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A NOTE ON THE GENERALIZATIONS OF JACOBTHAL AND JACOBTHAL-LUCAS SEQUENCES

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Abstract

In this paper, we present the generalization of Jacobsthal and Jacobsthal-Lucas sequences by the recurrence relations $J_n = 2aJ_{n-1} + (b-a^2)J_{n-2}$ & $j_n = 2aj_{n-1} + (b-a^2)j_{n-2}$, $n \geq 2$ with the initial conditions $J_0 = 0,\ J_1 = 1\ and\ j_0 = 2,\ j_1 = 2a$. We establish some of the interesting properties of involving them. Also we describe and derive sums, connection formulae and Generating function. We have used their Binet's formula to derive the identities.

Keywords: Generalized Jacobsthal sequence, Generalized Jacobsthal-Lucas sequence, Binet's formula and Generating function.

1. Introduction

Sequences have been fascinating topic for mathematicians for centuries. The Fibonacci sequence, Lucas sequence, Pell sequence, Pell-Lucas sequence, Jacobsthal sequence and Jacobsthal-Lucas sequence are most prominent examples of recursive sequences. The second order recurrence sequence has been generalized in two ways mainly, first by preserving the initial conditions and second by preserving the recurrence relation.

Kalman and Mena [10] generalize the Fibonacci sequence by

$$F_n = aF_{n-1} + bF_{n-2},\ n \geq 2 \quad with \quad F_0 = 0,\ F_1 = 1 \tag{1.1}$$

Horadam [8] defined generalized Fibonacci sequence $\{H_n\}$ by

$$H_n = H_{n-1} + H_{n-2},\ n \geq 3\ with\ H_1 = p,\ H_2 = p+q \tag{1.2}$$

where p and q are arbitrary integers.

The k-Fibonacci numbers defined by Falco'n and Plaza [4], for any positive real number k, the k-Fibonacci sequence is defined recurrently by

$$F_{k,n} = k\,F_{k,n-1} + F_{k,n-2},\ n \geq 2 \quad with \quad F_{k,0} = 0,\ F_{k,1} = 1 \tag{1.3}$$

The k-Lucas numbers defined by Falco'n [2],

$$L_{k,n} = k\,L_{k,n-1} + L_{k,n-2}\,,\; n \geq 2 \quad with \quad L_{k,0} = 2,\; L_{k,1} = k \tag{1.4}$$

Most of the authors introduced Fibonacci pattern based sequences in many ways which are known as Fibonacci-Like sequences and k-Fibonacci-like sequences [13, 17, 22, 28, 29].

Generalized Fibonacci sequence [7], is defined as

$$F_k = pF_{k-1} + qF_{k-2}\,,\; k \geq 2 \quad with \quad F_0 = a,\; F_1 = b \tag{1.5}$$

where p, q, a and b are positive integer.

(p,q) - Fibonacci numbers [19], is defined as

$$F_{p,q,n} = pF_{p,q,n} + bF_{p,q,n}\,,\; n \geq 2 \quad with \quad F_{p,q,0} = 0,\; F_{p,q,n} = 1 \tag{1.6}$$

(p,q) - Lucas numbers [20], is defined as

$$L_{p,q,n} = pL_{p,q,n} + bL_{p,q,n}\,,\; n \geq 2 \quad with \quad L_{p,q,0} = 2,\; L_{p,q,n} = p \tag{1.7}$$

Generalized (p,q) -Fibonacci-Like sequence [21], is defined by recurrence relation

$$S_{p,q,n} = pS_{p,q,n} + qS_{p,q,n}\,,\; n \geq 2 \quad with \quad S_{p,q,0} = 2k,\; S_{p,q,n} = 1 + kp \tag{1.8}$$

Goksal Bilgici [1], defined new generalizations of Fibonacci and Lucas sequences

$$f_k = 2af_{k-1} + (b - a^2)\,f_{k-2}\;,\; k \geq 2 \;\; with \;\; f_0 = 0,\; f_1 = 1 \tag{1.9}$$

$$l_k = 2al_{k-1} + (b - a^2)l_{k-2}\;,\; k \geq 2 \;\; with \;\; l_0 = 2,\; l_1 = 2a \tag{1.10}$$

Tulay Yagmur [30], defined generalizations of Pell and Pell-Lucas sequences

$$p_k = 2ap_{k-1} + (b - a^2)\,p_{k-2}\;,\; k \geq 2 \;\; with \;\; p_0 = 0,\; p_1 = 1 \tag{1.11}$$

$$q_k = 2aq_{k-1} + (b - a^2)q_{k-2}\;,\; k \geq 2 \;\; with \;\; q_0 = 2,\; q_1 = 2a \tag{1.12}$$

In this study, we present the generalization of Jacobsthal and Jacobsthal-Lucas sequences, in much the same way that Bilgici did for Fibonacci and Lucas sequences in [1] and Yagmur did for Pell and Pell-Lucas sequences in [30]. We prove the Catalan, Cassini, and d'Ocagne identities for this sequence. Moreover, we introduce the special sums of the generalized Jacobsthal and Jacobsthal-Lucas sequences and prove them using Binet's formula.

2. Generalized Jacobsthal and Jacobsthal-Lucas Sequences

In this section, we review basic definitions and introduce relevant facts.

For $n \geq 2$, The generalized Jacobsthal sequence is defined by

$$J_n = 2aJ_{n-1} + (b - a^2)J_{n-2} \tag{2.1}$$

with initial conditions $J_0 = 0$, $J_1 = 1$.

First few generalized Jacobsthal numbers are

$$\{J_n\} = \{0, 1, 2a, 3a^2 + b, 4a^3 + 4ab, 5a^4 + 10a^2b + b^2, ...\}$$

It is well known that the Jacobsthal and Jacobsthal-Lucas sequences are closely related.

