Solvable graphs of finite groups

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Abstract

Let $G$ be a finite non-solvable group with solvable radical $\text{Sol}(G)$. The solvable graph $\Gamma_s(G)$ of $G$ is a graph with vertex set $G \setminus \text{Sol}(G)$ and two distinct vertices $u$ and $v$ are adjacent if and only if $\langle u, v \rangle$ is solvable. We show that $\Gamma_s(G)$ is not a star graph, a tree, an $n$-partite graph for any positive integer $n \geq 2$ and not a regular graph for any non-solvable finite group $G$. We compute the girth of $\Gamma_s(G)$ and derive a lower bound of the clique number of $\Gamma_s(G)$. We prove the non-existence of finite non-solvable groups whose solvable graphs are planar, toroidal, double-toroidal, triple-toroidal or projective. We conclude the paper by obtaining a relation between $\Gamma_s(G)$ and the solvability degree of $G$.

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1. Introduction

Let $G$ be a finite group and $u \in G$. The solvabilizer of $u$, denoted by $\text{Sol}_G(u)$, is the set given by $\{v \in G : \langle u, v \rangle \text{ is solvable} \}$. Note that the centralizer $C_G(u) := \{v \in G : uv = vu\}$ is a subset of $\text{Sol}_G(u)$ and hence the center $Z(G) \subseteq \text{Sol}_G(u)$ for all $u \in G$. By [21, Proposition 2.13], $|C_G(u)|$ divides $|\text{Sol}_G(u)|$ for all $u \in G$ though $\text{Sol}_G(u)$ is not a subgroup of $G$ in general. A group $G$ is called a S-group if $\text{Sol}(G)$ is a subgroup of $G$ for all $u \in G$. A finite group $G$ is a S-group if and only if it is solvable (see [21, Proposition 2.22]). Many other properties of $\text{Sol}_G(u)$ can be found in [21]. We write $\text{Sol}(G) = \{u \in G : \langle u, v \rangle \text{ is solvable for all } v \in G\}$. It is easy to see that $\text{Sol}(G) = \bigcap_{u \in G} \text{Sol}_G(u)$. Also, $\text{Sol}(G)$ is the solvable radical of $G$ (see [18]). The solvable graph of a finite non-solvable group $G$ is a simple undirected graph whose vertex set is $G \setminus \text{Sol}(G)$, and two vertices $u$ and $v$ are adjacent if $\langle u, v \rangle$ is a solvable. We write $\Gamma_s(G)$ to denote this graph. It is worth mentioning that $\Gamma_s(G)$ is the complement of the non-solvable graph of $G$ considered in [4,21] and extension of commuting and nilpotent graphs of finite groups that are studied extensively in [1–3, 5, 6, 9–11, 13–16, 25, 26]. It is worth mentioning that the study of commuting graphs of finite groups is originated from a question posed by Erdös [23].

In this paper, we show that $\Gamma_s(G)$ is not a star graph, a tree, an $n$-partite graph for any positive integer $n \geq 2$ and not a regular graph for any non-solvable finite group $G$. In Section 2, we also show that the girth of $\Gamma_s(G)$ is 3 and the clique number of $\Gamma_s(G)$ is

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greater than or equal to 4. In Section 3, we first show that for a given non-negative integer \(k\), there are at the most finitely many finite non-solvable groups whose solvable graph have genus \(k\). We also show that there is no finite non-solvable group, whose solvable graph is planar, toroidal, double-toroidal, triple-toroidal or projective. We conclude the paper by obtaining a relation between \(\Gamma_s(G)\) and \(P_s(G)\) in Section 4, where \(P_s(G)\) is the probability that a randomly chosen pair of elements of \(G\) generate a solvable group (see [20]).

The reader may refer to [27] and [28] for various standard graph theoretic terminologies. For any subset \(X\) of the vertex set of a graph \(\Gamma\), we write \(\Gamma[X]\) to denote the induced subgraph of \(\Gamma\) on \(X\). The girth of \(\Gamma\) is the minimum of the lengths of all cycles in \(\Gamma\), and is denoted by \(\text{girth}(\Gamma)\). We write \(\omega(\Gamma)\) to denote the clique number of \(\Gamma\) which is the least upper bound of the sizes of all the cliques of \(\Gamma\). The smallest non-negative integer \(k\) is called the genus of a graph \(\Gamma\) if \(\Gamma\) can be embedded on the surface obtained by attaching \(k\) handles to a sphere. Let \(\gamma(\Gamma)\) be the genus of \(\Gamma\). Then, it is clear that \(\gamma(\Gamma) \geq \gamma(\Gamma_0)\) for any subgraph \(\Gamma_0\) of \(\Gamma\). Let \(K_n\) be the complete graph on \(n\) vertices and \(mK_n\) the disjoint union of \(m\) copies of \(K_n\). It was proved in [7, Corollary 1] that \(\gamma(\Gamma) \geq \gamma(K_m) + \gamma(K_n)\) if \(\Gamma\) has two disjoint subgraphs isomorphic to \(K_m\) and \(K_n\). Also, by [28, Theorem 6-38] we have

\[
\gamma(K_n) = \left\lfloor \frac{(n-3)(n-4)}{12} \right\rfloor \quad \text{if } n \geq 3. \tag{1.1}
\]

A graph \(\Gamma\) is called planar, toroidal, double-toroidal and triple-toroidal if \(\gamma(\Gamma) = 0, 1, 2\) and 3 respectively.

