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UNIVALENCE CRITERIA OF THE CERTAIN INTEGRAL OPERATORS

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Abstract. In this paper, we give some sufficient conditions for the univalence of some integral operators. For this, we use the Becker's and generalized version of the well known Ahlfor's and Becker's univalence criteria.

1. Introduction and preliminaries

Let A the class of analytic functions f in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$, normalized by $f(0) = 0 = f'(0) - 1$, of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + ... + a_n z^n + ... = z + \sum_{n=2}^{\infty} a_n z^n, a_n \in \mathbb{C}. \tag{1}$$

It is well-known that an analytic function $f : U \to \mathbb{C}$ is said to be univalent if the following condition is satisfied: $z_1 = z_2$ if $f(z_1) = f(z_2)$ or $f(z_1) \neq f(z_2)$ if $z_1 \neq z_2$.

We denote by S the subclass of A consisting of functions which are also univalent in U.

In recent years there have been many studies (see for example [2, 3, 5, 7]) on the univalence of the following integral operators

$$G_p(z) = \left\{ p \int_0^z t^{p-1} f'(t) dt \right\}^{1/p}, \tag{2}$$

$$G_{q_1,q_2,...,q_n,p}(z) = \left\{ p \int_0^z t^{p-1} \prod_{k=1}^{n} \left(\frac{f_k(t)}{t} \right)^{q_k} dt \right\}^{1/p}, \tag{3}$$

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and

$$G_q(z) = \left\{ q \int_0^z t^{q-1} \left(e^{f(t)} \right)^q dt \right\}^{1/q}, \tag{4}$$

where the functions f_k, $k = 1, 2, ..., n$ and f belong to the class A and the parameters $q_k, k = 1, 2, ..., n; p, q$ are complex numbers such that the integrals in (2) – (4) exist.

Furthermore, Breaz *et. al.* [4] have obtained various sufficient conditions for the univalence of the following integral operator

$$G_{n,\alpha}(z) = \left\{ (n\alpha + 1) \int_0^z \left(\prod_{k=1}^n f_k(t) \right)^\alpha dt \right\}^{1/(n\alpha+1)}, \tag{5}$$

where n is a natural number, α is a real number and functions $f_k \in A$, $k = 1, 2, ..., n$.

By Baricz and Frasin [1] was obtained some sufficient conditions for the univalence of the integral operators of type (3)-(5), when the functions f_k, $k = 1, 2, ..., n$ and f are the normalized Bessel functions of the first kind.

It is well known that, the Wright function is defined by the following infinite series:

$$W_{\lambda,\mu}(z) = \sum_{n=0}^\infty \frac{1}{\Gamma(\lambda n + \mu)} \frac{z^n}{n!}, \tag{6}$$

where Γ is Euler gamma function, $\lambda > -1, \mu \in \mathbb{C}$. This series is absolutely convergent in $\mathbb{C}$, when $\lambda > -1$ and absolutely convergent in open unit disk for $\lambda = -1$. Furthermore, for $\lambda > -1$ the Wright function $W_{\lambda,\mu}$ is an entire function. The Wright function was introduced by Wright in [13] and has appeared for the first time in the case $\lambda > 0$ in connection wit his investigation in the asymptotic theory of partitions.

Note that Wright function $W_{\lambda,\mu}$, defined by (6) does not belong to the class A. Thus, it is natural to consider the following two kinds of normalization of the Wright function

$$W^{(1)}_{\lambda,\mu}(z) := \Gamma(\mu)\, z W_{\lambda,\mu}(z) = \sum_{n=0}^\infty \frac{\Gamma(\mu)}{\Gamma(\lambda n + \mu)} \frac{z^{n+1}}{n!}, \lambda > -1, \mu > 0, z \in U$$

and

$$\begin{aligned} W^{(2)}_{\lambda,\mu}(z) \quad &: \quad = \Gamma(\lambda + \mu) \left[W_{\lambda,\mu}(z) - \frac{1}{\Gamma(\mu)} \right] = \sum_{n=1}^\infty \frac{\Gamma(\lambda + \mu)}{\Gamma(\lambda n + \lambda + \mu)} \frac{z^n}{n!}, \\ \lambda \quad &> \quad -1, \lambda + \mu > 0, z \in U \end{aligned}$$

Clearly can be written

$$W_{\lambda,\mu}^{(1)}(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\mu)}{\Gamma(\lambda(n-1)+\mu)} \frac{z^n}{(n-1)!}, z \in U, \lambda > -1, \mu > 0 \tag{7}$$

$$W_{\lambda,\mu}^{(2)}(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda(n-1)+\lambda+\mu)} \frac{z^n}{n!}, z \in U, \lambda > -1, \lambda+\mu > 0, \tag{8}$$

Note that

$$W_{1,p+1}^{(1)}(-z) = -J_p^{(1)}(z) = \Gamma(p+1) z^{1-p/2} J_p(2\sqrt{z}),$$

where J_p is Bessel function and J_p^1 is normalized Bessel function. It is well known that Bessel function first kind J_μ is defined as the particular solution of the second-order linear homogeneous differential equation (see, for example [11])

$$z^2 w''(z) + z w'(z) + (z^2 - \mu^2) w(z) = 0.$$

This is why, this equation is called Bessel differential equation.