For $n \geq 2$, The generalized Jacobsthal-Lucas sequence is defined by

$$j_n = 2a\, j_{n-1} + (b - a^2)\, j_{n-2} \tag{2.2}$$

with initial conditions $j_0 = 2$, $j_1 = 2a$.

First few generalized Jacobsthal-Lucas numbers are

$$\{j_n\} = \{2, 2a, 2a^2 + 2b, 2a^3 + 6ab, 2a^4 + 12a^2b + 2b^2, 2a^5 + 20a^3b + 10ab^2, ...\}$$

In (2.1) and (2.2), $a \,\&\, b$ are any nonzero real numbers.

If $a = \frac{1}{2}$ & $b = \frac{9}{4}$, then we obtained classical Jacobsthal and Jacobsthal-Lucas sequences,
If $a = \frac{1}{2}$ & $b = \frac{5}{4}$, then we obtained classical Fibonacci and Lucas sequences,
If $a = 1$ & $b = 3$, then we obtained classical Pell sequence and Pell-Lucas sequences,
If $a = \frac{3}{2}$ & $b = \frac{1}{4}$, then we obtained classical Mersenne and Fermat sequences.

For any positive integer k,
If $a = \frac{k}{2}$ & $b = \left(\frac{4 + k^2}{4}\right)$, then we obtained k-Fibonacci and k-Lucas sequences,
If $a = 1$ & $b = (1 + k)$, then we obtained k-Pell and k-Pell-Lucas sequences,
If $a = \frac{k}{2}$ & $b = \left(\frac{8 + k^2}{4}\right)$, then we obtained k- Jacobsthal and k- Jacobsthal-Lucas sequences.

2.1. Explicit sum formulae of generalized Jacobsthal and Jacobsthal-Lucas sequences

Theorem 2.1. Explicit sum Formula for generalized Jacobsthal sequence is given by

$$J_n = \sum_{i=0}^{\left[\frac{n-1}{2}\right]} \binom{n-i-1}{i} (2a)^{n-2i-1} (b - a^2)^i \tag{2.3}$$

Proof. Applying Binet's formula of generalized Jacobsthal sequence, the proof is clear.

Theorem 2.2. Explicit sum Formula for new generalized Jacobsthal-Lucas sequence is given by

$$j_n = 2\sum_{i=0}^{\left[\frac{n}{2}\right]} \binom{n-i}{i} (2a)^{n-2i} (b - a^2)^i - \sum_{i=0}^{\left[\frac{n-1}{2}\right]} \binom{n-i-1}{i} (2a)^{n-2i} (b - a^2)^i \tag{2.4}$$

Proof. Applying Binet's formula of generalized Jacobsthal-Lucas sequence, the proof is clear

2.2. Binet's formula of Generalized Jacobsthal and Jacobsthal-Lucas sequences

In the 19th century, the French mathematician Binet devised two remarkable analytical formulas for the Fibonacci and Lucas numbers. In our case, Binet's formula allows us to express the generalized Jacobsthal and Jacobsthal-Lucas sequences in function of the roots of the following characteristic equation, associated to the recurrence relation (2.1) & (2.2):

$$x^2 = 2ax + (b - a^2) \tag{2.5}$$

Theorem 2.3. ***(Binet's formula).*** The nth terms of the generalized Jacobsthal sequence is given by

$$J_n = \frac{\Re_1^n - \Re_2^n}{\Re_1 - \Re_2} \tag{2.6}$$

where $\Re_1$ & $\Re_2$ are the roots of the characteristic equation (2.5), with $\Re_1 = a + \sqrt{b}$, $\Re_2 = a - \sqrt{b}$.

Proof. We use the Principle of Mathematical Induction (PMI) on n. It is clear the result is true for $n = 0$ & $n = 1$ by hypothesis. Assume that it is true for i such that $0 \le i \le r + 1$, then

$$J_i = \frac{\Re_1^i - \Re_2^i}{\Re_1 - \Re_2}$$

It follows from definition generalized Jacobsthal sequence (2.1) and equation (2.6)

$$J_{r+2} = 2aJ_{r+1} + (b - a^2)J_r = \frac{\Re_1^{r+2} - \Re_2^{r+2}}{\Re_1 - \Re_2}$$

Thus, the formula is true for any positive integer n.

Theorem 2.4. ***(Binet's formula).*** The nth terms of the generalized Jacobsthal-Lucas sequence is given by

$$j_n = \Re_1^n + \Re_2^n \tag{2.7}$$

Proof. It can be proved same as Theorem 2.3.

Theorem 2.5. For every integer n , we have

(i) $$J_{-n} = \frac{-J_n}{(a^2 - b)^n} \tag{2.8}$$

(ii) $$j_{-n} = \frac{j_n}{(a^2 - b)^n} \tag{2.9}$$

Theorem 2.6. For every integer n , we have

$$J_n j_n = J_{2n} \tag{2.10}$$

Proof. Applying Binet's formula of generalized Jacobsthal sequence and generalized Jacobsthal-Lucas sequence, the proof is clear.

2.3. Identities of Generalized Jacobsthal and Jacobsthal-Lucas sequences

In this section, we introduce Catalan, Cassini and d'Ocagne identities for the generalized Jacobsthal and Jacobsthal-Lucas sequences and prove them using Binet's formula stated in the previous section.

2.3.1. Catalan's Identity

Catalan's identity for Fibonacci numbers was found in 1879 by Eugene Charles Catalan a Belgian mathematician who worked for the Belgian Academy of Science in the field of number theory:

Theorem 2.7. ***(Catalan's identity).*** For every integers $n \; and \; r$, we have

$$J_{n+r} J_{n+r} - J_n^2 = -(a^2 - b)^{n-r} J_r^2 \tag{2.11}$$

and

$$j_{n+r} j_{n+r} - j_n^2 = 4b(a^2 - b)^{n-r} j_r^2 \tag{2.12}$$

Proof. Applying Binet's formula of generalized Jacobsthal sequence and generalized Jacobsthal-Lucas sequence completes the proof of Catalan's identity.

