{(r+1)}\|_\infty = \mathcal{O}(h^4),$$

which shows that the proposed numerical scheme is of fourth-order convergent for linear BVPs.

4. Numerical experiments

For sufficiently small h (such that $B^{(r)}$ is invertible), and by taking α (which makes $A^{(r)}$ invertible and row sums of $A^{(r)}$ positive) that satisfies $|\alpha| < h^6$, we can compute the numerical solutions of BVPs.

The numerical solutions of the nonlinear boundary-value problems are computed as follows: For each fixed N , by taking the initial approximation $U_0^{(0)}, U_1^{(0)}, \dots, U_N^{(0)}$, we can compute the numerical solutions $U^{(r+1)} = (U_1^{(r+1)}, U_2^{(r+1)}, \dots, U_{N-1}^{(r+1)})$, $r = 0, 1, \dots$ of the discretized systems (2.11), (2.8), and (2.12).

We take sufficient number of iterations so that the maximum error between the two successive iterations satisfy the following tolerance bound:

$$\max_i |U_i^{(r+1)} - U_i^{(r)}| < \text{TOL}. \quad (4.1)$$

Once the criterion is satisfied, we consider $U^{(r+1)}$ is the numerical solution U of the nonlinear BVP. To compute the numerical solutions for each of the following nonlinear BVPs, we take $\text{TOL} = 10^{-15}$. For any fixed-value of N , the maximum point-wise error E^N will be calculated by

$$E^N = \max_i |u(x_i) - U_i|,$$

where $u(x_i)$ and U_i are respectively, the exact and numerical solutions of the continuous and discrete problems at $x = x_i$. Further, the order of convergence will be calculated by

$$p^N = \log_2 \left(\frac{E^N}{E^{2N}} \right).$$

From the differential equation given in (1.1), we get the approximation for $u_{xx}(x_i)$ as $M_i = f(x, U_i)$. Differentiating (1.1), we get

$$\begin{aligned} u_{xxxx}(x) = & -\frac{\partial^2 f(x, u(x))}{\partial x^2} - 2\frac{\partial^2 f(x, u(x))}{\partial x \partial u} u_x(x) \\ & - \frac{\partial^2 f(x, u(x))}{\partial u^2} (u_x(x))^2 - \frac{\partial f(x, u(x))}{\partial u} u_{xx}(x) \end{aligned}$$

and hence we get approximation of $u_{xxxx}(x_i)$ as

$$S_i = -\frac{\partial^2 f(x_i, U_i)}{\partial x^2} - 2\frac{\partial^2 f(x_i, U_i)}{\partial x \partial u} U_i' - \frac{\partial^2 f(x_i, U_i)}{\partial u^2} (U_i')^2 - \frac{\partial f(x_i, U_i)}{\partial u} M_i,$$

here U_i' is the approximation for $u_x(x_i)$ and U_i' , $i = 0, 1, \dots, N$ are computed as follows:

$$U_i' = \frac{-25U_i + 48U_{i+1} - 36U_{i+2} + 16U_{i+3} - 3U_{i+4}}{12h}, \quad i = 0, 1,$$

$$U_i' = \frac{U_{i-2} - 8U_{i-1} + 8U_{i+1} - U_{i+2}}{12h}, \quad i = 2, 3, \dots, N-2,$$

$$U_i' = \frac{25U_i - 48U_{i-1} + 36U_{i-2} - 16U_{i-3} + 3U_{i-4}}{12h}, \quad i = N-1, N.$$

We define E_2^N and E_4^N as

$$E_2^N = \max_i |u_{xx}(x_i) - M_i|, \quad E_4^N = \max_i |u_{xxxx}(x_i) - S_i|,$$

and

$$p_2^N = \log_2 \left(\frac{E_2^N}{E_2^{2N}} \right), \quad p_4^N = \log_2 \left(\frac{E_4^N}{E_4^{2N}} \right).$$

Once we get the values U_i , M_i , and S_i , we can compute the coefficients given in (2.1). Thus, we get the fractal quintic spline that passes through the solutions of the BVP given in (1.1). From (2.1), we can get the solution of the BVP (1.1) at any point in the interval.

4.1. Numerical schemes for comparison

We compare the numerical results of the proposed method with the finite difference method (FDM) and the Numerov's method (NM). We give brief detail about the finite difference method and the Numerov's method in the following subsections.

