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Research Article

Various spectra and energies of commuting graphs of finite rings

Walaa Nabil Taha Fasfous[1], Rajat Kanti Nath*[1], Reza Sharafdini[2]

[1] *Department of Mathematical Sciences, Tezpur University, Napaam-784028, Sonitpur, Assam, India*
[2] *Department of Mathematics, Faculty of Science, Persian Gulf University, Bushehr 75169-13817, Iran*

Abstract

The commuting graph of a non-commutative ring R with center $Z(R)$ is a simple undirected graph whose vertex set is $R \setminus Z(R)$ and two vertices x, y are adjacent if and only if $xy = yx$. In this paper, we compute various spectra and energies of commuting graphs of some classes of finite rings and study their consequences.

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1. Introduction

Let R be a non-commutative ring with center $Z(R)$. The commuting graph of R, denoted by Γ_R, is a simple undirected graph whose vertex set is $R \setminus Z(R)$ and two vertices x, y are adjacent if and only if $xy = yx$. In recent years, many mathematicians have considered commuting graph of different rings and studied various graph theoretic aspects (see [1, 3, 12, 13, 17, 19, 20, 23]). Some generalizations of Γ_R are also considered in [2, 9].

In Section 2, we compute spectrum, Laplacian spectrum and Signless Laplacian spectrum of commuting graphs of some classes of finite rings. Recall that the spectrum of a graph $\mathcal{G}$ denoted by Spec($\mathcal{G}$) is the set

$$\left\{\lambda_1^{k_1}, \lambda_2^{k_2}, \ldots, \lambda_n^{k_n}\right\},$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of the adjacency matrix of $\mathcal{G}$ with multiplicities $k_1, k_2, \ldots, k_n$ respectively. Let $A(\mathcal{G})$ and $D(\mathcal{G})$ denote the adjacency matrix and degree matrix of a graph $\mathcal{G}$ respectively. Then the Laplacian matrix and Signless Laplacian matrix of $\mathcal{G}$ are given by $L(\mathcal{G}) = D(\mathcal{G}) - A(\mathcal{G})$ and $Q(\mathcal{G}) = D(\mathcal{G}) + A(\mathcal{G})$ respectively. Let L-spec($\mathcal{G}$) and Q-spec($\mathcal{G}$) be the Laplacian spectrum and Signless Laplacian spectrum of $\mathcal{G}$ respectively. Then L-spec$(\mathcal{G}) = \left\{\beta_1^{b_1}, \beta_2^{b_2}, \ldots, \beta_m^{b_m}\right\}$ and Q-spec$(\mathcal{G}) = \{\gamma_1^{c_1}, \gamma_2^{c_2}, \ldots, \gamma_n^{c_n}\}$, where $\beta_1, \beta_2, \ldots, \beta_m$ are the eigenvalues of $L(\mathcal{G})$ with multiplicities $b_1, b_2, \ldots, b_m$ and $\gamma_1, \gamma_2, \ldots, \gamma_n$ are the eigenvalues of $Q(\mathcal{G})$ with multiplicities $c_1, c_2, \ldots, c_n$ respectively. The energy, Laplacian energy and Signless Laplacian energy of a graph $\mathcal{G}$ are given by

*Corresponding Author.
Email addresses: w.n.fasfous@gmail.com (W.N.T. Fasfous), rajatkantinath@yahoo.com (R.K. Nath), sharafdini@pgu.ac.ir (R. Sharafdini)
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$$\begin{aligned} E(\mathcal{G}) &= \sum_{\lambda \in \text{Spec}(\mathcal{G})} |\lambda|, \\ LE(\mathcal{G}) &= \sum_{\mu \in \text{L-spec}(\mathcal{G})} \left|\mu - \frac{2|e(\mathcal{G})|}{|v(\mathcal{G})|}\right| \text{ and} \\ LE^{+}(\mathcal{G}) &= \sum_{\nu \in \text{Q-spec}(\mathcal{G})} \left|\nu - \frac{2|e(\mathcal{G})|}{|v(\mathcal{G})|}\right|, \end{aligned} \tag{1.1}$$

where $v(\mathcal{G})$ and $e(\mathcal{G})$ are the set of vertices and edges of $\mathcal{G}$, respectively.

Throughout the paper R denotes a non-commutative finite ring and p, q denote distinct primes. $\frac{R}{Z(R)}$ denotes the additive quotient group. Also, K_n denotes a complete graph on n vertices and lK_n denotes the disjoint union of l copies of K_n.

2. Various spectra

In [19], various spectra of commuting graphs of some small order finite non-commutative rings have been computed. In this section we consider more classes of finite non-commutative rings. The following theorem is useful in computing various spectra of commuting graphs of finite rings.

Theorem 2.1 ([18, Theorem 2.1])**.** *If $\mathcal{G} = l_1K_{n_1} \sqcup l_2K_{n_2} \sqcup \cdots \sqcup l_mK_{n_m}$, then*

(a) $\text{Spec}(\mathcal{G}) = \left\{(-1)^{\sum_{i=1}^{m} l_i(n_i-1)}, (n_1-1)^{l_1}, (n_2-1)^{l_2}, \ldots, (n_m-1)^{l_m}\right\}$.

(b) $\text{L-spec}(\mathcal{G}) = \left\{0^{\sum_{i=1}^{m} l_i}, n_1^{l_1(n_1-1)}, n_2^{l_2(n_2-1)}, n_m^{l_m(n_m-1)}\right\}$.

(c) $\text{Q-spec}(\mathcal{G}) = \Big\{(2n_1-2)^{l_1}, (n_1-2)^{l_1(n_1-1)}, (2n_2-2)^{l_2}, (n_2-2)^{l_2(n_2-1)}, \ldots,$
$(2n_m-2)^{l_m}, (n_m-2)^{l_m(n_m-1)}\Big\}$.