2.3.2. Cassini's Identity

This is one of the oldest identities involving the Fibonacci numbers. It was discovered in 1680 by Jean-Dominique Cassini a French astronomer:

Theorem 2.8. ***(Cassini's identity).*** For every integers n, we have

$$J_{n+1} J_{n+1} - J_n^2 = -(a^2 - b)^{n-1} \tag{2.13}$$

and

$$j_{n+1} j_{n+1} - j_1^2 = 4b(a^2 - b)^{n-1} \tag{2.14}$$

Proof. Taking $r = 1$ in Catalan's identity (2.11) & (2.12) the proof is completed.

2.3.3. d'Ocagne's identity

Theorem 2.9. ***(d'Ocagne's identity).*** For every integers $n \; and \; m$, we have

$$J_m J_{n+1} - J_n J_{m+1} = (a^2 - b)^n J_{m-n} \tag{2.15}$$

and

$$j_m j_{n+1} - j_n j_{m+1} = -4ab(a^2 - b)^n J_{m-n} \tag{2.16}$$

Proof. Applying Binet's formula of generalized Jacobsthal sequence and generalized Jacobsthal-Lucas sequence completes the proof of d'Ocagne's identity.

3. The Sums of the Generalized Jacobsthal and Jacobsthal-Lucas Sequences

Binet's formula allows us to express the sum of generalized Jacobsthal and Jacobsthal-Lucas sequences.

3.1. Sums of Generailzed Jacobsthal Sequence

Theorem 3.1. For fixed integers $p,\ q\ with\ 0 \leq q \leq p-1$, the following equality holds

$$J_{p(n+2)+q} = j_p\, J_{p(n+1)+q} - (a^2-b)^p J_{pn+q} \tag{3.1}$$

Proof. From the the Binet's formula of generalized Jacobsthal and Jacobsthal-Lucas sequences,

$$\begin{aligned} j_p\, J_{p(n+1)+q} &= \left(\Re_1^p + \Re_2^p\right)\left(\frac{\Re_1^{p(n+1)+q} - \Re_2^{p(n+1)+q}}{\Re_1 - \Re_2}\right) \\ &= \frac{1}{\Re_1 - \Re_2}\left[\Re_1^{p(n+2)+q} + (a^2-b)^p \Re_1^{pn+q} - (a^2-b)^p \Re_2^{pn+q} - \Re_2^{p(n+2)+q}\right] \\ &= \frac{1}{\Re_1 - \Re_2}\left[\left\{\Re_1^{p(n+2)+q} - \Re_2^{p(n+2)+q}\right\} + (a^2-b)^p\left(\Re_1^{pn+q} - \Re_2^{pn+q}\right)\right] \\ &= J_{p(n+2)+q} + (a^2-b)^p J_{pn+q} \end{aligned}$$

then, the equality becomes,

$$J_{p(n+2)+q} = j_p\, J_{p(n+1)+q} - (a^2-b)^p J_{pn+q}$$

Theorem 3.2. For fixed integers $p,\ q\ with\ 0 \leq q \leq p-1$, the following equality holds

$$\sum_{i=0}^{n} J_{pi+q} = \frac{J_{p(n+1)+q} - (a^2-b)^q J_{p-q} - J_q - (a^2-b)^p J_{pn+q}}{j_p - (a^2-b)^p - 1} \tag{3.2}$$

Proof. From the the Binet's formula of generalized Jacobsthal sequence,

$$\begin{aligned} \sum_{i=0}^{n} J_{pi+q} &= \sum_{i=0}^{n} \frac{\Re_1^{pi+q} - \Re_2^{pi+q}}{\Re_1 - \Re_2} \\ &= \frac{1}{\Re_1 - \Re_2}\left[\sum_{i=0}^{n} \Re_1^{pi+q} - \sum_{i=0}^{n} \Re_2^{pi+q}\right] \\ &= \frac{1}{\Re_1 - \Re_2}\left[\frac{\Re_1^{pn+q+p} - \Re_1^q}{\Re_1^p - 1} - \frac{\Re_2^{pn+q+p} - \Re_2^q}{\Re_2^p - 1}\right] \\ &= \frac{1}{(a^2-b)^p - j_p + 1}\left[(a^2-b)^p J_{pn+q} - J_{p(n+1)+q} + J_q + (a^2-b)^q J_{p-q}\right] \\ &= \frac{J_{p(n+1)+q} - (a^2-b)^q J_{p-q} - J_q - (a^2-b)^p J_{pn+q}}{j_p - (a^2-b)^p - 1} \end{aligned}$$

This completes the proof.

Corollary 3.3. Sum of odd generalized Jacobsthal sequence, If $p = 2m+1$ then Eq. (3.2) is

$$\sum_{i=0}^{n} J_{(2m+1)i+q} = \frac{J_{(2m+1)(n+1)+q} - (a^2-b)^q J_{2m+1-q} - J_q - (a^2-b)^{(2m+1)} J_{(2m+1)n+q}}{j_{(2m+1)} - (a^2-b)^{(2m+1)} - 1} \tag{3.3}$$

For example

(1) If $m=0$ then $p=1$: $\sum_{i=0}^{n} J_{i+q} = \frac{J_{n+q+1} - (a^2-b)^q J_{1-q} - J_q - (a^2-b) J_{n+q}}{2a-(a^2-b)-1}$ (3.4)

(i) For $q=0$: $\sum_{i=0}^{n} J_i = \frac{J_{n+1} - 1 - (a^2-b) J_n}{2a-(a^2-b)-1}$

(2) If $m=1$ then $p=3$: $\sum_{i=0}^{n} J_{3i+q} = \frac{J_{3n+q+3} - (a^2-b)^q J_{3-q} - J_q - (a^2-b)^3 J_{3n+q}}{a^3(2-a^3)+3ab(2-ab)+b(3a^4+b^2)-1}$ (3.5)