4.1.1. Finite difference method. Let us consider the BVP given in (2.7). Let $U^{(r+1)}$ be the approximate solution for $u^{(r+1)}(x)$. Substituting

$$u_{xx}^{(r+1)}(x) \approx \frac{U_{i-1}^{(r+1)} - 2U_i^{(r+1)} + U_{i+1}^{(r+1)}}{h^2}$$

in (2.7) and after simplification, we get

$$U_{i-1}^{(r+1)} + \left[-2 + h^2 q_i^{(r)}\right] U_i^{(r+1)} + U_{i+1}^{(r+1)} = h^2 f_i^{(r)},$$

for $i = 1, 2, \dots, N-1$, where $q_i^{(r)}$ and $f_i^{(r)}$ are as given in Section 2.3. Here, $U_0^{(r+1)} = \eta_0$ and $U_N^{(r+1)} = \eta_1$.

Now, we can obtain the numerical solution using the condition (4.1).

4.1.2. Numerov's method. The Numerov's method for BVP (1.1) can be written as

$$-U_{i-1} + 2U_i - U_{i+1} = \frac{h^2}{12} [f_{i-1} + 10f_i + f_{i+1}]$$

where $f_i = f(x_i, U_i)$, $i = 0, 1, \dots, N$, $U_0 = \eta_0$ and $U_N = \eta_1$. For more details about the Numerov's method, one can see the references [13, 18]. We can use Newton's method to get the numerical solutions.

Example 4.1. Consider the nonlinear two-point BVP:

$$\begin{cases} u_{xx}(x) + \exp(-2 * u(x)) = 0, & x \in (0, 1), \\ u(0) = 0, & u(1) = \log(2). \end{cases}$$

The exact solution is $u(x) = \log(1+x)$. We calculate the numerical solutions for $N = 8, 16, 32, 64, 128$. The scaling factors $\alpha = 0.9999h^6$ are used to calculate the numerical solutions. The maximum point-wise error E^N , and the order of convergence p^N , E_2^N , p_2^N , E_4^N , and p_4^N are given in Table 1. We have compared the numerical results of the proposed method with the numerical results of finite difference method and Numerov's numerical method. Their results are given in Table 1.

Table 2 represents the solution corresponding to the quintic spline ($\alpha = 0$). Here, we take $U_0^{(0)} = 0$, $U_i^{(0)} = 1$, $i = 1, 2, \dots, N-1$, $U_N^{(0)} = \log(2)$ to compute the numerical solutions.

Figure 1 represents the numerical solution, the corresponding error, and the loglog plot of the error for Example 4.1. From the loglog plot (Figure 1(c)), it can be seen that the

developed method has fourth-order convergence.

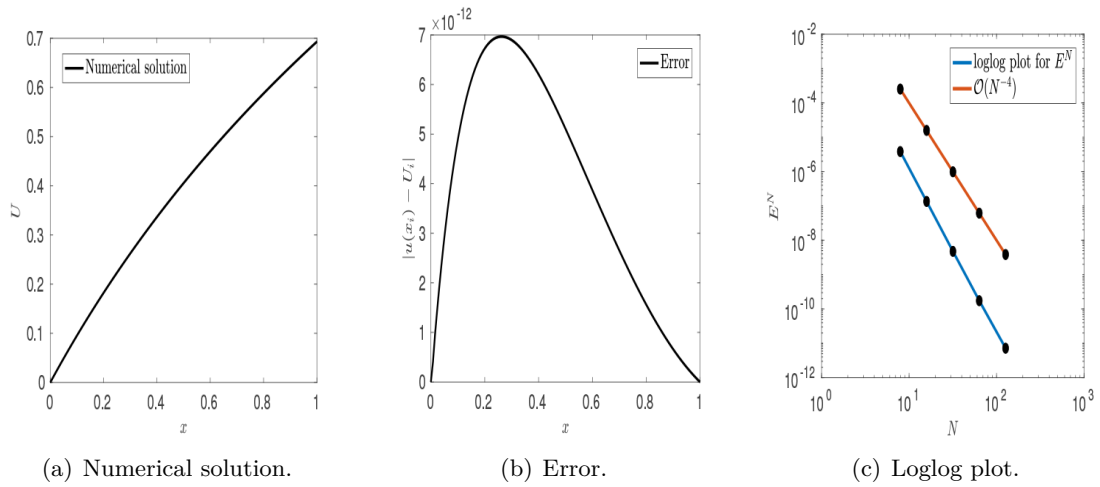


Figure 1. Numerical results of Example 4.1 for $N = 128$.