Theorem 2.2. *Let $|R| = p^4$ and R has unity.*

(a) *If $|Z(R)| = p$ then* $\text{Spec}(\Gamma_R) = \left\{(-1)^{(p^2+p+1)(p^2-p-1)}, (p^2-p-1)^{p^2+p+1}\right\}$,
$\text{L-spec}(\Gamma_R) = \left\{0^{p^2+p+1}, (p^2-p)^{(p^2+p+1)(p^2-p-1)}\right\}$ *and*
$\text{Q-spec}(\Gamma_R) = \left\{(2p^2-2p-2)^{p^2+p+1}, (p^2-p-2)^{(p^2+p+1)(p^2-p-1)}\right\}$; *or*
$\text{Spec}(\Gamma_R) = \left\{(-1)^{l_1(p^2-p-1)+l_2(p^3-p-1)}, (p^2-p-1)^{l_1}, (p^3-p-1)^{l_2}\right\}$,
$\text{L-spec}(\Gamma_R) = \left\{0^{l_1+l_2}, (p^2-p)^{l_1(p^2-p-1)}, (p^3-p)^{l_2(p^3-p-1)}\right\}$ *and* $\text{Q-spec}(\Gamma_R) =$
$\left\{(2p^2-2p-2)^{l_1}, (p^2-p-2)^{l_1(p^2-p-1)}, (2p^3-2p-2)^{l_2}, (p^3-p-2)^{l_2(p^3-p-1)}\right\}$,
where $l_1 + l_2(p+1) = p^2+p+1$.

(b) *If $|Z(R)| = p^2$ then* $\text{Spec}(\Gamma_R) = \left\{(-1)^{(p+1)(p^3-p^2-1)}, (p^3-p^2-1)^{p+1}\right\}$,
$\text{L-spec}(\Gamma_R) = \left\{0^{p+1}, (p^3-p^2)^{(p+1)(p^3-p^2-1)}\right\}$ *and*
$\text{Q-spec}(\Gamma_R) = \left\{(2p^3-2p^2-2)^{p+1}, (p^3-p^2-2)^{(p+1)(p^3-p^2-1)}\right\}$.

Proof. (a) If $|Z(R)| = p$ then, by Theorem 2.5 of [22], we have $\Gamma_R = (p^2+p+1)K_{(p^2-p)}$ or $l_1K_{(p^2-p)} \sqcup l_2K_{(p^3-p)}$, where $l_1 + l_2(p+1) = p^2+p+1$. Hence, the result follows from Theorem 2.1.

(b) If $|Z(R)| = p^2$ then, by Theorem 2.5 of [22], we have $\Gamma_R = (p+1)K_{(p^3-p^2)}$. Hence, the result follows from Theorem 2.1. □

Theorem 2.3. *Let $|R| = p^5$ with unity and $Z(R)$ is not a field.*

(a) *If* $|Z(R)| = p^2$ *then* $\text{Spec}(\Gamma_R) = \left\{(-1)^{(p^2+p+1)(p^3-p^2-1)}, (p^3-p^2-1)^{p^2+p+1}\right\}$, $\text{L-spec}(\Gamma_R) = \left\{0^{p^2+p+1}, (p^3-p^2)^{(p^2+p+1)(p^3-p^2-1)}\right\}$ *and* $\text{Q-spec}(\Gamma_R) = \left\{(2p^3-2p^2-2)^{p^2+p+1}, (p^3-p^2-2)^{(p^2+p+1)(p^3-p^2-1)}\right\}$; *or* $\text{Spec}(\Gamma_R) = \left\{(-1)^{l_1(p^3-p^2-1)+l_2(p^3-p-1)}, (p^3-p^2-1)^{l_1}, (p^3-p-1)^{l_2}\right\}$, $\text{L-spec}(\Gamma_R) = \left\{0^{l_1+l_2}, \ (p^3-p^2)^{l_1(p^3-p^2-1)}, \ (p^3-p)^{l_2(p^3-p-1)}\right\}$ *and* $\text{Q-spec}(\Gamma_R) = \left\{(2p^3-2p^2-2)^{l_1}, (p^3-p^2-2)^{l_1(p^3-p^2-1)}, (2p^3-2p-2)^{l_2}, (p^3-p-2)^{l_2(p^3-p-1)}\right\}$, *where* $l_1 + l_2(p+1) = p^2+p+1$.

(b) *If* $|Z(R)| = p^3$ *then* $\text{Spec}(\Gamma_R) = \left\{(-1)^{(p+1)(p^4-p^3-1)}, (p^4-p^3-1)^{p+1}\right\}$, $\text{L-spec}(\Gamma_R) = \left\{0^{p+1}, (p^4-p^3)^{(p+1)(p^4-p^3-1)}\right\}$ *and* $\text{Q-spec}(\Gamma_R) = \left\{(2p^4-2p^3-2)^{p+1}, (p^4-p^3-2)^{(p+1)(p^4-p^3-1)}\right\}$.

Proof. (a) If $|Z(R)| = p^2$ then, by Theorem 2.7 of [22], we have $\Gamma_R = (p^2+p+1)K_{(p^3-p^2)}$ or $l_1K_{(p^3-p^2)} \sqcup l_2K_{(p^3-p)}$, where $l_1 + l_2(p+1) = p^2+p+1$. Hence, the result follows from Theorem 2.1.

(b) If $|Z(R)| = p^3$ then, by Theorem 2.7 of [22], we have $\Gamma_R = (p+1)K_{(p^4-p^3)}$. Hence, the result follows from Theorem 2.1. □

Theorem 2.4. *Let $|R| = pq$ and $Z(R) = \{0\}$.*

(a) *If* $(p-1) \mid (pq-1)$ *then* $\text{Spec}(\Gamma_R) = \left\{(-1)^{\frac{(pq-1)(p-2)}{p-1}}, (p-2)^{\frac{pq-1}{p-1}}\right\}$, $\text{L-spec}(\Gamma_R) = \left\{0^{\frac{pq-1}{p-1}}, (p-1)^{\frac{(pq-1)(p-2)}{p-1}}\right\}$ *and* $\text{Q-spec}(\Gamma_R) = \left\{(2p-4)^{\frac{pq-1}{p-1}}, (p-3)^{\frac{(pq-1)(p-2)}{p-1}}\right\}$.