Furthermore, we observe that $W_{\lambda,\mu}^{(1)}$ and $W_{\lambda,\mu}^{(2)}$ are satisfying the following relations:

$$\lambda z \left(W_{\lambda,\mu}^{(1)}(z) \right)' = (\mu - 1) W_{\lambda,\mu-1}^{(1)}(z) + (\lambda - \mu + 1) W_{\lambda,\mu}^{(1)}(z), \tag{9}$$

$$\lambda z \left(W_{\lambda,\mu}^{(2)}(z) \right)' = (\lambda - \mu + 1) W_{\lambda,\mu-1}^{(2)}(z) - (\mu - 1) W_{\lambda,\mu-1}^{(2)}(z) \tag{10}$$

$$z \left(W_{\lambda,\mu}^{(2)}(z) \right)' = W_{\lambda,\lambda+\mu}^{(1)}(z). \tag{11}$$

It can be easily shown that the functions $W_{1,\mu}^{(1)}$ and $W_{1,\mu}^{(2)}$ are satisfying the following differential equations, respectively,

$$4z^2 w''(z) + 4(\mu - 2) z w'(z) + (5 - 3\mu + z^2) w(z) = 0$$

and

$$4z^2 w'''(z) + 4(\mu + 1) w''(z) + (\mu + 2 + z^2) w'(z) = 0.$$

To prove our main results, we shall require the following well known lemmas.

Lemma 1. *[9] Let p and c be complex numbers such that* $\mathrm{Re}(p) > 0$ *and* $|c| \leq 1, c \neq -1$. *If the function $f \in A$ satisfies the inequality*

$$\left| c\,|z|^{2p} + (1 - |z|^{2p}) \frac{z f''(z)}{p f'(z)} \right| \leq 1 \text{ for all } z \in U,$$

then the function $G_p : U \to \mathbb{C}$ *defined by (2) is univalent in* U.

Lemma 2. *[10]. Let* $q \in \mathbb{C}$ *and* $a \in \mathbb{R}$ *such that* $\mathrm{Re}(q) \geq 1, a > 1$ *and* $2a\,|q| \leq 3\sqrt{3}$. *If* $f \in A$ *satisfies the inequality* $|z f'(z)| \leq a$ *for all* $z \in U$, *then the function* $G_p : U \to \mathbb{C}$ *defined by (4) univalent in* U.

Lemma 3. *[8]. Let $q \in \mathbb{C}$ such that* $\operatorname{Re}(q) > 0$ *. If the function $f \in A$ satisfies the inequality*

$$\frac{1-|z|^{2\operatorname{Re}(q)}}{\operatorname{Re}(q)}\left|\frac{zf''(z)}{f'(z)}\right| \leq 1$$

for all $z \in U$, then for all $p \in \mathbb{C}$ such that $\operatorname{Re}(p) \geq \operatorname{Re}(q)$*, the function defined by (2) is univalent in U.*

Furthermore, we shall need the following results.

Lemma 4. *[6]. Let $\lambda \geq 1$ and $\mu > x_0$, where $x_0 \cong 1.2581$ is the numerical root of the equation*

$$2x-(x+1)e^{1/(x+1)}+1=0. \tag{12}$$

Then, the following inequalities hold true for all $z \in U$.

$$\left|\frac{z(W_{\lambda,\mu}^{(1)}(z))'}{W_{\lambda,\mu}^{(1)}(z)}-1\right| \leq \frac{e^{1/(\mu+1)}}{(2\mu+1)-(\mu+1)e^{1/(\mu+1)}}, \tag{13}$$

$$\left|z(W_{\lambda,\mu}^{(1)}(z))'\right| \leq 1+\frac{1}{\mu}\left\{(\mu+2)e^{1/(\mu+1)}-(\mu+1)\right\}. \tag{14}$$

Lemma 5. *[6] Let $\lambda \geq 1$ and $\lambda+\mu > x_0$ where $x_0 \cong 1.2581$ is the numerical root of the equation (12). Then, the following inequalities hold true for all $z \in U$.*

$$\left|\frac{z(W_{\lambda,\mu}^{(2)}(z))'}{W_{\lambda,\mu}^{(2)}(z)}-1\right| \leq \frac{(\lambda+\mu+1)(e^{1/(\mu+1)}-1)}{(\lambda+\mu)-(\lambda+\mu+1)(e^{1/(\mu+1)}-1)},$$

$$\left|z(W_{\lambda,\mu}^{(2)}(z))'\right| \leq 1+\frac{\lambda+\mu+1}{\lambda+\mu}(e^{1/(\mu+1)}-1).$$

In this paper, we give various sufficient conditions for the univalence on the open unit disk of the integral operators of type (2)-(5), when the functions f_k, $k=1,2,...,n$ and f are the normalized Wright functions. Also in this study, we would like to show that the univalence of integral operators, which involve normalized Wright functions can be derived easily via some well-known univalence criteria.