(i) For $q=0$: $\sum_{i=0}^{n} J_{3i} = \frac{J_{3n+3} - (3a^2+b) - (a^2-b)^3 J_{3n}}{a^3(2-a^3)+3ab(2-ab)+b(3a^4+b^2)-1}$

(ii) For $q=1$: $\sum_{i=0}^{n} J_{3i+1} = \frac{J_{3n+4} - 2a(a^2+b) - 1 - (a^2-b)^3 J_{3n+1}}{a^3(2-a^3)+3ab(2-ab)+b(3a^4+b^2)-1}$

(iii) For $q=2$: $\sum_{i=0}^{n} J_{3i+2} = \frac{J_{3n+5} - (a^2-b)^2 - 2a - (a^2-b)^3 J_{3n+2}}{a^3(2-a^3)+3ab(2-ab)+b(3a^4+b^2)-1}$

(3) If $m=2$ then $p=5$: $\sum_{i=0}^{n} J_{5i+q} = \frac{J_{5n+q+5} - (a^2-b)^q J_{5-q} - J_q - (a^2-b)^5 J_{5n+q}}{j_5 - (a^2-b)^5 - 1}$ (3.6)

(i) For $q=0$: $\sum_{i=0}^{n} J_{5i} = \frac{J_{5n+5} - J_5 - (a^2-b)^5 J_{5n}}{j_5 - (a^2-b)^5 - 1}$

(ii) For $q=1$: $\sum_{i=0}^{n} J_{5i+1} = \frac{J_{5n+6} - (a^2-b) J_4 - 1 - (a^2-b)^5 J_{5n+1}}{j_5 - (a^2-b)^5 - 1}$

(iii) For $q=2$: $\sum_{i=0}^{n} J_{5i+2} = \frac{J_{5n+7} - (a^2-b)^2 J_3 - 2a - (a^2-b)^5 J_{5n+2}}{j_5 - (a^2-b)^5 - 1}$

(iv) For $q=3$: $\sum_{i=0}^{n} J_{5i+3} = \frac{J_{5n+8} - (a^2-b)^3 2a - (3a^2+b) - (a^2-b)^5 J_{5n+3}}{j_5 - (a^2-b)^5 - 1}$

(v) For $q=4$: $\sum_{i=0}^{n} J_{5i+4} = \frac{J_{5n+9} - (a^2-b)^4 - 4a(a^2+b) - (a^2-b)^5 J_{5n+4}}{j_5 - (a^2-b)^5 - 1}$

(vi) For $q=5$: $\sum_{i=0}^{n} J_{5i+5} = \frac{J_{5n+10} - J_5 - (a^2-b)^5 J_{5n+5}}{j_5 - (a^2-b)^5 - 1}$

Corollary 3.4. Sum of even generalized Jacobsthal sequence, If $p=2m$ then Eq. (3.2) is

$$\sum_{i=0}^{n} J_{2mi+q} = \frac{J_{2m(n+1)+q} - (a^2-b)^q J_{2m-q} - J_q - (a^2-b)^{2m} J_{2mn+q}}{j_{2m} - (a^2-b)^{2m} - 1} \tag{3.7}$$

For example

(1) If $m=1$ then $p=2$: $\sum_{i=0}^{n} J_{2i+q} = \frac{J_{2n+2+q} - (a^2-b)^q J_{2-q} - J_q - (a^2-b)^2 J_{2n+q}}{j_2 - (a^2-b)^2 - 1}$ (3.8)

(i) For $q=0$: $\sum_{i=0}^{n} J_{2i} = \frac{J_{2n+2} - 2a - (a^2-b)^2 J_{2n}}{(2a^2+2b)-(a^2-b)^2-1}$

(ii) For $q=1$: $\sum_{i=0}^{n} J_{2i+1} = \frac{J_{2n+3} - (a^2-b) - 1 - (a^2-b)^2 J_{2n+1}}{(2a^2+2b)-(a^2-b)^2-1}$

(iii) For $q=2$: $\sum_{i=0}^{n} J_{2i+2} = \frac{J_{2n+4} - 2a - (a^2-b)^2 J_{2n+2}}{(2a^2+2b)-(a^2-b)^2-1}$

(2) If $m=2$ then $p=4$: $\sum_{i=0}^{n} J_{4i+q} = \frac{J_{4n+4+q} - (a^2-b)^q J_{4-q} - J_q - (a^2-b)^4 J_{4n+q}}{j_4 - (a^2-b)^4 - 1}$ (3.9)

(i) For $q=0$: $\sum_{i=0}^{n} J_{4i} = \frac{J_{4n+4} - J_4 - (a^2-b)^4 J_{4n}}{j_4 - (a^2-b)^4 - 1}$

(ii) For $q=1$: $\sum_{i=0}^{n} J_{4i+1} = \frac{J_{4n+5} - (a^2-b) J_3 - 1 - (a^2-b)^4 J_{4n+1}}{j_4 - (a^2-b)^4 - 1}$

(iii) For $q=2$: $\sum_{i=0}^{n} J_{4i+2} = \frac{J_{4n+6} - (a^2-b)^2 2a - 2a - (a^2-b)^4 J_{4n+2}}{j_4 - (a^2-b)^4 - 1}$

(iv) For $q=3$: $\sum_{i=0}^{n} J_{4i+3} = \frac{J_{4n+7} - (a^2-b)^3 - J_3 - (a^2-b)^4 J_{4n+3}}{j_4 - (a^2-b)^4 - 1}$

(v) For $q=4$: $\sum_{i=0}^{n} J_{4i+4} = \frac{J_{4n+8} - J_4 - (a^2-b)^4 J_{4n+4}}{j_4 - (a^2-b)^4 - 1}$