(b) *If* $(q-1) \mid (pq-1)$ *then* $\text{Spec}(\Gamma_R) = \left\{(-1)^{\frac{(pq-1)(q-2)}{q-1}}, (q-2)^{\frac{pq-1}{q-1}}\right\}$, $\text{L-spec}(\Gamma_R) = \left\{0^{\frac{pq-1}{q-1}}, (q-1)^{\frac{(pq-1)(q-2)}{q-1}}\right\}$ *and* $\text{Q-spec}(\Gamma_R) = \left\{(2q-4)^{\frac{pq-1}{q-1}}, (q-3)^{\frac{(pq-1)(q-2)}{q-1}}\right\}$.

(c) *If* $l_1(p-1) + l_2(q-1) = pq-1$ *then* $\text{Spec}(\Gamma_R) = \left\{(-1)^{l_1(p-2)+l_2(q-2)}, (p-2)^{l_1}, (q-2)^{l_2}\right\}$, $\text{L-spec}(\Gamma_R) = \left\{0^{l_1+l_2}, (p-1)^{l_1(p-2)}, (q-1)^{l_2(q-2)}\right\}$ *and* $\text{Q-spec}(\Gamma_R) = \left\{(2p-4)^{l_1}, (p-3)^{l_1(p-2)}, (2q-4)^{l_2}, (q-3)^{l_2(q-2)}\right\}$.

Proof. It was shown in [23, Theorem 2.8] that

$$\Gamma_R = \begin{cases} \frac{pq-1}{p-1}K_{p-1}, & \text{if } (p-1) \mid (pq-1) \\ \frac{pq-1}{q-1}K_{q-1}, & \text{if } (q-1) \mid (pq-1) \\ l_1K_{p-1} \sqcup l_2K_{q-1}, & \text{if } l_1(p-1) + l_2(q-1) = pq-1. \end{cases}$$

Hence, the result follows from Theorem 2.1. □

Theorem 2.5. *Let $|R| = p^2q$ and $Z(R) = \{0\}$.*

(a) *If* $t \in \{p, q, p^2, pq\}$ *and* $(t-1) \mid (p^2q-1)$ *then* $\text{Spec}(\Gamma_R) = \left\{(-1)^{\frac{(p^2q-1)(t-2)}{t-1}}, (t-2)^{\frac{p^2q-1}{t-1}}\right\}$, $\text{L-spec}(\Gamma_R) = \left\{0^{\frac{p^2q-1}{t-1}}, (t-1)^{\frac{(p^2q-1)(t-2)}{t-1}}\right\}$ *and* $\text{Q-spec}(\Gamma_R) = \left\{(2t-4)^{\frac{p^2q-1}{t-1}}, (t-3)^{\frac{(p^2q-1)(t-2)}{t-1}}\right\}$.

(b) *If* $l_1(p-1) + l_2(q-1) + l_3(p^2-1) + l_4(pq-1) = p^2q-1$ *then* $\text{Spec}(\Gamma_R) = \left\{(-1)^{l_1(p-2)+l_2(q-2)+l_3(p^2-2)+l_4(pq-2)}, (p-2)^{l_1}, (q-2)^{l_2}, (p^2-2)^{l_3}, (pq-2)^{l_4}\right\}$,

$$\text{L-spec}(\Gamma_R) = \left\{0^{l_1+l_2+l_3+l_4}, (p-1)^{l_1(p-2)}, (q-1)^{l_2(q-2)}, (p^2-1)^{l_3(p^2-2)}, (pq-1)^{l_4(pq-2)}\right\} \quad \textit{and} \quad \text{Q-spec}(\Gamma_R) = \left\{(2p-4)^{l_1}, (p-3)^{l_1(p-2)}, (2q-4)^{l_2}, (q-3)^{l_2(q-2)}, (2p^2-4)^{l_3}, (p^2-3)^{l_3(p^2-2)}, (2pq-4)^{l_4}, (pq-3)^{l_4(pq-2)}\right\}.$$

Proof. (a) By [23, Theorem 2.9], we have $\Gamma_R = \frac{p^2q-1}{t-1}K_{t-1}$ if $t \in \{p, q, p^2, pq\}$ and $(t-1) \mid (p^2q-1)$. Hence, the result follows from Theorem 2.1.

(b) By [23, Theorem 2.9], we also have $\Gamma_R = l_1K_{p-1} \sqcup l_2K_{q-1} \sqcup l_3K_{p^2-1} \sqcup l_4K_{pq-1}$ if $l_1(p-1) + l_2(q-1) + l_3(p^2-1) + l_4(pq-1) = p^2q-1$. Hence, the result follows from Theorem 2.1. □

Theorem 2.6. *Let $|R| = p^3q$ and R has unity. If $|Z(R)| = pq$ then*

$$\begin{aligned}
\text{Spec}(\Gamma_R) &= \left\{(-1)^{(p+1)(p^2q-pq-1)}, (p^2q-pq-1)^{p+1}\right\},\\
\text{L-spec}(\Gamma_R) &= \left\{0^{p+1}, (p^2q-pq)^{(p+1)(p^2q-pq-1)}\right\} \quad \textit{and}\\
\text{Q-spec}(\Gamma_R) &= \left\{(2p^2q-2pq-2)^{p+1}, (p^2q-pq-2)^{(p+1)(p^2q-pq-1)}\right\}.
\end{aligned}$$

Proof. If $|Z(R)| = pq$ then, by [23, Theorem 2.12], we have $\Gamma_R = (p+1)K_{p^2q-pq}$. Hence, the result follows from Theorem 2.1. □

We conclude this section with the following result.