2. Univalence of the integral operators involving normalized Wright functions

In this section of the paper, our main aim are give sufficient conditions for the integral operators of type (3) - (5), when the functions f_k, $k=1,2,...,n$ are normalized Wright functions with various parameters to be univalent in the open unit disk.

Let

$$G^{p,q_1,q_2,\ldots,q_n}_{\lambda,\mu_1,\mu_2,\ldots,\mu_n}(z)=\left\{p\int_0^z t^{p-1}\prod_{k=1}^{n}\left(\frac{W^{(1)}_{\lambda,\mu_k}(t)}{t}\right)^{q_k}dt\right\}^{1/p},z\in U, \tag{15}$$

$$\lambda>-1,\mu_k>0,k=1,2,...,n,$$

where the functions $W^{(1)}_{\lambda,\mu_k}$ are normalized Wright functions defined by (7).

On the univalence of the function $G^{p,q_1,q_2,\ldots,q_n}_{\lambda,\mu_1,\mu_2,\ldots,\mu_n}$, we give the following theorem.

Theorem 6. *Let $\lambda\geq 1$, n be a natural number and $q_1,q_2,...,q_n$ are nonzero complex numbers, $\mu>x_0$, where $\mu=\min\{\mu_k:k=1,2,...,n\}$, $x_0\cong 1.2581$ is numerical root of the equation (12) and p, c are complex numbers with* $\operatorname{Re}(p)>0$,

$$|c|\leq 1-\frac{e^{1/(\mu+1)}}{|p|\left[(2\mu+1)-(\mu+1)e^{1/(\mu+1)}\right]}\sum_{k=1}^{n}|q_k|\,.$$

Then, the function $G^{p,q_1,q_2,\ldots,q_n}_{\lambda,\mu_1,\mu_2,\ldots,\mu_n}:U\to\mathbb{C}$ defined by (15) is univalent in U.

Proof. Firstly, we define the function $G^{q_1,q_2,\ldots,q_n}_{\lambda,\mu_1,\mu_2,\ldots,\mu_n}:U\to\mathbb{C}$ by

$$G^{q_1,q_2,\ldots,q_n}_{\lambda,\mu_1,\mu_2,\ldots,\mu_n}(z)=\int_0^z\prod_{k=1}^{n}\left(\frac{W^{(1)}_{\lambda,\mu_k}(t)}{t}\right)^{q_k}dt. \tag{16}$$

We observe that, since for all $k=1,2,...,n$ we have $W^{(1)}_{\lambda,\mu_k}\in A$, clearly $G^{q_1,q_2,\ldots,q_n}_{\lambda,\mu_1,\mu_2,\ldots,\mu_n}\in A$. Also, from (16) it is easy to see that

$$\left(G^{q_1,q_2,\ldots,q_n}_{\lambda,\mu_1,\mu_2,\ldots,\mu_n}(z)\right)'=\prod_{k=1}^{n}\left(\frac{W^{(1)}_{\lambda,\mu_k}(z)}{z}\right)^{q_k}. \tag{17}$$

In this case, the integral operator (15), can be written as follows

$$G^{p,q_1,q_2,\ldots,q_n}_{\lambda,\mu_1,\mu_2,\ldots,\mu_n}(z)=\left\{p\int_0^z t^{p-1}\left(G^{q_1,q_2,\ldots,q_n}_{\lambda,\mu_1,\mu_2,\ldots,\mu_n}(t)\right)'dt\right\}^{1/p},z\in U, \tag{18}$$

$$\lambda>-1,\mu_k>0,k=1,2,...,n,$$

From the equation (17), by simple computation, we have

$$\frac{\left(G^{q_1,q_2,\ldots,q_n}_{\lambda,\mu_1,\mu_2,\ldots,\mu_n}(z)\right)''}{\left(G^{q_1,q_2,\ldots,q_n}_{\lambda,\mu_1,\mu_2,\ldots,\mu_n}(z)\right)'}=\sum_{k=1}^{n}q_k\frac{z\left(W^{(1)}_{\lambda,\mu_k}(z)\right)'-W^{(1)}_{\lambda,\mu_k}(z)}{zW^{(1)}_{\lambda,\mu_k}(z)}$$

and

$$\frac{z\left(G^{q_1,q_2,\ldots,q_n}_{\lambda,\mu_1,\mu_2,\ldots,\mu_n}(z)\right)''}{\left(G^{q_1,q_2,\ldots,q_n}_{\lambda,\mu_1,\mu_2,\ldots,\mu_n}(z)\right)'} = \sum_{k=1}^{n} q_k \left(\frac{z\left(W^{(1)}_{\lambda,\mu_k}(z)\right)'}{W^{(1)}_{\lambda,\mu_k}(z)} - 1\right). \tag{19}$$