Let \(N_k\) be the connected sum of \(k\) projective planes. A simple graph which can be embedded in \(N_k\) but not in \(N_{k-1}\), is called a graph of crosscap \(k\). The notation \(\bar{\gamma}(\Gamma)\) stand for the crosscap of a graph \(\Gamma\). It is easy to see that \(\bar{\gamma}(\Gamma) \geq \gamma(\Gamma_0)\) for any subgraph \(\Gamma_0\) of \(\Gamma\). It was shown in [8] that

\[
\bar{\gamma}(K_n) = \begin{cases} \left\lfloor \frac{1}{6} (n-3)(n-4) \right\rfloor & \text{if } n \geq 3 \text{ and } n \neq 7, \\ 3 & \text{if } n = 7. \end{cases} \tag{1.2}
\]

A graph \(\Gamma\) is called a projective graph if \(\bar{\gamma}(\Gamma) = 1\). It is worth mentioning that \(2K_5\) is not projective graph (see [17]).

2. Graph realization

We begin with the following lemma.

**Lemma 2.1.** For every \(u \in G \setminus \text{Sol}(G)\) we have

\[
\text{deg}(u) = |\text{Sol}_G(u)| - |\text{Sol}(G)| - 1.
\]

**Proof.** Note that \(\text{deg}(u)\) represents the number of vertices from \(G \setminus \text{Sol}(G)\) which are adjacent to \(u\). Since \(u \in \text{Sol}_G(u)\), therefore \(|\text{Sol}_G(u)| - 1\) represents the number of vertices which are adjacent to \(u\). Since we are excluding \(\text{Sol}(G)\) from the vertex set therefore \(\text{deg}(u) = |\text{Sol}_G(u)| - |\text{Sol}(G)| - 1\). \(\square\)

**Proposition 2.2.** \(\Gamma_s(G)\) is not a star.

**Proof.** Suppose for a contradiction \(\Gamma_s(G)\) is a star. Let \(|G| - |\text{Sol}(G)| = n\). Then there exists \(u \in G \setminus \text{Sol}(G)\) such that \(\text{deg}(u) = n - 1\). Therefore, by Lemma 2.1, \(|\text{Sol}_G(u)| = |G|\). This gives \(u \in \text{Sol}(G)\), a contradiction. Hence, the result follows. \(\square\)

**Proposition 2.3.** \(\Gamma_s(G)\) is not complete bipartite.

**Proof.** Let \(\Gamma_s(G)\) be complete bipartite. Suppose that \(A_1\) and \(A_2\) are parts of the bipartition. Then, by Proposition 2.2, \(|A_1| \geq 2\) and \(|A_2| \geq 2\). Let \(u \in A_1, v \in A_2\). If \(|\langle u, v \rangle \setminus \text{Sol}(G)| > 2\), then there exists \(y \in \langle u, v \rangle \setminus \text{Sol}(G)\) such that \(\langle u, y \rangle\) and \(\langle v, y \rangle\) are both solvable. But then \(y \notin A_1\) and \(y \notin A_2\), a contradiction.
It follows that \(|\langle u, v \rangle \text{Sol}(G) \setminus \text{Sol}(G)| = 2\). In particular, \(\text{Sol}(G) = 1\) and \(\langle u, v \rangle\) is cyclic of order 3 or \(|\text{Sol}(G)| = 2\) and \(v = uz\) for \(z\) an involution in \(\text{Sol}(G)\). Now the neighbours of \(u \in A_1\) is just \(u^2 \in A_2\) or \(uz\) in the respective cases. Hence \(|A_2| = |A_1| = 1\), a contradiction. Hence, the result follows. \(\square\)

Following similar arguments as in the proof of Proposition 2.3 we get the following result.