Theorem 2.7. *Let $|R| = p^3q$, R has unity and $|Z(R)| = p^2$.*

(a) *If $(p-1) \mid (pq-1)$ then* $\text{Spec}(\Gamma_R) = \left\{(-1)^{\frac{(pq-1)(p^3-p^2-1)}{p-1}}, (p^3-p^2-1)^{\frac{pq-1}{p-1}}\right\}$,

$$\text{L-spec}(\Gamma_R) = \left\{0^{\frac{pq-1}{p-1}}, (p^3-p^2)^{\frac{(pq-1)(p^3-p^2-1)}{p-1}}\right\} \quad \textit{and}$$
$$\text{Q-spec}(\Gamma_R) = \left\{(2p^3-2p^2-2)^{\frac{pq-1}{p-1}}, (p^3-p^2-2)^{\frac{(pq-1)(p^3-p^2-1)}{p-1}}\right\}.$$

(b) *If $(q-1) \mid (pq-1)$ then* $\text{Spec}(\Gamma_R) = \left\{(-1)^{\frac{(pq-1)(p^2q-p^2-1)}{q-1}}, (p^2q-p^2-1)^{\frac{pq-1}{q-1}}\right\}$,

$$\text{L-spec}(\Gamma_R) = \left\{0^{\frac{pq-1}{q-1}}, (p^2q-p^2)^{\frac{(pq-1)(p^2q-p^2-1)}{q-1}}\right\} \quad \textit{and}$$
$$\text{Q-spec}(\Gamma_R) = \left\{(2p^2q-2p^2-2)^{\frac{pq-1}{q-1}}, (p^2q-p^2-2)^{\frac{(pq-1)(p^2q-p^2-1)}{q-1}}\right\}.$$

(c) *If $l_1(p-1) + l_2(q-1) = pq-1$ then*

$$\text{Spec}(\Gamma_R) = \left\{(-1)^{l_1(p^3-p^2-1)+l_2(p^2q-p^2-1)}, (p^3-p^2-1)^{l_1}, (p^2q-p^2-1)^{l_2}\right\},$$
$$\text{L-spec}(\Gamma_R) = \left\{0^{l_1+l_2}, (p^3-p^2)^{l_1(p^3-p^2-1)}, (p^2q-p^2)^{l_2(p^2q-p^2-1)}\right\} \quad \textit{and}$$
$$\text{Q-spec}(\Gamma_R) = \left\{(2p^3-2p^2-2)^{l_1}, (p^3-p^2-2)^{l_1(p^3-p^2-1)}, (2p^2q-2p^2-2)^{l_2}, (p^2q-p^2-2)^{l_2(p^2q-p^2-1)}\right\}.$$

Proof. If $|Z(R)| = p^2$ then, by [23, Theorem 2.12], we have

$$\Gamma_R = \begin{cases} \frac{pq-1}{p-1}K_{p^3-p^2}, & \text{if } (p-1) \mid (pq-1)\\ \frac{pq-1}{q-1}K_{p^2q-p^2}, & \text{if } (q-1) \mid (pq-1)\\ l_1K_{p^3-p^2} \sqcup l_2K_{p^2q-p^2}, & \text{if } l_1(p-1) + l_2(q-1) = pq-1. \end{cases}$$

Hence, the result follows from Theorem 2.1. □

3. Various energies

If $\frac{R}{Z(R)}$ is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$, then it was shown in Theorem 3.1 of [19] that

$$E(\Gamma_R) = LE(\Gamma_R) = LE^+(\Gamma_R) = 2(p^2 - 1)|Z(R)| - 2(p + 1). \tag{3.1}$$

As a consequence of (3.1), in the following results, we compute various energies of commuting graphs of several well-known classes of finite rings.

Theorem 3.1. *If* $|R| = p^2$*, then* $E(\Gamma_R) = LE(\Gamma_R) = LE^+(\Gamma_R) = 2(p^2 - p - 2)$.

Proof. If R is a non-commutative ring of order p^2, then $Z(R)$ has only one element. Therefore, the additive group $\frac{R}{Z(R)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Hence the result follows from (3.1). □

Theorem 3.2. *If* $|R| = p^3$ *and* R *has unity, then*

$$E(\Gamma_R) = LE(\Gamma_R) = LE^+(\Gamma_R) = 2(p^3 - 2p - 1).$$

Proof. If R is a non-commutative ring with unity of order p^3, then $Z(R)$ has p elements. Therefore, the additive group $\frac{R}{Z(R)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Hence the result follows from (3.1). □

A ring R is called an n-centralizer ring if $|\operatorname{Cent}(R)| = n$, where $\operatorname{Cent}(R) = \{C_R(x) : x \in R\}$. Various properties of n-centralizer rings can be found in [5, 10, 11]. In the following results we compute various energies of some finite n-centralizer rings.

Theorem 3.3. *If* $|\operatorname{Cent}(R)| = 4$*, then* $E(\Gamma_R) = LE(\Gamma_R) = LE^+(\Gamma_R) = 6|Z(R)| - 6$.

Proof. It was shown in [11, Theorem 3.2] that the additive quotient group $\frac{R}{Z(R)} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ if R is a finite 4-centralizer ring. Hence, the result follows from (3.1) putting $p = 2$. □

Theorem 3.4. *If* $|\operatorname{Cent}(R)| = 5$*, then* $E(\Gamma_R) = LE(\Gamma_R) = LE^+(\Gamma_R) = 16|Z(R)| - 8$.