Using triangly inequality to the (19), then applying the inequality (13) for each $\mu_k : k = 1, 2, ..., n$, we obtain

$$\begin{aligned}\left|\frac{z\left(G^{q_1,q_2,\ldots,q_n}_{\lambda,\mu_1,\mu_2,\ldots,\mu_n}(z)\right)''}{\left(G^{q_1,q_2,\ldots,q_n}_{\lambda,\mu_1,\mu_2,\ldots,\mu_n}(z)\right)'}\right| &\leq \sum_{k=1}^{n} |q_k| \left|\frac{z\left(W^{(1)}_{\lambda,\mu_k}(z)\right)'}{W^{(1)}_{\lambda,\mu_k}(z)} - 1\right| \\ &\leq \sum_{k=1}^{n} |q_k| \frac{e^{1/(\mu_k+1)}}{(2\mu_k+1) - (\mu_k+1)e^{1/(\mu_k+1)}}.\end{aligned}$$

Now, we define the function $g : (1.2581, +\infty) \to \mathbb{R}$ as follows:

$$g(x) = \frac{e^{1/(x+1)}}{(2x+1) - (x+1)e^{1/(x+1)}}. \tag{20}$$

It can be easily see that the function $g : (1.2581, +\infty) \to \mathbb{R}$, defined by (20) is decreasing. Consequently for all μ_k , $k = 1, 2, ..., n$, we have

$$\frac{e^{1/(\mu_k+1)}}{(2\mu_k+1) - (\mu_k+1)e^{1/(\mu_k+1)}} \leq \frac{e^{1/(\mu+1)}}{(2\mu+1) - (\mu+1)e^{1/(\mu+1)}}. \tag{21}$$

where $\mu = \min\{\mu_k : k = 1, 2, ..., n\}$.

By using triangle inequality, we obtain the following inequality

$$\begin{aligned}&\left|c|z|^{2p} + \left(1 - |z|^{2p}\right) \frac{z\left(G^{q_1,q_2,\ldots,q_n}_{\lambda,\mu_1,\mu_2,\ldots,\mu_n}(z)\right)''}{p\left(G^{q_1,q_2,\ldots,q_n}_{\lambda,\mu_1,\mu_2,\ldots,\mu_n}(z)\right)'}\right| \\ \leq\ & |c| + \frac{e^{1/(\mu+1)}}{(2\mu+1) - (\mu+1)e^{1/(\mu+1)}} \sum_{k=1}^{n} \left|\frac{q_k}{p}\right|.\end{aligned}$$

The right hand side expression in the above inequality is bounded by 1 if and only if the hypothesis of the theorem is satisfied. Hence,

$$\left|c|z|^{2p} + \left(1 - |z|^{2p}\right) \frac{z\left(G^{q_1,q_2,\ldots,q_n}_{\lambda,\mu_1,\mu_2,\ldots,\mu_n}(z)\right)''}{p\left(G^{q_1,q_2,\ldots,q_n}_{\lambda,\mu_1,\mu_2,\ldots,\mu_n}(z)\right)'}\right| \leq 1$$

under hypothesis of theorem. Thus, according to Lemma 1, the function defined by (18); so, the integral operator (15) is univalent in U.

With this the proof of Theorem 6 is completed. □

By setting $q_1 = q_2 = ... = q_n = q$ in Theorem 6, we obtain the following corollary.

Corollary 7. *Let* $\lambda \geq 1$ n *be a natural number and* q *be a nonzero complex number and* $\mu > x_0$*, where* $\mu = \min\{\mu_k : k = 1, 2, ..., n\}$, $x_0 \cong 1.2581$ *is the numerical root of the equation (12),* p *and* c *are complex numbers such that* $\mathrm{Re}(p) > 0$,$|c| < 1$ *and the following condition is satisfied:*

$$|c| \leq 1 - \frac{|q|\, n e^{1/(\mu+1)}}{|p| \left[(2\mu+1) - (\mu+1)e^{1/(\mu+1)}\right]}.$$

Then, the integral operator $G^{p,q}_{\lambda,\mu_1,\mu_2,...,\mu_n} : U \to \mathbb{C}$ *defined by*

$$G^{p,q}_{\lambda,\mu_1,\mu_2,...,\mu_n}(z) = \left\{ p \int\limits_0^z t^{p-1} \prod_{k=1}^{n} \left(\frac{W^{(1)}_{\lambda,\mu_k}(t)}{t} \right)^q dt \right\}^{1/p}$$

is univalent in U.