**Proposition 2.4.** \(\Gamma_s(G)\) is not complete \(n\)-partite.

**Proposition 2.5.** For any finite non-solvable group \(G\), \(\Gamma_s(G)\) has no isolated vertex.

**Proof.** Suppose \(x\) is an isolated vertex of \(\Gamma_s(G)\). Then \(|\text{Sol}(G)| = 1\); otherwise \(x\) is adjacent to \(xz\) for any \(z \in \text{Sol}(G) \setminus \{1\}\). Thus it follows that \(o(x) = 2\); otherwise \(x\) is adjacent to \(x^2\). Let \(y \in G\). Then \(\langle x, x^y \rangle\) is dihedral and so \(x = x^y\) as \(x\) is isolated. Hence \(x \in Z(G)\) and so \(x \in Z(G) \leq \text{Sol}(G)\), a contradiction. Hence, \(\Gamma_s(G)\) has no isolated vertex. \(\square\)

The following lemma is useful in proving the next two results as well as some results in subsequent sections.

**Lemma 2.6.** Let \(G\) be a finite non-solvable group. Then there exist \(x \in G\) such that \(x, x^2 \notin \text{Sol}(G)\).

**Proof.** Suppose that for all \(x \in G\), we have \(x^2 \in \text{Sol}(G)\). Therefore, \(G/\text{Sol}(G)\) is elementary abelian and hence solvable. Also, \(\text{Sol}(G)\) is solvable. It follows that \(G\) is solvable, a contradiction. Hence, the result follows. \(\square\)

**Theorem 2.7.** Let \(G\) be a finite non-solvable group. Then \(\text{girth}(\Gamma_s(G)) = 3\).

**Proof.** Suppose for a contradiction that \(\Gamma_s(G)\) has no 3-cycle. Let \(x \in G\) such that \(x, x^2 \notin \text{Sol}(G)\) (by Lemma 2.6). Suppose \(|\text{Sol}(G)| \geq 2\). Let \(z \in \text{Sol}(G), z \neq 1\), then \(x, x^2\) and \(xz\) form a 3-cycle, which is a contradiction. Thus \(|\text{Sol}(G)| = 1\). In this case, every element of \(G\) has order 2 or 3; otherwise, \(\{x, x^2, x^3\}\) forms a 3-cycle in \(\Gamma_s(G)\) for all \(x \in G\) with \(o(x) > 3\). Therefore, \(|G| = 2^m3^n\) for some non-negative integers \(m\) and \(n\). By Burnside’s Theorem, it follows that \(G\) is solvable; a contradiction. Hence, \(\text{girth}(\Gamma_s(G)) = 3\). \(\square\)

**Theorem 2.8.** Let \(G\) be a finite non-solvable group. Then \(\omega(\Gamma_s(G)) \geq 4\).

**Proof.** Suppose for a contradiction that \(G\) is a finite non-solvable group with \(\omega(\Gamma_s(G)) \leq 3\). Let \(x \in G \setminus \text{Sol}(G)\) such that \(x^2 \notin \text{Sol}(G)\) according to Lemma 2.6. Suppose \(|\text{Sol}(G)| \geq 2\). Let \(z \in \text{Sol}(G), z \neq 1\), then \(\{x, x^2, xz, x^2z\}\) is a clique which is a contradiction. Thus \(|\text{Sol}(G)| = 1\). In this case every element of \(G \setminus \text{Sol}(G)\) has order 2, 3 or 4 otherwise \(\{x, x^2, x^3, x^4\}\) is a clique with \(o(x) > 4\), which is a contradiction. Therefore \(|G| = 2^m3^n\) where \(m, n\) are non-negative integers. Again, by Burnside’s Theorem, it follows that \(G\) is solvable; a contradiction. This completes the proof. \(\square\)

As a consequence of Theorem 2.7 and Theorem 2.8 we have the following corollary.

**Corollary 2.9.** The solvable graph of a finite non-solvable group is not a tree.

We conclude this section with the following result.

**Proposition 2.10.** \(\Gamma_s(G)\) is not regular.

**Proof.** Follows from [21, Corollary 3.17], noting the fact that a graph is regular if and only if its complement is regular. \(\square\)
3. Genus and diameter

We begin this section with the following useful lemma.

**Lemma 3.1.** Let $G$ be a finite group and $H$ a solvable subgroup of $G$. Then $\langle H, \text{Sol}(G) \rangle$ is a solvable subgroup of $G$.

**Proposition 3.2.** Let $G$ be a finite non-solvable group such that $\gamma(\Gamma_s(G)) = m$.

(a) If $S$ is a nonempty subset of $G \setminus \text{Sol}(G)$ such that $\langle x, y \rangle$ is solvable for all $x, y \in S$, then $|S| \leq \left\lfloor \frac{7 + \sqrt{1 + 48m}}{2} \right\rfloor$.

(b) $|\text{Sol}(G)| \leq \frac{7 + \sqrt{1 + 48m}}{2}$, where $t = \max\{o(x) \text{Sol}(G)) | x \text{Sol}(G) \in G/\text{Sol}(G)\}$.