Proof. It was shown in [11, Theorem 4.3] that the additive quotient group $\frac{R}{Z(R)} \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ if R is a finite 5-centralizer ring. Hence, the result follows from (3.1). □

Theorem 3.5. *If* R *is a finite* p*-ring and* $|\operatorname{Cent}(R)| = (p + 2)$*, then*

$$E(\Gamma_R) = LE(\Gamma_R) = LE^+(\Gamma_R) = 2(p^2 - 1)|Z(R)| - 2(p + 1).$$

Proof. It was shown in [11, Theorem 2.12] that the additive quotient group $\frac{R}{Z(R)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$ if R is a finite $(p + 2)$-centralizer p-ring. Hence, the result follows from (3.1). □

In 1976, MacHale [16] initiated the study of commutativity degree of a finite ring R denoted by $\Pr(R)$. Recall that the commutativity degree of R is the probability that a randomly chosen pair of elements of R commute. Recent results on $\Pr(R)$ can be found in [4, 6–8]. In the following theorem we compute various energies of Γ_R for some given values of $\Pr(R)$.

Theorem 3.6. *Let* p *be the smallest prime dividing* $|R|$*. If* $\Pr(R) = \frac{p^2+p-1}{p^3}$ *then*

$$E(\Gamma_R) = LE(\Gamma_R) = LE^+(\Gamma_R) = 2(p^2 - 1)|Z(R)| - 2(p + 1).$$

Proof. By Theorem 2 of [16] we have $\frac{R}{Z(R)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Hence, the result follows from (3.1). □

We have the following corollary to the above theorem.

Corollary 3.7. *If* $\Pr(R) = \frac{5}{8}$ *then*

$$E(\Gamma_R) = LE(\Gamma_R) = LE^+(\Gamma_R) = 6|Z(R)| - 6.$$

Now we compute various energies of Γ_R for the rings considered in Section 2. Note that one can do this using Theorems 2.1 - 2.7 and (1.1). However, using the following theorem one can also compute various energies.

Theorem 3.8. *If $\mathcal{G} = l_1K_{n_1} \sqcup l_2K_{n_2}$, then $E(\mathcal{G}) = 2l_1(n_1-1) + 2l_2(n_2-1)$. Further, if $n_1 = n_2 = n$ then*

$$E(lK_n) = LE(lK_n) = LE^+(lK_n) = 2l(n-1),$$

where $l = l_1 + l_2$.

Proof. By Theorem 2.1(a) we have

$$\text{Spec}(\mathcal{G}) = \left\{(-1)^{\sum_{i=1}^{2} l_i(n_i-1)},\ (n_1-1)^{l_1},\ (n_2-1)^{l_2}\right\}.$$

Therefore, (1.1) gives

$$\begin{aligned}E(\mathcal{G}) &= |-1|\sum_{i=1}^{2} l_i(n_i-1) + l_1|n_1-1| + l_2|n_2-1| \\ &= l_1(n_1-1) + l_2(n_2-1) + l_1(n_1-1) + l_2(n_2-1) \\ &= 2l_1(n_1-1) + 2l_2(n_2-1).\end{aligned}$$

If $n_1 = n_2 = n$ and $l = l_1 + l_2$ then $\mathcal{G} = lK_n$ and so $E(lK_n) = 2l(n-1)$. In this case, we also have $|v(lK_n)| = ln$, $|e(lK_n)| = \frac{ln(n-1)}{2}$ and so $\frac{2|e(lK_n)|}{|v(lK_n)|} = n-1$.

By Theorem 2.1(b) we have L-spec$(lK_n) = \left\{0^l, n^{l(n-1)}\right\}$. Therefore

$$\left|0 - \frac{2|e(lK_n)|}{|v(lK_n)|}\right| = n-1 \text{ and } \left|n - \frac{2|e(lK_n)|}{|v(lK_n)|}\right| = 1.$$

Hence, (1.1) gives

$$LE(lK_n) = (n-1)l + l(n-1) = 2l(n-1).$$

Again, by Theorem 2.1(c) we also have Q-spec$(lK_n) = \left\{(2n-2)^l, (n-2)^{l(n-1)}\right\}$. Therefore

$$\left|2n-2 - \frac{2|e(lK_n)|}{|v(lK_n)|}\right| = n-1 \text{ and } \left|n-2 - \frac{2|e(lK_n)|}{|v(lK_n)|}\right| = 1.$$

Hence, (1.1) gives

$$LE^+(lK_n) = (n-1)l + l(n-1) = 2l(n-1).$$

This completes the proof. □

Theorem 3.9. *Let R have unity and $|R| = p^4$.*

(a) *If $|Z(R)| = p$ then $E(\Gamma_R) = LE(\Gamma_R) = LE^+(\Gamma_R) = 2(p^2+p+1)(p^2-p-1)$ or $E(\Gamma_R) = 2l_1(p^2-p-1) + 2l_2(p^3-p-1)$, where $l_1 + l_2(p+1) = p^2+p+1$.*
(b) *If $|Z(R)| = p^2$ then $E(\Gamma_R) = LE(\Gamma_R) = LE^+(\Gamma_R) = 2(p+1)(p^3-p^2-1)$.*

Proof. By Theorem 2.5 of [22], we have $\Gamma_R = (p^2+p+1)K_{(p^2-p)}$ or $l_1K_{(p^2-p)} \sqcup l_2K_{(p^3-p)}$ (where $l_1 + l_2(p+1) = p^2+p+1$) if $|Z(R)| = p$ and $(p+1)K_{(p^3-p^2)}$ if $|Z(R)| = p^2$. Hence, the result follows from Theorem 3.8. □

Theorem 3.10. *Let R have unity, $|R| = p^5$ and $Z(R)$ is not a field.*

(a) *If $|Z(R)| = p^2$ then $E(\Gamma_R) = LE(\Gamma_R) = LE^+(\Gamma_R) = 2(p^2+p+1)(p^3-p^2-1)$ or $E(\Gamma_R) = 2l_1(p^3-p^2-1) + 2l_2(p^3-p-1)$, where $l_1 + l_2(p+1) = p^2+p+1$.*
(b) *If $|Z(R)| = p^3$ then $E(\Gamma_R) = LE(\Gamma_R) = LE^+(\Gamma_R) = 2(p+1)(p^4-p^3-1)$.*