By taking $n = 1$ in Theorem 6, we immediately obtain the following result.

Corollary 8. *Let* $\lambda \geq 1$*and* $\mu > x_0$ *where* $x_0 \cong 1.2581$ *is numerical root of the equation (12). Moreover, suppose that* p, q *and* c *are complex numbers such that* $\mathrm{Re}(p) > 0$,$|c| < 1$ *and the following condition is satisfied:*

$$|c| \leq 1 - \frac{|q|\, e^{1/(\mu+1)}}{|p| \left[(2\mu+1) - (\mu+1)e^{1/(\mu+1)}\right]}.$$

Then, the integral operator $G^{p,q}_{\lambda,\mu} : U \to \mathbb{C}$ *defined by*

$$G^{p,q}_{\lambda,\mu}(z) = \left\{ p \int\limits_0^z t^{p-1} \left(\frac{W^{(1)}_{\lambda,\mu}(t)}{t} \right)^q dt \right\}^{1/p}$$

is univalent in U.

By taking $q = 1$ in Corollary 8, we have the following corollary.

Corollary 9. *Let* $\lambda \geq 1$*and* $\mu > x_0$*, where* $x_0 \cong 1.2581$ *is numerical root of the equation (12). Moreover, suppose that* p *and* c *are complex numbers such that* $\mathrm{Re}(p) > 0$,$|c| < 1$ *and the following condition is satisfied:*

$$|c| \leq 1 - \frac{e^{1/(\mu+1)}}{|p| \left[(2\mu+1) - (\mu+1)e^{1/(\mu+1)}\right]}.$$

Then, the integral operator $G^{p}_{\lambda,\mu} : U \to \mathbb{C}$ *defined by*

$$G^{p}_{\lambda,\mu}(z) = \left\{ p \int\limits_0^z t^{p-2} W^{(1)}_{\lambda,\mu}(t) dt \right\}^{1/p} \tag{22}$$

is univalent in U.

Remark 10. *Note that, recently the function* $G^{p}_{\lambda,\mu} : U \to \mathbb{C}$ *defined by (22) was investigated by Prajapat [12] and obtained some sufficient conditions for the univalence of this function.*

On the univalence of the integral operator

$$F^{p,q_1,q_2,...,q_n}_{\lambda,\mu_1,\mu_2,...,\mu_n}(z) = \left\{ p \int\limits_0^z t^{p-1} \prod_{k=1}^{n} \left(\frac{W^{(2)}_{\lambda,\mu_k}(t)}{t} \right)^{q_k} dt \right\}^{1/p}, \tag{23}$$

$$\lambda > -1, \lambda + \mu_k > 0, k = 1, 2, ..., n, \ z \in U,$$

where the functions $W^{(2)}_{\lambda,\mu_k}(z)$ are normalized Wright functions defined by (8).

Theorem 11. *Let* $\lambda \geq 1$*,* n *be a natural number and* $q_1, q_2, ..., q_n$ *are nonzero complex numbers,* $\lambda + \mu > x_0$*, where* $\mu = \min\{\mu_k : k = 1, 2, ..., n\}$*,* $x_0 \cong 1.2581$ *is numerical root of the equation (12). Moreover, suppose that* p *and* c *are complex numbers such that* $\mathrm{Re}(p) > 0$*,* $|c| < 1$ *and the following condition is satisfied:*

$$|c| \leq 1 - \frac{(\lambda + \mu + 1)(e^{1/(\mu+1)} - 1)}{|p| \left[(\lambda + \mu) - (\lambda + \mu + 1)(e^{1/(\mu+1)} - 1) \right]} \sum_{k=1}^{n} |q_k| .$$

Then, the integral operator $F^{p,q_1,q_2,...,q_n}_{\lambda,\mu_1,\mu_2,...,\mu_n} : U \to \mathbb{C}$ defined by (23) is univalent in U.

Proof. The proof of Theorem 11 is similar to the proof of Theorem 6. Hence, the details of the proof of this theorem may be omitted. □

By setting $q_1 = q_2 = ... = q_n = q$ in Theorem 11, we arrive at the following corollary.