(c) If $H$ is a solvable subgroup of $G$, then $|H| \leq \frac{7 + \sqrt{1 + 48m}}{2} + |\text{Sol}(G)|$.

**Proof.** We have $\Gamma_s(G)[S] \cong K|S|$ and $\gamma(K|S|) = \gamma(\Gamma_s(G)[S]) \leq \gamma(\Gamma_s(G))$. Therefore, if $m = 0$ then $\gamma(K|S|) = 0$. This gives $|S| \leq 4$, otherwise $K|S|$ will have a subgraph $K_5$ having genus 1. If $m > 0$ then, by Heawood’s formula [27, Theorem 6.3.25], we have

$$|S| = 4\gamma(\Gamma_s(G)[S]) \leq \omega(\Gamma_s(G)) \leq \chi(\Gamma_s(G)) \leq \left\lfloor \frac{7 + \sqrt{1 + 48m}}{2} \right\rfloor$$

where $\chi(\Gamma_s(G))$ is the chromatic number of $\Gamma_s(G)$. Hence part (a) follows.

Part (b) follows from Lemma 3.1 and part (a) considering $S = \bigcup_{i=1}^{t-1} y^i \text{Sol}(G)$, where $y \in G \setminus \text{Sol}(G)$ such that $o(y) \text{Sol}(G)) = t$.

Part (c) follows from part (a) noting that $H = (H \setminus \text{Sol}(G)) \cup (H \cap \text{Sol}(G))$. □

**Theorem 3.3.** Let $G$ be a finite non-solvable group. Then $|G|$ is bounded above by a function of $\gamma(\Gamma_s(G))$.

**Proof.** Let $\gamma(\Gamma_s(G)) = m$ and $h_m = \left\lfloor \frac{7 + \sqrt{1 + 48m}}{2} \right\rfloor$. By Lemma 3.1, we have $\Gamma_s(G)[x \text{Sol}(G)] \cong K|\text{Sol}(G)|$, where $x \in G \setminus \text{Sol}(G)$. Therefore by Proposition 3.2(a), $|\text{Sol}(G)| \leq h_m$.

Let $P$ be a Sylow $p$-subgroup of $G$ for any prime $p$ dividing $|G|$ having order $p^n$ for some positive integer $n$. Then $P$ is a solvable. Therefore, by Proposition 3.2(c), we have $|P| \leq h_m + |\text{Sol}(G)| \leq 2h_m$. Hence, $|G| < (2h_m)^h_m$ noting that the number of primes less than $2h_m$ is at most $h_m$. This completes the proof. □

As an immediate consequence of Theorem 3.3 we have the following corollary.

**Corollary 3.4.** Let $n$ be a non-negative integer. Then there are at the most finitely many finite non-solvable groups $G$ such that $\gamma(\Gamma_s(G)) = n$.

The following two lemmas are essential in proving the main results of this section.

**Lemma 3.5.** [24, Lemma 3.4] Let $G$ be a finite group.

(a) If $|G| = 7m$ and the Sylow 7-subgroup is normal in $G$, then $G$ has an abelian subgroup of order at least 14 or $|G| \leq 42$.

(b) If $|G| = 9m$, where $3 \nmid m$ and the Sylow 3-subgroup is normal in $G$, then $G$ has an abelian subgroup of order at least 18 or $|G| \leq 72$.

**Lemma 3.6.** If $G$ is a non-solvable group of order not exceeding 120 then $\Gamma_s(G)$ has a subgraph isomorphic to $K_{11}$ and $\gamma(\Gamma_s(G)) \geq 5$.

**Proof.** If $G$ is a non-solvable group and $|G| \leq 120$ then $G$ is isomorphic to $A_5$, $A_5 \times Z_2$, $S_5$ or $SL(2, 5)$. Note that $|\text{Sol}(A_5)| = |\text{Sol}(S_5)| = 1$ and $|\text{Sol}(A_5 \times Z_2)| = |\text{Sol}(SL(2, 5))| = 2$. Also, $A_5$ has a solvable subgroup of order 12 and $S_5$, $A_5 \times Z_2$, $SL(2, 5)$ have solvable subgroups of order 24. It follows that $\Gamma_s(G)$ has a subgraph isomorphic to $K_{11}$. Therefore, by (1.1), $\gamma(\Gamma_s(G)) \geq \gamma(K_{11}) = 5$. □
The solvable graph of a finite non-solvable group is neither planar, toroidal, double-toroidal nor triple-toroidal.