Proof. By Theorem 2.7 of [22], we have $\Gamma_R = (p^2+p+1)K_{(p^3-p^2)}$ or $l_1K_{(p^3-p^2)} \sqcup l_2K_{(p^3-p)}$ (where $l_1 + l_2(p+1) = p^2+p+1$) if $|Z(R)| = p^2$ and $(p+1)K_{(p^4-p^3)}$ if $|Z(R)| = p^3$. Hence, the result follows from Theorem 3.8. □

Theorem 3.11. *Let $|R| = pq$ and $Z(R) = \{0\}$.*

(a) *If $(p-1) \mid (pq-1)$ then $E(\Gamma_R) = LE(\Gamma_R) = LE^+(\Gamma_R) = \frac{2(pq-1)(p-2)}{p-1}$.*

(b) *If $(q-1) \mid (pq-1)$ then $E(\Gamma_R) = LE(\Gamma_R) = LE^+(\Gamma_R) = \frac{2(pq-1)(q-2)}{q-1}$.*

(c) *If $l_1(p-1) + l_2(q-1) = pq-1$ then $E(\Gamma_R) = 2l_1(p-2) + 2l_2(q-2)$.*

Proof. It was shown in [23, Theorem 2.8] that

$$\Gamma_R = \begin{cases} \frac{pq-1}{p-1} K_{p-1}, & \text{if } (p-1) \mid (pq-1) \\ \frac{pq-1}{q-1} K_{q-1}, & \text{if } (q-1) \mid (pq-1) \\ l_1 K_{p-1} \sqcup l_2 K_{q-1}, & \text{if } l_1(p-1) + l_2(q-1) = pq-1. \end{cases}$$

Hence, the result follows from Theorem 3.8. □

Theorem 3.12. *Let $|R| = p^2 q$ and $Z(R) = \{0\}$.*

(a) *If $t \in \{p, q, p^2, pq\}$ and $(t-1) \mid (p^2q-1)$ then*

$$E(\Gamma_R) = LE(\Gamma_R) = LE^+(\Gamma_R) = \frac{2(p^2q-1)(t-2)}{t-1}.$$

(b) *If $l_1(p-1) + l_2(q-1) + l_3(p^2-1) + l_4(pq-1) = p^2q-1$ then*

$$E(\Gamma_R) = 2\left(p^2q - 1 - (l_1 + l_2 + l_3 + l_4\right).$$

Proof. (a) By [23, Theorem 2.9], we have $\Gamma_R = \frac{p^2q-1}{t-1} K_{t-1}$ if $t \in \{p, q, p^2, pq\}$ and $(t-1) \mid (p^2q-1)$. Hence, the result follows from Theorem 3.8.

(b) By [23, Theorem 2.9], we also have $\Gamma_R = l_1 K_{p-1} \sqcup l_2 K_{q-1} \sqcup l_3 K_{p^2-1} \sqcup l_4 K_{pq-1}$ if $l_1(p-1) + l_2(q-1) + l_3(p^2-1) + l_4(pq-1) = p^2q-1$. Hence, the result follows from Theorem 2.5 and (1.1). □

Theorem 3.13. *Let R have unity and $|R| = p^3 q$. If $|Z(R)| = pq$ then*

$$E(\Gamma_R) = LE(\Gamma_R) = LE^+(\Gamma_R) = 2(p+1)(p^2q - pq - 1).$$

Proof. If $|Z(R)| = pq$ then, by [23, Theorem 2.12], we have $\Gamma_R = (p+1)K_{p^2q-pq}$. Hence the result follows from Theorem 3.8. □

Theorem 3.14. *Let R have unity, $|R| = p^3 q$ and $|Z(R)| = p^2$.*

(a) *If $(p-1) \mid (pq-1)$ then $E(\Gamma_R) = LE(\Gamma_R) = LE^+(\Gamma_R) = \frac{2(pq-1)(p^3-p^2-1)}{p-1}$.*

(b) *If $(q-1) \mid (pq-1)$ then*

$$E(\Gamma_R) = LE(\Gamma_R) = LE^+(\Gamma_R) = \frac{2(pq-1)(p^2q - p^2 - 1)}{q-1}.$$

(c) *If $l_1(p-1) + l_2(q-1) = pq-1$ then*

$$E(\Gamma_R) = 2l_1(p^3 - p^2 - 1) + 2l_2(p^2q - p^2 - 1).$$

Proof. If $|Z(R)| = p^2$ then, by [23, Theorem 2.12], we have

$$\Gamma_R = \begin{cases} \frac{pq-1}{p-1} K_{p^3-p^2}, & \text{if } (p-1) \mid (pq-1) \\ \frac{pq-1}{q-1} K_{p^2q-p^2}, & \text{if } (q-1) \mid (pq-1) \\ l_1 K_{p^3-p^2} \sqcup l_2 K_{p^2q-p^2}, & \text{if } l_1(p-1) + l_2(q-1) = pq-1. \end{cases}$$

Hence, the result follows from Theorem 3.8. □

Note that the rings considered above are CC-rings. Recall that a non-commutative ring R is called a CC-ring if all the centralizers of its non-central elements are commutative. In other words, $C_R(x)$ for all $x \in R \setminus Z(R)$ is commutative, where $C_R(x) := \{y \in R : xy = yx\}$ is the centralizer of x. The study of CC-rings was initiated by Erfanian et al. in [13]. In the following theorem we compute energy of a CC-ring.