Corollary 12. *Let* $\lambda \geq 1$*,* n *be a natural number and* q *be a nonzero complex number,* $\lambda + \mu > x_0$*, where* $\mu = \min\{\mu_k : k = 1, 2, ..., n\}$*,* $x_0 \cong 1.2581$ *is numerical root of the equation (12). Moreover, suppose that* p *and* c *are complex numbers such that* $\mathrm{Re}(p) > 0$*,* $|c| < 1$ *and the following condition is satisfied:*

$$|c| \leq 1 - \frac{|q|\, n(\lambda + \mu + 1)(e^{1/(\mu+1)} - 1)}{|p| \left[(\lambda + \mu) - (\lambda + \mu + 1)(e^{1/(\mu+1)} - 1) \right]}.$$

Then, the integral operator $F^{p,q}_{\lambda,\mu_1,\mu_2,...,\mu_n} : U \to \mathbb{C}$ *defined by*

$$F^{p,q}_{\lambda,\mu_1,\mu_2,...,\mu_n}(z) = \left\{ p \int\limits_0^z t^{p-1} \prod_{k=1}^{n} \left(\frac{W^{(2)}_{\lambda,\mu_k}(t)}{t} \right)^{q} dt \right\}^{1/p}$$

is univalent in U.

Taking $n = 1$ in Theorem 11, we immediately obtain the following result.

Corollary 13. *Let* $\lambda \geq 1$ *and* $\lambda + \mu > x_0$*, where* $x_0 \cong 1.2581$ *is numerical root of the equation (12). Moreover, suppose that* p*,* q *and* c *are complex numbers such that* $\mathrm{Re}(p) > 0$*,* $|c| < 1$ *and the following condition is satisfied:*

$$|c| \leq 1 - \frac{|q|\,(\lambda + \mu + 1)(e^{1/(\mu+1)} - 1)}{|p|\left[(\lambda + \mu) - (\lambda + \mu + 1)(e^{1/(\mu+1)} - 1)\right]}.$$

Then, the integral operator $F^{p,q}_{\lambda,\mu} : U \to \mathbb{C}$ *defined by*

$$F^{p,q}_{\lambda,\mu}(z) = \left\{ p \int_0^z t^{p-1} \left(\frac{W^{(2)}_{\lambda,\mu}(t)}{t} \right)^q dt \right\}^{1/p}$$

is univalent in U*.*

By taking $q = 1$, from the Corollary 13, we have the following corollary.

Corollary 14. *Let* $\lambda \geq 1$ *and* $\lambda + \mu > x_0$*, where* $x_0 \cong 1.2581$ *is numerical root of the equation (12). Moreover, suppose that* p *and* c *are complex numbers such that* $\mathrm{Re}(p) > 0$*,* $|c| < 1$ *and the following condition is satisfied:*

$$|c| \leq 1 - \frac{(\lambda + \mu + 1)(e^{1/(\mu+1)} - 1)}{|p|\left[(\lambda + \mu) - (\lambda + \mu + 1)(e^{1/(\mu+1)} - 1)\right]}.$$

Then, the integral operator $F^{p}_{\lambda,\mu} : U \to \mathbb{C}$ *defined by*

$$F^{p}_{\lambda,\mu}(z) = \left\{ p \int_0^z t^{p-2} W^{(2)}_{\lambda,\mu}(t) dt \right\}^{1/p}$$

is univalent in U*.*

Now let

$$H^{n,\alpha}_{\lambda,\mu_1,\mu_2,\ldots,\mu_n}(z) = \left\{ (n\alpha + 1) \int_0^z \prod_{k=1}^{n} \left(W^{(1)}_{\lambda,\mu_k}(t) \right)^{\alpha} dt \right\}^{1/(n\alpha+1)}, \qquad (24)$$

$$\lambda > -1, \mu_k > 0, k = 1, 2, ..., n, z \in U,$$

where the functions $W^{(1)}_{\lambda,\mu_k}$ are normalized Wright functions defined by (7).

The following theorem contains sufficient condition for the integral operator (24) to be univalent in the open unit disk U.

Theorem 15. *Let* $\lambda \geq 1$*,* n *be a natural number,* $\mu > x_0$*, where* $\mu = \min\{\mu_k : k = 1, 2, ..., n\}$ *and* x_0 *is numerical root of the equation (12). Moreover, suppose that* α *be a complex numbers such that* $\mathrm{Re}(p) > 0$ *and the following condition is satisfied:*

$$|\alpha| \leq \frac{(2\mu + 1) - (\mu + 1)e^{1/(\mu+1)}}{n e^{1/(\mu+1)}} \mathrm{Re}(\alpha).$$

Then, the integral operator $H^{n,\alpha}_{\lambda,\mu_1,\mu_2,\ldots,\mu_n} : U \to \mathbb{C}$ *defined by (24) is univalent in* U.