(3) If $m=3$ then $p=6$: $\sum_{i=0}^{n} J_{6i+q} = \frac{J_{6n+6+q} - (a^2-b)^q J_{6-q} - J_q - (a^2-b)^6 J_{6n+q}}{j_6 - (a^2-b)^6 - 1}$ (3.10)

(i) For $q=0$: $\sum_{i=0}^{n} J_{6i} = \frac{J_{6n+6} - J_6 - (a^2-b)^6 J_{6n}}{j_6 - (a^2-b)^6 - 1}$

(ii) For $q=1$: $\sum_{i=0}^{n} J_{6i+1} = \frac{J_{6n+7} - (a^2-b) J_5 - 1 - (a^2-b)^6 J_{6n+1}}{j_6 - (a^2-b)^6 - 1}$

(iii) For $q=2$: $\sum_{i=0}^{n} J_{6i+2} = \frac{J_{6n+8} - (a^2-b)^2 J_4 - 2a - (a^2-b)^6 J_{6n+2}}{j_6 - (a^2-b)^6 - 1}$

(iv) For $q=3$: $\sum_{i=0}^{n} J_{6i+3} = \frac{J_{6n+9} - (a^2-b)^3 J_3 - J_3 - (a^2-b)^6 J_{6n+3}}{j_6 - (a^2-b)^6 - 1}$

Theorem 3.5. For fixed integers p, q *with* $0 \le q \le p-1$, the following equality holds

$$\sum_{i=0}^{n} (-1)^i J_{pi+q} = \frac{(-1)^n J_{p(n+1)+q} + (-1)^n (a^2-b)^p J_{pn+q} - (a^2-b)^q J_{p-q} + J_q}{j_p + (a^2-b)^p + 1} \quad (3.11)$$

Proof. Applying Binet's formula of generalized Jacobsthal sequence, the proof is clear.

For different values of $p \& q$:

(i) $$\sum_{i=0}^{n}(-1)^{i}J_{i}=\frac{(-1)^{n}J_{n+1}+(-1)^{n}(a^{2}-b)J_{n}-1}{2a+a^{2}-b+1}$$

(ii) $$\sum_{i=0}^{n}(-1)^{i}J_{2i}=\frac{(-1)^{n}J_{2n+2}+(-1)^{n}(a^{2}-b)^{2}J_{2n}-2a}{(2a^{2}+2b)+(a^{2}-b)^{2}+1}$$

(iii) $$\sum_{i=0}^{n}(-1)^{i}J_{2i+1}=\frac{(-1)^{n}J_{2n+3}+(-1)^{n}(a^{2}-b)^{2}J_{2n+1}-(a^{2}-b)+1}{(2a^{2}+2b)+(a^{2}-b)^{2}+1}$$

(iv) $$\sum_{i=0}^{n}(-1)^{i}J_{4i}=\frac{(-1)^{n}J_{4n+4}+(-1)^{n}(a^{2}-b)^{4}J_{4n}-J_{4}}{j_{4}+(a^{2}-b)^{4}+1}$$

(v) $$\sum_{i=0}^{n}(-1)^{i}J_{4i+1}=\frac{(-1)^{n}J_{4n+5}+(-1)^{n}(a^{2}-b)^{4}J_{4n+1}-(a^{2}-b)J_{3}+1}{j_{4}+(a^{2}-b)^{4}+1}$$

(vi) $$\sum_{i=0}^{n}(-1)^{i}J_{4i+2}=\frac{(-1)^{n}J_{4n+6}+(-1)^{n}(a^{2}-b)^{4}J_{4n+2}-(a^{2}-b)^{2}2a+2a}{j_{4}+(a^{2}-b)^{4}+1}$$

(vii) $$\sum_{i=0}^{n}(-1)^{i}J_{4i+3}=\frac{(-1)^{n}J_{4n+7}+(-1)^{n}(a^{2}-b)^{4}J_{4n+3}-(a^{2}-b)^{3}+J_{3}}{j_{4}+(a^{2}-b)^{4}+1}$$

3.2. Sums of Generailzed Jacobsthal-Lucas Sequence

Theorem 3.6. For fixed integers p, q *with* $0\le q\le p-1$, the following equality holds

$$j_{p(n+1)+q}=j_{p}j_{pn+q}-(a^{2}-b)^{p}j_{p(n-1)+q} \tag{3.12}$$

Theorem 3.7. For fixed integers p, q *with* $0\le q\le p-1$, the following equality holds

$$\sum_{i=0}^{n}j_{pi+q}(x,y)=\frac{j_{p(n+1)+q}+(a^{2}-b)^{q}j_{p-q}-j_{q}-(a^{2}-b)^{p}j_{pn+q}}{j_{p}-(a^{2}-b)^{p}-1} \tag{3.13}$$

Corollary 3.8. Sum of odd generalized Jacobsthal-Lucas sequence, If $p=2m+1$ then Eq. (3.13) is

$$\sum_{i=0}^{n}j_{(2m+1)i+q}=\frac{j_{(2m+1)(n+1)+q}+(a^{2}-b)^{q}j_{2m+1-q}-j_{q}-(a^{2}-b)^{(2m+1)}j_{(2m+1)n+q}}{j_{(2m+1)}-(a^{2}-b)^{(2m+1)}-1} \tag{3.14}$$

For example

(1) If $m=0$ then $p=1$: $$\sum_{i=0}^{n}j_{i+q}=\frac{j_{n+q+1}+(a^{2}-b)^{q}j_{1-q}-j_{q}-(a^{2}-b)j_{n+q}}{2a-(a^{2}-b)-1} \tag{3.15}$$

(i) For $q=0$: $$\sum_{i=0}^{n}j_{i}=\frac{j_{n+1}+2a-2-(a^{2}-b)j_{n}}{2a-(a^{2}-b)-1}$$