Theorem 3.15. *Let R be a finite CC-ring with distinct centralizers $S_1, S_2, \ldots, S_n$ of non-central elements of R. Then $E(\Gamma_R) = 2(|R| - |Z(R)| - n)$.*

Proof. By Theorem 2.1 of [12] we have

$$\operatorname{Spec}(\Gamma_R) = \{(-1)^{\sum_{i=1}^{n}|S_i| - n(|Z(R)|+1)}, (|S_1| - |Z(R)| - 1)^1, \ldots, (|S_n| - |Z(R)| - 1)^1\}.$$

Therefore

$$\begin{aligned} E(\Gamma_R) &= \sum_{i=1}^{n}|S_i| - n(|Z(R)|+1) + (|S_1| - |Z(R)| - 1) + \cdots \\ &\qquad + (|S_n| - |Z(R)| - 1) \\ &= 2\sum_{i=1}^{n}|S_i| - 2n|Z(R)| - 2n. \end{aligned}$$

Since $\sum_{i=1}^{n}|S_i| = |R| + (n-1)|Z(R)|$, we get the required expression for $E(\Gamma_R)$. □

Corollary 3.16. *Let R be a finite CC-ring and A be any finite commutative ring. Then $E(\Gamma_{R\times A}) = 2(|R||A| - |Z(R)||A| - n)$, where $n = |\operatorname{Cent}(R)| - 1$.*

Proof. Follows from Theorem 3.15 noting that $R \times A$ is a CC-ring, $|\operatorname{Cent}(R)| = |\operatorname{Cent}(R \times A)|$ and $Z(R \times A) = Z(R) \times A$. □

4. Some consequences

A finite non-commutative ring R is called super integral if spectrum, Laplacian spectrum and Signless Laplacian spectrum of Γ_R contain only integers. The notion of super integral ring was introduced in [19]. It can be seen that all the rings considered in Section 2 are super integral.

A finite graph $\mathcal{G}$ is called hyperenergetic and borderenergetic if $E(\mathcal{G}) > E(K_{|v(\mathcal{G})|})$ and $E(\mathcal{G}) = E(K_{|v(\mathcal{G})|})$ respectively. Similarly, $\mathcal{G}$ is called L-hyperenergetic and L-borderenergetic if $LE(\mathcal{G}) > LE(K_{|v(\mathcal{G})|})$ and $LE(\mathcal{G}) = LE(K_{|v(\mathcal{G})|})$ respectively; $\mathcal{G}$ is called Q-hyperenergetic and Q-borderenergetic if $LE^+(\mathcal{G}) > LE^+(K_{|v(\mathcal{G})|})$ and $LE^+(\mathcal{G}) = LE^+(K_{|v(\mathcal{G})|})$ respectively. The study of hyperenergetic graph was initiated by Walikar et al. [24] and Gutman [15] in 1999. The concepts of borderenergetic and L-borderenergetic graphs were introduced by Gong et al. [14] and Tura [21] in the years 2015 and 2017 respectively.

A finite graph $\mathcal{G}$ is called super hyperenergetic if it is hyperenergetic, L-hyperenergetic and Q-hyperenergetic. Similarly, we define super borderenergetic graph. In this section, we show that the commuting graphs of the rings considered in Section 3 are neither super hyperenergetic nor super borderenergetic.

Theorem 4.1. *If $\frac{R}{Z(R)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$ then Γ_R is neither super hyperenergetic nor super borderenergetic.*

Proof. We have $|v(\Gamma_R)| = |Z(R)|(p^2-1)$, since $|R| = p^2|Z(R)|$ and $|v(\Gamma_R)| = |R| - |Z(R)|$. Therefore

$$E(K_{|v(\Gamma_R)|}) = LE(K_{|v(\Gamma_R)|}) = LE^+(K_{|v(\Gamma_R)|}) = 2(|Z(R)|(p^2-1) - 1).$$

Since $2(|Z(R)|(p^2-1)) - (p+1) < 2(|Z(R)|(p^2-1) - 1)$, by (3.1) the result follows. □

Corollary 4.2. *Γ_R is neither super hyperenergetic nor super borderenergetic if*

(a) *R is of order p^2.*
(b) *R is of order p^3 with unity.*
(c) *R is a 4-centralizer ring.*

(d) *R is a 5-centralizer ring.*
(e) *R is a $(p+2)$-centralizer p-ring.*
(f) *p is the smallest prime dividing $|R|$ and $\Pr(R) = \frac{p^2+p-1}{p^3}$.*
(g) $\Pr(R) = \frac{5}{8}$.

Proof. In any of the above cases, the additive quotient group $\frac{R}{Z(R)}$ is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$ for some prime p. Hence, the result follows from Theorem 4.1. □

Theorem 4.3. *If R is a non-commutative ring with unity of order p^4 then Γ_R is neither super hyperenergetic nor super borderenergetic.*

Proof. If $|Z(G)| = p$ then $|v(\Gamma_R)| = p^4 - p$. Therefore

$$\begin{aligned} E(K_{|v(\Gamma_R)|}) = LE(K_{|v(\Gamma_R)|}) = LE^+(K_{|v(\Gamma_R)|}) &= 2(p^4 - p - 1) \\ &= 2((p^2+p+1)(p^2-p) - 1). \end{aligned}$$

We have $2(p^2+p+1)(p^2-p-1) < 2(p^4-p-1)$. Also

$$2l_1(p^2-p-1) + 2l_2(p^3-p-1) < 2((p^2+p+1)(p^2-p) - 1)$$

if l_1, l_2 are positive integers such that $l_1 + l_2(p+1) = p^2+p+1$. Hence, the result follows Theorem 3.9.