**Proof.** Let $G$ be a finite non-solvable group. Note that it is enough to show $\gamma(\Gamma_s(G)) \geq 4$ to complete the proof. Suppose that $\gamma(\Gamma_s(G)) \leq 3$. Let $x \in G \setminus \text{Sol}(G)$ such that $x^2 \not\in \text{Sol}(G)$. Such element exists by Lemma 2.6. Since any two elements of the set $A = x \text{Sol}(G) \cup x^2 \text{Sol}(G)$ generate a solvable group, by Proposition 3.2(a), we have $2|\text{Sol}(G)| = |A| \leq \frac{7+\sqrt{1+48}}{2} = 9$. Thus $|\text{Sol}(G)| \leq 4$. Let $p$ be a prime divisor of $|G|$ and $P$ is a Sylow $p$-subgroup of $G$. Since $P$ is solvable, by Proposition 3.2(c), we get $|P| \leq 9 + |P \cap \text{Sol}(G)| \leq 13$. If $|P| = 11$ or $13$ then $|P \cap \text{Sol}(G)| = 1$. Therefore, $\Gamma_s(G)[P \setminus \text{Sol}(G)] \cong K_{10}$ or $K_{12}$. Using (1.1), we get $\gamma(\Gamma_s(G)[P \setminus \text{Sol}(G)]) = 4$ or $6$. Therefore, $\gamma(\Gamma_s(G)) \geq \gamma(\Gamma_s(G)[P \setminus \text{Sol}(G)]) \geq 4$, a contradiction. Thus $|P| \leq 9$ and hence $p \leq 7$. This shows that $|G|$ divides $2^7 \cdot 3^2 \cdot 5 \cdot 7$.

We consider the following cases.

**Case 1.** $|\text{Sol}(G)| = 4.$

If $H$ is a Sylow $p$-subgroup of $G$ where $p = 5$ or $7$ then $\langle H, \text{Sol}(G) \rangle$ is solvable since $H$ is solvable (by Lemma 3.1). We have $|H \cap \text{Sol}(G)| = 1$ and $|\langle H, \text{Sol}(G) \rangle| = 20, 28$ according as $p = 5, 7$ respectively. Therefore $\Gamma_s(G)[\langle H, \text{Sol}(G) \rangle \setminus \text{Sol}(G)] \cong K_{16}$ or $K_{24}$. By (1.1) we get $\gamma(\Gamma_s(G)) \geq \gamma(\Gamma_s(G)[\langle H, \text{Sol}(G) \rangle \setminus \text{Sol}(G)]) \geq 13$, which is a contradiction.

Thus $|G|$ is a divisor of 72. Therefore, by Lemma 3.6 we have $\gamma(\Gamma_s(G)) \geq 5$, a contradiction.

**Case 2.** $|\text{Sol}(G)| = 3.$

If $H$ is a Sylow $p$-subgroup of $G$ where $p = 5$ or $7$ then $\langle H, \text{Sol}(G) \rangle$ is solvable. We have $|H \cap \text{Sol}(G)| = 1$ and $|\langle H, \text{Sol}(G) \rangle| = 15, 21$ according as $p = 5, 7$ respectively. Therefore $\Gamma_s(G)[\langle H, \text{Sol}(G) \rangle \setminus \text{Sol}(G)] \cong K_{12}$ or $K_{18}$. By (1.1) we get $\gamma(\Gamma_s(G)) \geq \gamma(\Gamma_s(G)[\langle H, \text{Sol}(G) \rangle \setminus \text{Sol}(G)]) \geq 6$, which is a contradiction.

Thus $|G|$ is a divisor of 72. Therefore, by Lemma 3.6 we have $\gamma(\Gamma_s(G)) \geq 5$, a contradiction.

**Case 3.** $|\text{Sol}(G)| = 2.$

If $H$ is a Sylow 7-subgroup of $G$ then $\langle H, \text{Sol}(G) \rangle$ is solvable. We have $|H \cap \text{Sol}(G)| = 1$ and $|\langle H, \text{Sol}(G) \rangle| = 14$. So, $\Gamma_s(G)[\langle H, \text{Sol}(G) \rangle \setminus \text{Sol}(G)] \cong K_{12}$. By (1.1) we get $\gamma(\Gamma_s(G)) \geq \gamma(\Gamma_s(G)[\langle H, \text{Sol}(G) \rangle \setminus \text{Sol}(G)]) \geq 6$, which is a contradiction. Let $K$ be a Sylow 3-subgroup of $G$. If $|K| = 9$ then $\langle K, \text{Sol}(G) \rangle$ is solvable since $K$ is solvable (by Lemma 3.1). We have $|K \cap \text{Sol}(G)| = 1$ and $|\langle K, \text{Sol}(G) \rangle| = 18$. So, $\Gamma_s(G)[\langle K, \text{Sol}(G) \rangle \setminus \text{Sol}(G)] \cong K_{16}$. By (1.1) we get $\gamma(\Gamma_s(G)) \geq \gamma(\Gamma_s(G)[\langle K, \text{Sol}(G) \rangle \setminus \text{Sol}(G)]) = 13$, which is a contradiction.