(2) If $m=1$ then $p=3$: $$\sum_{i=0}^{n}j_{3i+q}=\frac{j_{3n+q+3}+(a^{2}-b)^{q}j_{3-q}-j_{q}-(a^{2}-b)^{3}j_{3n+q}}{2(a^{3}+3ab)-(a^{2}-b)^{3}-1} \tag{3.16}$$

(3) If $m=2$ then $p=5$: $$\sum_{i=0}^{n}j_{5i+q}=\frac{j_{5n+q+5}+(a^{2}-b)^{q}j_{5-q}-j_{q}-(a^{2}-b)^{5}j_{5n+q}}{j_{5}-(a^{2}-b)^{5}-1} \tag{3.17}$$

Corollary 3.9. Sum of even generalized Jacobsthal-Lucas sequence, If $p=2m$ then Eq. (3.13) is

$$\sum_{i=0}^{n} j_{2mi+q}=\frac{j_{2m(n+1)+q}+(a^2-b)^q j_{2m-q}-j_q-(a^2-b)^{2m} j_{2mn+q}}{j_{2m}-(a^2-b)^{2m}-1} \tag{3.18}$$

For example

(1) If $m=1$ then $p=2$: $\sum_{i=0}^{n} j_{2i+q}=\frac{j_{2n+2+q}+(a^2-b)^q j_{2-q}-j_q-(a^2-b)^2 j_{2n+q}}{j_2-(a^2-b)^2-1}$ (3.19)

(2) If $m=2$ then $p=4$: $\sum_{i=0}^{n} j_{4i+q}=\frac{j_{4n+4+q}+(a^2-b)^q j_{4-q}-j_q-(a^2-b)^4 j_{4n+q}}{j_4-(a^2-b)^4-1}$ (3.20)

(3) If $m=3$ then $p=6$: $\sum_{i=0}^{n} j_{6i+q}=\frac{j_{6n+6+q}+(a^2-b)^q j_{6-q}-j_q-(a^2-b)^6 j_{6n+q}}{j_6-(a^2-b)^6-1}$ (3.21)

Theorem 3.10. For fixed integers p, q *with* $0\le q\le p-1$, the following equality holds

$$\sum_{i=0}^{n}(-1)^i j_{pi+q}=\frac{(-1)^n j_{p(n+1)+q}+(-1)^n(a^2-b)^p j_{pn+q}+(a^2-b)^q j_{p-q}+j_q}{j_p+(a^2-b)^p+1} \tag{3.22}$$

Proof. Applying Binet's formula of generalized Jacobsthal-Lucas sequence, the proof is clear. For different values of $p\,\&\,q$:

(i) $$\sum_{i=0}^{n}(-1)^i j_i=\frac{(-1)^n\{j_{n+1}+(a^2-b)j_n\}+2(a+1)}{2a+(a^2-b)+1}$$

(ii) $$\sum_{i=0}^{n}(-1)^i j_{2i}=\frac{(-1)^n\{j_{2n+2}+(a^2-b)^2 j_{2n}\}+2(a^2+b+1)}{2(a^2+b)+(a^2-b)^2+1}$$

(iii) $$\sum_{i=0}^{n}(-1)^i j_{2i+1}=\frac{(-1)^n\{j_{2n+3}+(a^2-b)^2 j_{2n+1}\}+2a(a^2-b+1)}{2(a^2+b)+(a^2-b)^2+1}$$

(iv) $$\sum_{i=0}^{n}(-1)^i j_{4i}=\frac{(-1)^n\{j_{4n+4}+(a^2-b)^4 j_{4n}\}+j_4+2}{j_4+(a^2-b)^4+1}$$

(v) $$\sum_{i=0}^{n}(-1)^i j_{4i+1}=\frac{(-1)^n\{j_{4n+5}+(a^2-b)^4 j_{4n+1}\}+(a^2-b)j_3+2a}{j_4+(a^2-b)^4+1}$$

(vi) $$\sum_{i=0}^{n}(-1)^i j_{4i+2}=\frac{(-1)^n\{j_{4n+6}+(a^2-b)^4 j_{4n+2}\}+\{(a^2-b)^2+1\}(2a^2+2b)}{j_4+(a^2-b)^4+1}$$

4. Generalized Identities of the Product of Generalized Jacobsthal and Jacobsthal-Lucas Sequences

Thongmoon [24, 25], defined various identities of Fibonacci and Lucas numbers. Singh, Bhadouria and Sikhwal [16], present some generalized identities involving common factors of Fibonacci and Lucas numbers. Gupta and Panwar [6], present identities involving common factors of generalized Fibonacci, Jacobsthal and jacobsthal-Lucas numbers. Panwar, Singh and Gupta ([14, 15]), present Generalized Identities Involving Common factors of generalized Fibonacci, Jacobsthal and jacobsthal-Lucas numbers. Singh, Sisodiya and Ahmed [18], investigate some products of k-Fibonacci and k-Lucas numbers, also present some generalized identities on the products of k-Fibonacci and k-Lucas numbers to establish connection formulas between them with the help of Binet's formula. Pakapongpun [12], present Jacobsthal-like sequence ([7]), also provide generalized identities on the products of Jacobsthal-like and Jacobsthal-Lucas numbers. Thongkam, Butsuwan and Bunya [23], present the investigation of products of (p, q)-Fibonacci-like and (p, q)-Lucas numbers. In this section, we present identities involving product of generalized Jacobsthal and Jacobsthal-Lucas sequences and related identities.