Proof. Let us define the function $H^{\alpha}_{\lambda,\mu_1,\mu_2,\ldots,\mu_n} : U \to \mathbb{C}$ by

$$H^{\alpha}_{\lambda,\mu_1,\mu_2,\ldots,\mu_n}(z) = \int_0^z \prod_{k=1}^{n} \left(\frac{W^{(1)}_{\lambda,\mu_k}(t)}{t} \right)^{\alpha} dt. \tag{25}$$

We can easily see that $H^{\alpha}_{\lambda,\mu_1,\mu_2,\ldots,\mu_n} \in A$. Also, from (25) it is easy to see that

$$\left(H^{\alpha}_{\lambda,\mu_1,\mu_2,\ldots,\mu_n}(z) \right)' = \prod_{k=1}^{n} \left(\frac{W^{(1)}_{\lambda,\mu_k}(z)}{z} \right)^{\alpha}. \tag{26}$$

By simple computation, it follows from (26) that

$$\frac{z\left(H^{\alpha}_{\lambda,\mu_1,\mu_2,\ldots,\mu_n}(z) \right)''}{\left(H^{\alpha}_{\lambda,\mu_1,\mu_2,\ldots,\mu_n}(z) \right)'} = \sum_{k=1}^{n} \alpha \left(\frac{z\left(W^{(1)}_{\lambda,\mu_k}(z) \right)' - zW^{(1)}_{\lambda,\mu_k}(z)}{zW^{(1)}_{\lambda,\mu_k}(z)} \right).$$

That is,

$$\frac{z\left(H^{\alpha}_{\lambda,\mu_1,\mu_2,\ldots,\mu_n}(z) \right)''}{\left(H^{\alpha}_{\lambda,\mu_1,\mu_2,\ldots,\mu_n}(z) \right)'} = \sum_{k=1}^{n} \alpha \left(\frac{z\left(W^{(1)}_{\lambda,\mu_k}(z) \right)'}{W^{(1)}_{\lambda,\mu_k}(z)} - 1 \right).$$

Using in the last equality firstly triangle inequality and then applying the inequality (13) gives the following inequality

$$\begin{aligned} \left| \frac{z\left(H^{\alpha}_{\lambda,\mu_1,\mu_2,\ldots,\mu_n}(z) \right)''}{\left(H^{\alpha}_{\lambda,\mu_1,\mu_2,\ldots,\mu_n}(z) \right)'} \right| &\leq \sum_{k=1}^{n} |\alpha| \left(\frac{z\left(W^{(1)}_{\lambda,\mu_k}(z) \right)'}{W^{(1)}_{\lambda,\mu_k}(z)} - 1 \right) \\ &\leq \sum_{k=1}^{n} |\alpha| \frac{e^{1/(\mu_k+1)}}{(2\mu_k+1) - (\mu_k+1)e^{1/(\mu_k+1)}}. \end{aligned}$$

On the other hand, by using (21), we obtain that

$$\frac{1 - |z|^{2\,\mathrm{Re}(\alpha)}}{\mathrm{Re}(\alpha)} \left| \frac{z\left(H^{\alpha}_{\lambda,\mu_1,\mu_2,\ldots,\mu_n}(z) \right)''}{\left(H^{\alpha}_{\lambda,\mu_1,\mu_2,\ldots,\mu_n}(z) \right)'} \right| \leq \frac{|\alpha|\, n}{\mathrm{Re}(\alpha)} \frac{e^{1/(\mu+1)}}{(2\mu+1) - (\mu+1)e^{1/(\mu+1)}}$$

for all $z \in U$.

Under hypothesis of theorem right hand side expression of the above inequality is bounded by 1. Since the function $H^{n,\alpha}_{\lambda,\mu_1,\mu_2,\ldots,\mu_n}$ can be rewritten in the form

$$H^{n,\alpha}_{\lambda,\mu_1,\mu_2,\ldots,\mu_n}(z) = \left\{ (n\alpha+1) \int_0^z t^{na} \left(H^{\alpha}_{\lambda,\mu_1,\mu_2,\ldots,\mu_n}(z) \right)' dt \right\}^{1/(na+1)}$$

and $\mathrm{Re}(n\alpha+1) > \mathrm{Re}(\alpha)$, applying Lemma 3, we obtain the required result. Thus, the proof of Theorem 15 is completed. □

By setting $n = 1$ in Theorem 15, we arrive at the following corollary.

Corollary 16. *Let $\lambda \geq 1$ and $\mu > x_0$, where x_0 is numerical root of the equation (12). Moreover, suppose that α be a complex number with $\mathrm{Re}(\alpha) > 0$ and the following condition is satisfied: $e^{1/(\mu+1)} |\alpha| \leq \left((2\mu+1) - (\mu+1)e^{1/(\mu+1)}\right) \mathrm{Re}(\alpha)$. Then, the integral operator $H^{\alpha}_{\lambda,\mu} : U \to \mathbb{C}$ defined by*

$$H^{\alpha}_{\lambda,\mu}(z) = \left\{ (\alpha+1) \int_0^z \left(W^{(1)}_{\lambda,\mu}(t) \right)^{\alpha} dt \right\}^{1/(\alpha+1)}$$

is univalent in U.