Thus $|G|$ is a divisor of 120. Therefore, by Lemma 3.6 we have $\gamma(\Gamma_s(G)) \geq 5$, a contradiction.

**Case 4.** $|\text{Sol}(G)| = 1.$

In this case, first we shall show that $7 \nmid |G|$. On the contrary, assume that $7 \mid |G|$. Let $n$ be the number of Sylow 7-subgroups of $G$. Then $n \mid 2^7 \cdot 3^2 \cdot 5$ and $n \equiv 1 \mod 7$. If $n \neq 1$ then $n \geq 8$. Let $H_1, \ldots, H_8$ be the eight distinct Sylow 7-subgroups of $G$. Then the induced subgraphs $\Gamma_s(G)[H_i \setminus \text{Sol}(G)]$ for each $1 \leq i \leq 8$ contribute $\gamma(\Gamma_s(G)[H_i \setminus \text{Sol}(G)]) = 1$ to the genus of $\Gamma_s(G)$. Thus

$$\gamma(\Gamma_s(G)) \geq \sum_{i=1}^{8} \gamma(\Gamma_s(G)[H_i \setminus \text{Sol}(G)]) = 8,$$

a contradiction. Therefore, Sylow 7-subgroup of $G$ is unique and hence normal. Since we have started with a non-solvable group, by Lemma 3.5, it follows that $G$ has an abelian subgroup of order at least 14. Therefore, by (1.1) we have $\gamma(\Gamma_s(G)) \geq \gamma(K_{13}) = 8$, a contradiction. Hence, $|G|$ is a divisor of $2^7 \cdot 3^2 \cdot 5$.

Now, we shall show that $9 \nmid |G|$. Assume that, on the contrary, $9 \mid |G|$. If Sylow 3-subgroup of $G$ is not normal in $G$, then the number of Sylow 3-subgroups is greater than
or equal to 4. Let \( H_1, H_2, H_3 \) be the three Sylow 3-subgroups of \( G \). Then the induced subgraph \( \Gamma_S(G)[H_1 \setminus \Sol(G)] \cong K_8 \) and so it contributes \( \gamma(\Gamma_S(G)[H_1 \setminus \Sol(G)]) = 2 \) to the genus of \( \Gamma_S(G) \). If \(|H_1 \cap H_2| = 1\), then the induced subgraph \( \Gamma_S(G)[H_2 \setminus \Sol(G)] \cong K_8 \) and so it contributes +2 to the genus \( \Gamma_S(G) \). Thus
\[
\gamma(\Gamma_S(G)) \geq \gamma(\Gamma_S(G)[(H_1 \cup H_2) \setminus \Sol(G)]) = 4
\]
which is a contradiction. So assume that \(|H_1 \cap H_2| = 3\). Similarly \(|H_1 \cap H_3| = 3\) and \(|H_2 \cap H_3| = 3\). Let \( M = H_2 \setminus H_1 \). Then \(|M| = 6\). Also note that if \( L = H_1 \cup H_2 \) and \( K = H_3 \setminus L \), then \(|K| \geq 4\). Also \( H_1 \cap M = H_1 \cap K = M \cap K = \emptyset \).

If \(|K| \geq 5\) then \( H_1 \) contribute +2 to genus of \( \Gamma_S(G) \), \( M \) and \( K \) each contribute +1 to genus of \( \Gamma_S(G) \). Hence genus of \( \Gamma_S(G) \) is greater than or equal to 4, a contradiction.

Assume that \(|K| = 4\). In this case \(|M \cap H_3| = 2\). Let \( x \in M \cap H_3 \). Then \( H_1 \) contribute +2 to genus of \( \Gamma_S(G) \), \( M \setminus \{x\} \) and \( K \cup \{x\} \) each contribute +1 to genus of \( \Gamma_S(G) \). Hence genus of \( \Gamma_S(G) \) is greater than or equal to 4, a contradiction.

These show that Sylow 3-subgroup of \( G \) is unique and hence normal in \( G \). Therefore, by Lemma 3.5 and Lemma 3.6, \( G \) has an abelian subgroup \( A \) of order at least 18. Hence,
\[
\gamma(\Gamma_S(G)) \geq \gamma(\Gamma_S(G)[A \setminus \Sol(G)]) \geq \gamma(K_{17}) = 16
\]
which is a contradiction.

It follows that \(|G| \neq |K_3|\) and \( |G| \) is a divisor of 120. Therefore, by Lemma 3.6 we get \( \gamma(\Gamma_S(G)) \geq 5 \), a contradiction. Hence, \( \gamma(\Gamma_S(G)) \geq 4 \) and the result follows. \( \Box \)

The above theorem gives that \( \gamma(\Gamma_S(G)) \geq 4 \). Usually, genera of solvable graphs of finite non-solvable groups are very large. For example, if \( G \) is the smallest non-solvable group \( A_5 \) then \( \Gamma_S(G) \) has 59 vertices and 571 edges. Also \( \gamma(\Gamma_S(G)) \geq 571/6 - 59/2 + 1 = 68 \) (follows from [28, Corollary 6–14]). The following theorem shows that the crosscap number of the solvable graph of a finite non-solvable group is greater than 1.