Theorem 4.1. If J_k *and* j_k are generalized Jacobsthal and Jacobsthal-Lucas sequences, then

$$J_{2k+p}j_{2k+1} = J_{4k+p+1} + (a^2-b)^{2k+1}J_{p-1}\ ,\ \textit{where } k \geq 0\ \&\ p \leq 0 \tag{4.1}$$

Proof. Applying Binet's formula of generalized Jacobsthal and Jacobsthal-Lucas sequences,

$$J_{2k+p}j_{2k+1} = \left(\frac{\Re_1^{2k+p}-\Re_2^{2k+p}}{\Re_1-\Re_2}\right)\left(\Re_1^{2k+1}+\Re_2^{2k+1}\right) \tag{4.2}$$

$$= \left(\frac{\Re_1^{4k+p+1}-\Re_2^{4k+p+1}}{\Re_1-\Re_2}\right) + \frac{\left(\Re_1\Re_2\right)^{2k}}{\left(\Re_1-\Re_2\right)}\left(\Re_1^p\Re_2-\Re_2^p\Re_1\right)$$

$$= \left(\frac{\Re_1^{4k+p+1}-\Re_2^{4k+p+1}}{\Re_1-\Re_2}\right) + \left(\Re_1\Re_2\right)^{2k}(a^2-b)\left(\frac{\Re_1^{p-1}-\Re_2^{p-1}}{\Re_1-\Re_2}\right)$$

$$= J_{4k+p+1} + (a^2-b)^{2k+1}J_{p-1}$$

This completes the proof.

Corollary 4.2. *For different values of p, (4.1) can be expressed as:*

(*i*) *If* $p=-3$, *then*: $J_{2k-3}j_{2k+1} = J_{4k-2} - (a^2-b)^{2k-3}(4a^3+4ab)$ (4.3)

(*ii*) *If* $p=-2$, *then*: $J_{2k-2}j_{2k+1} = J_{4k-1} - (a^2-b)^{2k-2}(3a^2+b)$ (4.4)

(*iii*) *If* $p=-1$, *then*: $J_{2k+2}j_{2k+1} = J_{4k} - 2a(a^2-b)^{2k-1}$ (4.5)

Following theorems can be solved by Binet's formula of new generalization of Fibonacci and Lucas numbers.

Theorem 4.3. $J_{2k+p}j_{2k+2} = J_{4k+p+2} + (a^2-b)^{2k+2}J_{p-2}\ ,\ \textit{where } k \geq 0\ \&\ p \leq 0$ (4.6)

Corollary 4.4. *For different values of p, (4.6) can be expressed as:*

(i) *If* $p=-3$, *then*: $J_{2k-3}j_{2k+2}=J_{4k-1}-(a^2-b)^{2k-3}J_5$ (4.7)

(ii) *If* $p=-2$, *then*: $J_{2k-2}j_{2k+2}=J_{4k}-(a^2-b)^{2k-2}J_4$ (4.8)

(iii) *If* $p=-1$, *then*: $J_{2k-1}j_{2k+2}=J_{4k+1}-(a^2-b)^{2n-1}J_3$ (4.9)

Theorem 4.5. $J_{2k+p}j_{2k}=J_{4k+p}+(a^2-b)^{2k}J_p$, *where* $k\geq 0$ & $p\leq 0$ (4.10)

Corollary 4.6. *For different values of p, (4.10) can be expressed as:*

(i) *If* $p=-3$, *then*: $J_{2k-3}j_{2k}=J_{4k-3}-(a^2-b)^{2n-3}J_3$ (4.11)

(ii) *If* $p=-2$, *then*: $J_{2k-2}j_{2k}=J_{4k-3}-2a(a^2-b)^{2k-2}$ (4.12)

(iii) *If* $p=-1$, *then*: $J_{2k-1}j_{2k}=J_{4k-1}-(a^2-b)^{2k-1}$ (4.13)

Theorem 4.7. $J_{2k-p}j_{2k+1}=J_{4k-p+1}+(a^2-b)^{2k+1}J_{-p-1}$, *where* $k\geq 0$ & $p\leq 0$ (4.14)

Corollary 4.8. *For different values of p, (4.14) can be expressed as:*

(i) *If* $p=-3$, *then*: $J_{2k+3}j_{2k+1}=J_{4k+4}-2a(a^2-b)^{2k-1}$ (4.15)

(ii) *If* $p=-2$, *then*: $J_{2k+2}j_{2k+1}=J_{4k+3}-(a^2-b)^{2k}$ (4.16)

(iii) *If* $p=-1$, *then*: $J_{2k+1}j_{2k+1}=J_{4k+2}$ (4.17)

Theorem 4.9. $J_{2k-p}j_{2k-1}=J_{4k-p-1}+(a^2-b)^{2k-1}J_{1-p}$, *where* $k\geq 0$ & $p\leq 0$ (4.18)

Corollary 4.10. *For different values of p, (4.18) can be expressed as:*

(i) *If* $p=-3$, *then*: $J_{2k+3}j_{2k-1}=J_{4k+2}+(a^2-b)^{2k-1}(4a^3+4ab)$ (4.19)

(ii) *If* $p=-2$, *then*: $J_{2k+2}j_{2k-1}=J_{4k+1}+(a^2-b)^{2k-1}(3a^2+b)$ (4.20)

(iii) *If* $p=-1$, *then*: $J_{2k+1}j_{2k-1}=J_{4k}+2a(a^2-b)^{2k-1}$ (4.21)

Theorem 4.11. $J_{2k-p}j_{2k}=J_{4k-p}+(a^2-b)^{2k}J_{-p}$, *where* $k\geq 0$ & $p\leq 0$ (4.22)

Corollary 4.12. *For different values of p, (4.22) can be expressed as:*

(i) *If* $p=-3$, *then*: $J_{2k+3}j_{2k}=J_{4k+3}+(a^2-b)^{2k}(3a^2+b)$ (4.23)

(ii) *If* $p=-2$, *then*: $J_{2k+2}j_{2k}=J_{4k+2}+2a(a^2-b)^{2k}$ (4.24)

(iii) *If* $p=-1$, *then*: $J_{2k+1}j_{2k}=J_{4k+1}+(a^2-b)^{2k}$ (4.25)

Theorem 4.13. $J_{2k} j_{2k+p} = J_{4k+p} - (a^2 - b)^{2k} J_p$, *where* $k \geq 0$ & $p \leq 0$ (4.26)