If $|Z(G)| = p^2$ then $|v(\Gamma_R)| = p^4 - p^2$. Therefore

$$E(K_{|v(\Gamma_R)|}) = LE(K_{|v(\Gamma_R)|}) = LE^+(K_{|v(\Gamma_R)|}) = 2(p^4 - p^2 - 1).$$

We have

$$2(p+1)(p^3-p^2-1) < 2(p^4-p^2-1).$$

Hence, the result follows Theorem 3.9. □

Theorem 4.4. *If R is a non-commutative ring with unity of order p^5 such that $Z(R)$ is not a field then Γ_R is neither super hyperenergetic nor super borderenergetic.*

Proof. If $|Z(R)| = p^2$ then $|v(\Gamma_R)| = p^5 - p^2$. Therefore

$$\begin{aligned} E(K_{|v(\Gamma_R)|}) = LE(K_{|v(\Gamma_R)|}) = LE^+(K_{|v(\Gamma_R)|}) &= 2(p^5 - p^2 - 1) \\ &= 2((p^3-p^2)(p^2+p+1) - 1). \end{aligned}$$

We have $2(p^2+p+1)(p^3-p^2-1) < 2(p^5-p^2-1)$. Also

$$2l_1(p^3-p^2-1) + 2l_2(p^3-p-1) < 2((p^3-p^2)(p^2+p+1) - 1)$$

if l_1, l_2 are positive integers such that $l_1 + l_2(p+1) = p^2+p+1$. Hence, the result follows from Theorem 3.10.

If $|Z(R)| = p^3$ then $|v(\Gamma_R)| = p^5 - p^3$. Therefore

$$E(K_{|v(\Gamma_R)|}) = LE(K_{|v(\Gamma_R)|}) = LE^+(K_{|v(\Gamma_R)|}) = 2(p^5 - p^3 - 1).$$

We have $2(p+1)(p^4-p^3-1) < 2(p^5-p^3-1)$. Hence, the result follows from Theorem 3.10. □

Theorem 4.5. *Let R be a non-commutative ring of order pq such that $Z(R) = \{0\}$. Then Γ_R is neither super hyperenergetic nor super borderenergetic.*

Proof. We have $|v(\Gamma_R)| = pq - 1$. Therefore

$$E(K_{|v(\Gamma_R)|}) = LE(K_{|v(\Gamma_R)|}) = LE^+(K_{|v(\Gamma_R)|}) = 2(pq - 2).$$

If $t \in \{p, q\}$ and $(t-1) \mid (pq-1)$ then $\frac{2(pq-1)(t-2)}{t-1} < 2(pq-2)$. Also $2l_1(p-2) + 2l_2(q-2) < 2(pq-2)$ if l_1, l_2 are positive integers and $l_1(p-1) + l_2(q-1) = pq - 1$. Hence, the result follows from Theorem 3.11. □

Theorem 4.6. *Let R be a non-commutative ring of order p^2q such that $Z(R) = \{0\}$. Then Γ_R is neither super hyperenergetic nor super borderenergetic.*

Proof. We have $|v(\Gamma_R)| = p^2q - 1$. Therefore

$$E(K_{|v(\Gamma_R)|}) = LE(K_{|v(\Gamma_R)|}) = LE^+(K_{|v(\Gamma_R)|}) = 2(p^2q - 2).$$

If $t \in \{p, q, p^2, pq\}$ and $(t-1) \mid (p^2q - 1)$ then we have $\frac{2(p^2q-1)(t-2)}{t-1} < 2(p^2q - 2)$. Also, $2\,(p^2q - 1 - (l_1 + l_2 + l_3 + l_4) < 2(p^2q - 2)$ if l_1, l_2 are positive integers such that $l_1(p-1) + l_2(q-1) + l_3(p^2-1) + l_4(pq-1) = p^2q - 1$. Hence, the result follows from Theorem 3.12. □

Theorem 4.7. *Let R be a non-commutative ring with unity having order p^3q. If $|Z(R)|$ is not a prime then Γ_R is neither super hyperenergetic nor super borderenergetic.*

Proof. If $|Z(R)| = pq$ then $|v(\Gamma_R)| = p^3q - pq$. Therefore

$$E(K_{|v(\Gamma_R)|}) = LE(K_{|v(\Gamma_R)|}) = LE^+(K_{|v(\Gamma_R)|}) = 2(p^3q - pq - 1).$$

We have $2(p+1)(p^2q - pq - 1) < 2(p^3q - pq - 1)$. Hence, the result follows from Theorem 3.13.

If $|Z(R)| = p^2$ then $|v(\Gamma_R)| = p^3q - p^2$. Therefore

$$E(K_{|v(\Gamma_R)|}) = LE(K_{|v(\Gamma_R)|}) = LE^+(K_{|v(\Gamma_R)|}) = 2(p^3q - p^2 - 1).$$

We have $\frac{2(pq-1)(p^3-p^2-1)}{p-1} < 2(p^3q - p^2 - 1)$, $\frac{2(pq-1)(p^2q-p^2-1)}{q-1} < 2(p^3q - p^2 - 1)$ and $2l_1(p^3 - p^2 - 1) + 2l_2(p^2q - p^2 - 1) < 2(p^3q - p^2 - 1)$ if $(p-1) \mid (pq-1)$, $(q-1) \mid (pq-1)$ and $l_1(p-1) + l_2(q-1) = pq - 1$ respectively. Hence, the result follows from Theorem 3.14. □

We conclude this paper with the following general result.

Theorem 4.8. *If R is finite CC-ring then Γ_R is neither super hyperenergetic nor super borderenergetic.*

Proof. We have $|v(\Gamma_R)| = |R| - |Z(R)|$. Therefore

$$E(K_{|v(\Gamma_R)|}) = LE(K_{|v(\Gamma_R)|}) = LE^+(K_{|v(\Gamma_R)|}) = 2(|R| - |Z(R)| - 1).$$

Also, $2(|R| - |Z(R)| - n) < 2(|R| - |Z(R)| - 1)$, where n is the number of distinct centralizers of non-central elements of R. Hence, the results follows from Theorem 3.15. □

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