By taking $\alpha = 1$ in Corollary 16, we have the following result.

Corollary 17. *Let $\lambda \geq 1$ and $\mu > x_0$, where x_0 is numerical root of the equation (12). Then, the integral operator $H_{\lambda,\mu} : U \to \mathbb{C}$ defined by*

$$H_{\lambda,\mu}(z) = \sqrt{2} \left\{ \int_0^z W^{(1)}_{\lambda,\mu}(t) dt \right\}^{1/2}$$

is univalent in U.

Notation 18. *For the integral operator (24), when the function $W^{(1)}_{\lambda,\mu_k}$ is normalized Wright function $W^{(2)}_{\lambda,\mu_k}$ defined by (8) can be proved similar result.*

Finally, we give the following theorem, which contain another sufficient conditions for the integral operator

$$\mathbb{Q}^q_{\lambda,\mu}(z) = \left\{ q \int_0^z t^{q-1} \left(e^{W^{(1)}_{\lambda,\mu}(t)} \right)^q dt \right\}^{1/q}, \lambda > -1, \lambda+\mu > 0, \ z \in U \quad (27)$$

to be univalent in the open unit disk U.

Theorem 19. *Let $q \in \mathbb{C}$, $\lambda \geq 1$ and $\mu > x_0$, where $x_0 \cong 1.2581$ is numerical root of the equation (12). If $\mathrm{Re}(q) > 0$ and the following condition is satisfied*

$$|q| \leq \frac{3\sqrt{3}\mu}{2\left[(\mu+2)e^{1/(\mu+1)} - 1\right]}, \quad (28)$$

then the function $\mathbb{Q}^q_{\lambda,\mu}: U \to \mathbb{C}$ defined by (27) is univalent in U.

Proof. It follows from (14) that

$$\left| z\left(W^{(1)}_{\lambda,\mu}(z)\right)' \right| \leq 1+\frac{1}{\mu}\left\{(\mu+2)e^{1/(\mu+1)}-(\mu+1)\right\}$$

for all $z \in U$.

Taking

$$\alpha = 1+\frac{1}{\mu}\left\{(\mu+2)e^{1/(\mu+1)}-(\mu+1)\right\},$$

we easily see that $2\alpha|q| \leq 3\sqrt{3}$ if provided (28). Thus, under hypothesis of theorem, all hypothesis of the Lemma 2 is provided.

Hence, the proof of Theorem 19 is completed. □

B setting $q=1$ in Theorem 19, we have the following result.

Corollary 20. *Let $\lambda \geq 1$ and $\mu > x_1$, where $x_1 \cong 1.6692$ is the numerical root of the equation*

$$3\sqrt{3}x - 2(x+2)e^{1/(x+1)} + 2 = 0. \tag{29}$$

Then, the function $\mathbb{Q}_{\lambda,\mu}: U \to \mathbb{C}$ defined by

$$\mathbb{Q}_{\lambda,\mu}(z) = \int_0^z e^{W^{(1)}_{\lambda,\mu}(t)}dt$$

is univalent in U.

Now, let

$$D^q_{\lambda,\mu}(z) = \left\{ q\int_0^z t^{q-1}\left(e^{W^{(2)}_{\lambda,\mu}(t)}\right)^q dt \right\}^{1/q}, \lambda > -1, \lambda+\mu > 0,\ z \in U. \tag{30}$$

For the function (30), we give the following theorem which will be proved similarly to Theorem 19.

Theorem 21. *Let $q \in \mathbb{C}$, $\lambda \geq 1$ and $\lambda+\mu > x_0$, where $x_0 \cong 1.2581$ is the root of the equation (12). If $\operatorname{Re}(q) > 1$ and the following condition is satisfied*

$$|q| \leq \frac{3\sqrt{3}(\lambda+\mu)}{2\left[(\lambda+\mu+1)e^{1/(\lambda+\mu+1)}-1\right]},$$

then the function $D^q_{\lambda,\mu}: U \to \mathbb{C}$ defined by (30) is univalent in U.

By setting $q=1$ in Theorem 21, we obtain the following corollary.

Corollary 22. *Let* $\lambda \geq 1$ *and* $\lambda + \mu > x_2$*, where* $x_2 \cong 0.83232$ *is numerical root of the equation*

$$3\sqrt{3}x - 2(x+1)e^{1/(x+1)} + 2 = 0.$$

Then the function $D^q_{\lambda,\mu} : U \to \mathbb{C}$ *defined by*

$$D_{\lambda,\mu}(z) = \int_0^z e^{W^{(2)}_{\lambda,\mu}(t)} dt$$

is univalent in U.

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