Table 1. Maximum point-wise error E^N , order of convergence p^N , E_2^N , p_2^N , E_4^N , and p_4^N corresponding to Example 4.1.

| N | 8 | 16 | 32 | 64 | 128 |
|-------------|------------|------------|------------|------------|------------|
| E^N | 3.7039e-06 | 1.3093e-07 | 4.6024e-09 | 1.6823e-10 | 6.9676e-12 |
| p^N | 4.8222 | 4.8303 | 4.7739 | 4.5936 | |
| E_2^N | 3.6039e-06 | 1.3560e-07 | 5.1526e-09 | 2.0841e-10 | 9.3744e-12 |
| p_2^N | 4.7321 | 4.7179 | 4.6278 | 4.4746 | |
| E_4^N | 3.8032e-03 | 3.5970e-04 | 2.8433e-05 | 2.0142e-06 | 1.3430e-07 |
| p_4^N | 3.4024 | 3.6611 | 3.8193 | 3.9066 | |
| (FDM) E^N | 2.2281e-04 | 5.6130e-05 | 1.4060e-05 | 3.5166e-06 | 8.7940e-07 |
| p^N | 1.9890 | 1.9972 | 1.9993 | 1.9996 | |
| (NM) E^N | 2.0165e-06 | 1.2867e-07 | 8.0669e-09 | 5.0513e-10 | 3.1574e-11 |
| p^N | 3.9701 | 3.9955 | 3.9973 | 3.9999 | |

Table 2. Maximum point-wise error E^N , order of convergence p^N , E_2^N , p_2^N , E_4^N , and p_4^N corresponding to Example 4.1 with quintic spline ($\alpha = 0$).

| N | 8 | 16 | 32 | 64 | 128 |
|---------|------------|------------|------------|------------|------------|
| E^N | 6.0415e-07 | 2.9388e-08 | 2.4273e-09 | 1.6374e-10 | 1.0446e-11 |
| p^N | 4.3616 | 3.5978 | 3.8898 | 3.9704 | |
| E_2^N | 9.5470e-07 | 3.2934e-08 | 2.9499e-09 | 2.0374e-10 | 1.3077e-11 |
| p_2^N | 4.8574 | 3.4809 | 3.8559 | 3.9616 | |
| E_4^N | 3.6735e-03 | 3.5614e-04 | 2.8356e-05 | 2.0140e-06 | 1.3444e-07 |
| p_4^N | 3.3666 | 3.6507 | 3.8155 | 3.9051 | |

Example 4.2. Consider the BVP

$$\begin{cases} u_{xx}(x) - \frac{(2-x)\exp(2u(x)) + (1/(1+x))}{3} = 0, & x \in (0, 1), \\ u(0) = 0, \quad u(1) = \log(1/2). \end{cases}$$

The exact solution is $u(x) = \log(1/(1+x))$. We calculate the numerical solutions for $N = 8, 16, 32, 64, 128$. The scaling factors $\alpha = 0.9999h^6$ are used to calculate the numerical solutions. The maximum point-wise error E^N , and the order of convergence p^N , E_2^N , p_2^N , E_4^N , and p_4^N are given in Table 3. We have compared the numerical results of developed method with the numerical results of finite difference method and the Numerov's method. Their results are given in Table 3. Table 4 represents the solution corresponding to $\alpha = 0$, i.e., for the classical quintic spline case. Here, we take $U_0^{(0)} = 0$, $U_i^{(0)} = 1$, $i = 1, 2, \dots, N - 1$, $U_N^{(0)} = \log(1/2)$ to compute the numerical solutions. Figure 2 represents the numerical solution, the corresponding error and the loglog plot of the error for Example 4.2. Figure 2(c) reveals that the developed method has fourth-order convergence.