Corollary 4.14. *For different values of p, (4.26) can be expressed as:*

(i) *If* $p = -3$, *then*: $J_{2k} j_{2k-3} = J_{4n-3} + (a^2 - b)^{2k-3}(3a^2 + b)$ (4.27)

(ii) *If* $p = -2$, *then*: $J_{2k} j_{2k-2} = J_{4k-2} + 2a(a^2 - b)^{2k-2}$ (4.28)

(iii) *If* $p = -3$, *then*: $J_{2k} j_{2k-1} = J_{4k-1} + (a^2 - b)^{2k-1}$ (4.29)

Theorem 4.15. $4bJ_{2k} J_{2k+p} = j_{4k+p} - (a^2 - b)^{2k} j_p$, *where* $k \geq 0$ & $p \leq 0$ (4.30)

Corollary 4.16. *For different values of p, (4.30) can be expressed as:*

(i) *If* $p = -3$, *then*: $4bJ_{2k} J_{2k-3} = j_{4k-3} - (2a^3 + 6ab)(a^2 - b)^{2k-3}$ (4.31)

(ii) *If* $p = -2$, *then*: $4bJ_{2k} J_{2k-2} = j_{4k-2} - (2a^2 + 2b)(a^2 - b)^{2k-2}$ (4.32)

(iii) *If* $p = -1$, *then*: $4bJ_{2k} J_{2k-1} = j_{4k-1} - 2a(a^2 - b)^{2k-1}$ (4.33)

Theorem 4.17. $j_{2k} j_{2k+p} = j_{4k+p} + (a^2 - b)^{2k} j_p$, *where* $k \geq 0$ & $p \leq 0$ (4.34)

Corollary 4.18. *For different values of p, (4.34) can be expressed as:*

(i) *If* $p = -3$, *then*: $j_{2k} j_{2k-3} = j_{4k-3} + (2a^3 + 6ab)(a^2 - b)^{2k-3}$ (4.35)

(ii) *If* $p = -2$, *then*: $j_{2k} j_{2k-2} = j_{4k-2} + (2a^2 + 2b)(a^2 - b)^{2k-2}$ (4.36)

(iii) *If* $p = -1$, *then*: $j_{2k} j_{2k-1} = j_{4k-1} + 2a(a^2 - b)^{2k-1}$ (4.37)

5. Generating function of the Generalized Jacobsthal and Jacobsthal-Lucas Sequences

Generating functions provide a powerful technique for solving linear homogeneous recurrence relations. Even though generating functions are typically used in conjunction with linear recurrence relations with constant coefficients, we will systematically make use of them for linear recurrence relations with non constant coefficients. In this paragraph, the generating function for generalized Jacobsthal and Jacobsthal-Lucas Sequences are given. As a result, generalized Jacobsthal and Jacobsthal-Lucas Sequences are seen as the coefficients of the corresponding generating function. Function defined in such a way is called the generating function of the generalized Jacobsthal and Jacobsthal-Lucas Sequences. So,

Theorem 5.1. The generating functions of the generalized Jacobsthal and Jacobsthal-Lucas sequences are given, respectively, by

(i) $$J(x) = \sum_{n=0}^{\infty} J_n x^n = \frac{x}{1 - 2ax - (b - a^2)x^2} \quad (5.1)$$

(ii) $$j(x)=\sum_{n=0}^{\infty} j_n x^n=\frac{2-2ax}{1-2ax-(b-a^2)x^2} \tag{5.2}$$

Proof. Applying the generating functions $J(x)$ and $j(x)$ can be written as $J(x)=\sum_{n=0}^{\infty} J_n x^n$ and $j(x)=\sum_{n=0}^{\infty} j_n x^n$. Then, we write

$$J(x)=J_0+xJ_1+x^2J_2+x^3J_3+...+x^nJ_n+...$$

and then $$2axJ(x)=2axJ_0+2ax^2J_1+2ax^3J_2+2ax^4J_3+...+2ax^{n+1}J_n+...$$

$$(b-a^2)x^2J(x)=(b-a^2)x^2J_0+(b-a^2)x^3J_1+(b-a^2)x^4J_2+...+(b-a^2)x^{n+2}J_n+...$$

$$\rightarrow \quad \{1-2ax-(b-a^2)x^2\}J(x)=x$$

$$\rightarrow \quad J(x)=\sum_{n=0}^{\infty} J_n x^n=\frac{x}{1-2ax-(b-a^2)x^2}$$

Similarly, we have

$$j(x)=j_0+xj_1+x^2j_2+x^3j_3+...+x^nj_n+...$$

and then $$2axj(x)=2axj_0+2ax^2j_1+2ax^3j_2+2ax^4j_3+...+2ax^{n+1}j_n+...$$

$$(b-a^2)x^2J(x)=(b-a^2)x^2j_0+(b-a^2)x^3j_1+(b-a^2)x^4j_2+...+(b-a^2)x^{n+2}j_n+...$$

$$\rightarrow \quad \{1-2ax-(b-a^2)x^2\}j(x)=2-2ax$$

$$\rightarrow \quad j(x)=\sum_{n=0}^{\infty} j_n x^n=\frac{2-2ax}{1-2ax-(b-a^2)x^2}$$

This completes the proof.

6. Conclusion

In this paper generalized Jacobsthal and Jacobsthal-Lucas sequences have been studied. Many of the properties of these sequences like Catalan’s identity, Cassini’s identity or Simpson’s identity, d’ocagnes’s identity are proved by simple algebra and Binet’s formula. We describe sums of generalized Jacobsthal and Jacobsthal-Lucas sequences. This enables us to give in a straightforward way several formulas for the sums of such generalized numbers. These identities can be used to develop new identities of numbers and polynomials. Also we present some generalized identities involving product of Jacobsthal and Jacobsthal-Lucas sequences. Finally we present the generating function of Jacobsthal and Jacobsthal-Lucas sequences.

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