**Proposition 3.8.** The solvable graph of a finite non-solvable group is not projective.

**Proof.** Suppose \( G \) is a finite non-solvable group whose solvable graph is projective. Note that if \( \Gamma_S(G) \) has a subgraph isomorphic to \( K_n \) then, by (1.2), we must have \( n \leq 6 \). Let \( x \in G \), such that \( x, x^2 \notin \Sol(G) \). Then
\[
\Gamma_S(G)[x \Sol(G) \cup x^2 \Sol(G)] \cong K_{2|\Sol(G)|}
\]
Therefore, \( 2|\Sol(G)| \leq 6 \) and hence \( |\Sol(G)| \leq 3 \).

Let \( p \mid |G| \) be a prime and \( P \) be a Sylow \( p \)-subgroup of \( G \). Then \( \Gamma_S(G)[P \setminus \Sol(G)] \cong K_{|P|/\Sol(G)|} \) since \( P \) is solvable. Therefore, \(|P \setminus \Sol(G)| = |P| - |P \cap \Sol(G)| \leq 6 \) and hence \(|P| \leq 9 \). This shows that \(|G|\) is a divisor of \( 2^33^35 \).

If \(|P| \geq 7\mid |G|\) then Sylow 7-subgroup of \( G \) is unique and hence normal in \( G \); otherwise, let \( H \) and \( K \) be two Sylow 7-subgroups of \( G \). Then \(|H \cap K| = |H \setminus \Sol(G)| = |K \setminus \Sol(G)| = 1\). Therefore, \( \Gamma_S(G)[(H \cup K) \setminus \Sol(G)] \) has a subgraph isomorphic to \( 2K_3 \). Hence, \( \Gamma_S(G) \) has a subgraph isomorphic to \( 2K_5 \), which is a contradiction. Similarly, if \(|G| \leq 6\mid |G|\), then the Sylow 3-subgroup of \( G \) is normal in \( G \). Therefore, by Lemma 3.5, it follows that \(|G| \leq 72 \) or \(|G| \) is a divisor of \( 2^33^5 \). In the both cases, by Lemma 3.6, \( \Gamma_S(G) \) has complete subgraphs isomorphic to \( K_{11} \), which is a contradiction. This completes the proof. \( \Box \)

We conclude this section, by an observation and a couple of problems regarding the diameter and connectedness of \( \Gamma_S(G) \). Using the following programme in GAP [29], we see that the solvable graph of the groups \( A_5, S_5, A_5 \times Z_2, SL(2, 5), PSL(3, 2) \) and \( GL(2, 4) \) are connected with diameter 2. The solvable graphs of \( S_6 \) and \( A_6 \) are connected with diameters greater than 2.

\[\text{P. Bhowal, D. Nongsiang and R.K. Nath}\]
g:=PSL(3,2);
sol:=RadicalGroup(g);
L:=[ ];
gsol:=Difference(g,sol);
for x in gsol do
  AddSet(L,[x]);
  for y in Difference(gsol,L) do
    if IsSolvable(Subgroup(g,[x,y]))=true then
      break;
    fi;
    i:=0;
    for z in gsol do
      if IsSolvable(Subgroup(g,[x,z]))=true and
           IsSolvable(Subgroup(g,[z,y]))=true
      then
        i:=1;
        break;
      fi;
    od;
    if i=0 then
      Print("Diameter>2");
      Print(x, " ", y);
    fi;
  od;
od;

In this connection, we have the following problems.

Problem 3.1. Is $\Gamma_s(G)$ connected for any finite non-solvable group $G$?

Problem 3.2. Is there any finite bound for the diameter of $\Gamma_s(G)$ when $\Gamma_s(G)$ is connected?

4. Relations with solvability degree

The solvability degree of a finite group $G$ is defined by the following ratio

$$P_s(G) := \frac{|\{(u,v) \in G \times G : \langle u, v \rangle \text{ is solvable}\}|}{|G|^2}.$$  

Using the solvability criterion (see [12, Section 1]),

“A finite group is solvable if and only if every pair of its elements generates a solvable group”

for finite groups we have $G$ is solvable if and only if its solvability degree is 1. It was shown in [20, Theorem A] that $P_s(G) \leq \frac{11}{30}$ for any finite non-solvable group $G$. In this section, we study a few properties of $P_s(G)$ and derive a connection between $P_s(G)$ and $\Gamma_s(G)$ for finite non-solvable groups $G$. We begin with the following lemma.