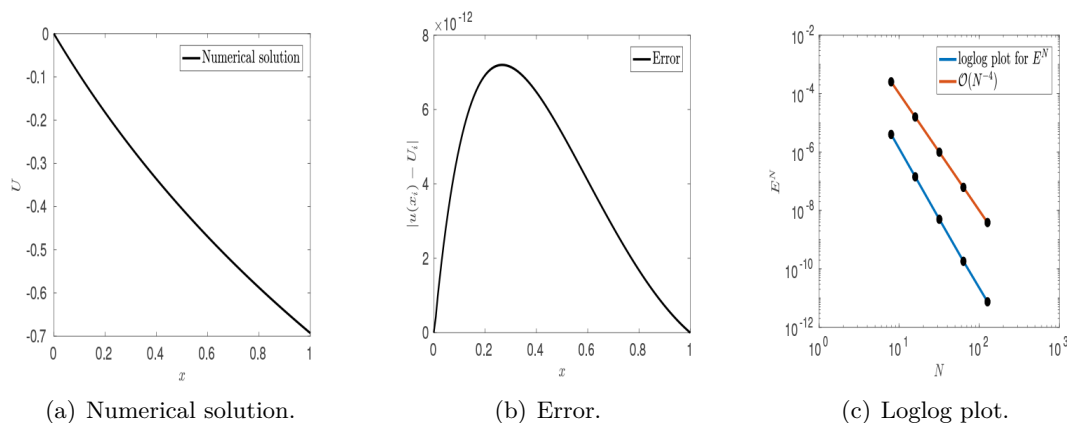


Figure 2. Numerical results of Example 4.2 with $N = 128$.

Table 3. Maximum point-wise error E^N , order of convergence p^N , E_2^N , p_2^N , E_4^N , and p_4^N corresponding to Example 4.2.

| N | 8 | 16 | 32 | 64 | 128 |
|-------------|------------|------------|------------|------------|------------|
| E^N | 3.8662e-06 | 1.3680e-07 | 4.8082e-09 | 1.7524e-10 | 7.2015e-12 |
| p^N | 4.8207 | 4.8304 | 4.7781 | 4.6048 | |
| E_2^N | 2.1004e-06 | 8.0168e-08 | 3.1240e-09 | 1.2878e-10 | 5.8584e-12 |
| p_2^N | 4.7115 | 4.6816 | 4.6004 | 4.4583 | |
| E_4^N | 3.1728e-03 | 2.9988e-04 | 2.3700e-05 | 1.6787e-06 | 1.1193e-07 |
| p_4^N | 3.4033 | 3.6615 | 3.8195 | 3.9067 | |
| (FDM) E^N | 2.3261e-04 | 5.8573e-05 | 1.4670e-05 | 3.6702e-06 | 9.1777e-07 |
| p^N | 1.9896 | 1.9974 | 1.9989 | 1.9997 | |
| (NM) E^N | 2.1034e-06 | 1.3382e-07 | 8.4017e-09 | 5.2577e-10 | 3.2865e-11 |
| p^N | 3.9743 | 3.9935 | 3.9982 | 3.9998 | |

Table 4. Maximum point-wise error E^N , order of convergence p^N, E_2^N, p_2^N, E_4^N , and p_4^N corresponding to Example 4.2 with quintic spline ($\alpha = 0$).

| N | 8 | 16 | 32 | 64 | 128 |
|---------|------------|------------|------------|------------|------------|
| E^N | 6.0740e-07 | 3.0589e-08 | 2.5274e-09 | 1.7045e-10 | 1.0880e-11 |
| p^N | 4.3116 | 3.5973 | 3.8903 | 3.9696 | |
| E_2^N | 5.9990e-07 | 1.9926e-08 | 1.8133e-09 | 1.2596e-10 | 8.1067e-12 |
| p_2^N | 4.9120 | 3.4579 | 3.8477 | 3.9576 | |
| E_4^N | 3.0610e-03 | 2.9681e-04 | 2.3633e-05 | 1.6785e-06 | 1.1204e-07 |
| p_4^N | 3.3664 | 3.6506 | 3.8155 | 3.9051 | |

Example 4.3. Consider the following nonlinear two-point BVP:

$$\begin{cases} u_{xx}(x) - \frac{25x^8 \exp(u(x)) - 20x^3}{4 + x^5} = 0, & x \in (0, 1), \\ u(0) = -\log(4), & u(1) = -\log(5). \end{cases}$$

The exact solution is $u(x) = -\log(4 + x^5)$. We calculate the numerical solutions for $N = 8, 16, 32, 64, 128$. The scaling factors $\alpha = 0.85h^6$ are used to calculate the numerical solutions. The maximum point-wise error E^N , order of convergence p^N, E_2^N, p_2^N, E_4^N , and p_4^N are given in Table 5. We have compared the numerical results of developed method with the numerical results of finite difference method and the Numerov’s method. Their results are given in Table 5. Table 6 represents the solution corresponding to $\alpha = 0$. We take $U_0^{(0)} = -\log(4), U_i^{(0)} = 1, i = 1, 2, \dots, N - 1, U_N^{(0)} = -\log(5)$ to compute the numerical solutions. Figure 3 represents the numerical solution, the corresponding error, and the loglog plot of the error for Example 4.3. Figure 3(c) reveals that the developed method has fourth-order convergence.

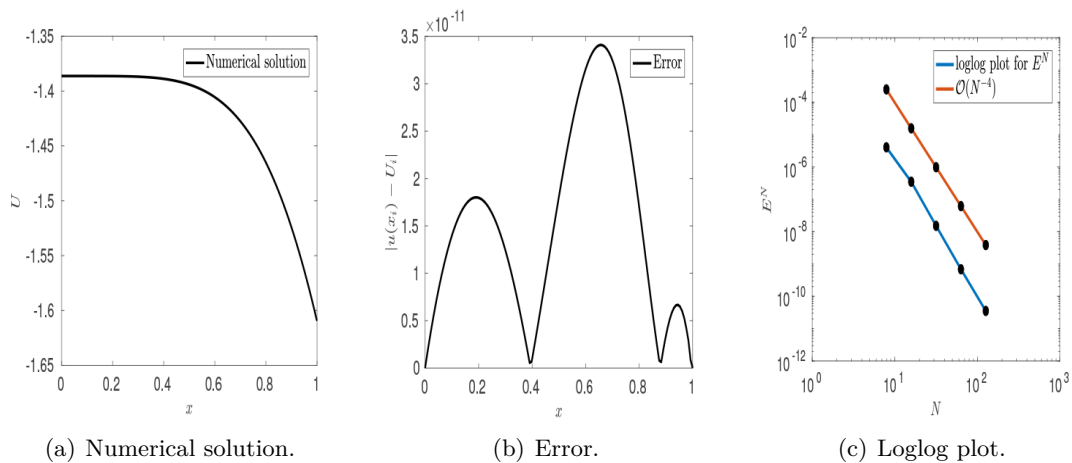


Figure 3. Numerical results of Example 4.3 with $N = 128$.

Table 5. Maximum point-wise error E^N , order of convergence p^N , E_2^N , p_2^N , E_4^N , and p_4^N corresponding to Example 4.3.

| N | 8 | 16 | 32 | 64 | 128 |
|-------------|------------|------------|------------|------------|------------|
| E^N | 3.9439e-6 | 3.3929e-07 | 1.4653e-08 | 6.6424e-10 | 3.4105e-11 |
| p^N | 3.5391 | 4.5332 | 4.4633 | 4.2837 | |
| E_2^N | 4.8434e-07 | 1.0201e-07 | 2.2710e-09 | 8.3241e-11 | 4.8099e-12 |
| p_2^N | 2.2473 | 5.4893 | 4.7699 | 4.1132 | |
| E_4^N | 6.8887e-02 | 4.8907e-03 | 2.5980e-04 | 1.4220e-05 | 7.8036e-07 |
| p_4^N | 3.8161 | 4.2346 | 4.1914 | 4.1876 | |
| (FDM) E^N | 1.1795e-03 | 2.9324e-04 | 7.3024e-05 | 1.8265e-05 | 4.5652e-06 |
| p^N | 2.0080 | 2.0056 | 1.9992 | 2.0004 | |
| (NM) E^N | 3.0070e-05 | 1.8480e-06 | 1.1585e-07 | 7.2337e-09 | 4.5198e-10 |
| p^N | 4.0242 | 3.9956 | 4.0014 | 4.0004 | |

Table 6. Maximum point-wise error E^N , order of convergence p^N , E_2^N , p_2^N , E_4^N , and p_4^N corresponding to Example 4.3 with quintic spline ($\alpha = 0$).

| N | 8 | 16 | 32 | 64 | 128 |
|---------|------------|------------|------------|------------|------------|
| E^N | 1.0857e-05 | 7.0970e-07 | 4.0662e-08 | 2.4478e-09 | 1.5124e-10 |
| p^N | 3.9352 | 4.1255 | 4.0541 | 4.0165 | |
| E_2^N | 1.7413e-06 | 1.6518e-07 | 7.0721e-09 | 3.9525e-10 | 2.3900e-11 |
| p_2^N | 3.3981 | 4.5458 | 4.1613 | 4.0477 | |
| E_4^N | 6.9280e-02 | 4.9109e-03 | 2.6117e-04 | 1.4299e-05 | 7.8595e-07 |
| p_4^N | 3.8184 | 4.2329 | 4.1910 | 4.1854 | |

5. Conclusion

In this article, with the help of fractal quintic spline a new numerical scheme is proposed for solving nonlinear two-point BVPs. By using the quasilinearization technique, we convert the nonlinear BVP into a sequence of linear BVPs. Then, we obtain the numerical solutions of these linear BVPs using the proposed numerical method. Convergence analysis of the numerical method is derived. The proposed numerical scheme has fourth-order convergence. Numerical examples are presented in supporting the theoretical error estimates.

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