Lemma 4.1. Let $G$ be a finite group. Then $P_s(G) = \frac{1}{|G|^2} \sum_{u \in G} |\text{Sol}_G(u)|$.

Proof. Let $S = \{(u,v) \in G \times G : \langle u, v \rangle \text{ is solvable}\}$. Then

$$S = \bigcup_{u \in G} \left( \{u\} \times \{v \in G : \langle u, v \rangle \text{ is solvable}\} \right) = \bigcup_{u \in G} \{u\} \times \text{Sol}_G(u).$$

Therefore, $|S| = \sum_{u \in G} |\text{Sol}_G(u)|$. Hence, the result follows. \qed
Corollary 4.2. \(|G|P_s(G)\) is an integer for any finite group \(G\).

Proof. By Proposition 2.16 of [21] we have that \(|G|\) divides \(\sum_{u \in G} |\text{Sol}_G(u)|\). Hence, the result follows from Lemma 4.1.

We have the following lower bound for \(P_s(G)\).

Theorem 4.3. For any finite group \(G\),

\[
P_s(G) \geq \frac{|\text{Sol}(G)|}{|G|} + \frac{2(|G| - |\text{Sol}(G)|)}{|G|^2}.
\]

Proof. By Lemma 4.1, we have

\[
|G|^2 P_s(G) = \sum_{u \in \text{Sol}(G)} |\text{Sol}_G(u)| + \sum_{u \in G \setminus \text{Sol}(G)} |\text{Sol}_G(u)|
\]

\[
= |G||\text{Sol}(G)| + \sum_{u \in G \setminus \text{Sol}(G)} |\text{Sol}_G(u)|.
\]

By Proposition 2.13 of [21], \(|C_G(u)|\) is a divisor of \(|\text{Sol}_G(u)|\) for all \(u \in G\) where \(C_G(u) = \{v \in G : uv = vu\}\), the centralizer of \(u \in G\). Since \(|C_G(u)| \geq 2\) for all \(u \in G\) we have \(|\text{Sol}_G(u)| \geq 2\) for all \(u \in G\). Therefore

\[
\sum_{u \in G \setminus \text{Sol}(G)} |\text{Sol}_G(u)| \geq 2(|G| - |\text{Sol}(G)|).
\]

Hence, the result follows from (4.1).

The following theorem shows that \(P_s(G) > \Pr(G)\) for any finite non-solvable group where \(\Pr(G)\) is the commuting probability of \(G\) (see [19]).

Theorem 4.4. Let \(G\) be a finite group. Then \(P_s(G) \geq \Pr(G)\) with equality if and only if \(G\) is a solvable group.

Proof. The result follows from Lemma 4.1 and the fact that \(\Pr(G) = \frac{1}{|G|^2} \sum_{u \in G} |C_G(u)|\) noting that \(C_G(u) \subseteq \text{Sol}_G(u)\) and so \(|\text{Sol}_G(u)| \geq |C_G(u)|\) for all \(u \in G\).

The equality holds if and only if \(C_G(u) = \text{Sol}_G(u)\) for all \(u \in G\), that is \(\text{Sol}_G(u)\) is a subgroup of \(G\) for all \(u \in G\). Hence, by Proposition 2.22 of [21], the equality holds if and only if \(G\) is solvable.

Let \(|E(\Gamma_s(G))|\) be the number of edges of the non-solvable graph \(\Gamma_s(G)\) of \(G\). The following theorem gives a relation between \(P_s(G)\) and \(|E(\Gamma_s(G))|\).

Theorem 4.5. Let \(G\) be a finite non-solvable group. Then

\[
2|E(\Gamma_s(G))| = |G|^2 P_s(G) + |\text{Sol}(G)|^2 + |\text{Sol}(G)| - |G|(2|\text{Sol}(G)| + 1).
\]

Proof. We have

\[
2|E(\Gamma_s(G))| = |\{(x, y) \in (G \setminus \text{Sol}(G)) \times (G \setminus \text{Sol}(G)) : \langle x, y \rangle \text{ is solvable}\}| - |G| + |\text{Sol}(G)|.
\]

Also

\[
S = \{(x, y) \in G \times G : \langle x, y \rangle \text{ is solvable}\}
\]

\[
= \text{Sol}(G) \times \text{Sol}(G) \quad \succeq \quad \text{Sol}(G) \times (G \setminus \text{Sol}(G)) \quad \succeq \quad \text{Sol}(G) \times (G \setminus \text{Sol}(G)) \times \text{Sol}(G)
\]

\[
\succeq \quad \{(x, y) \in (G \setminus \text{Sol}(G)) \times (G \setminus \text{Sol}(G)) : \langle x, y \rangle \text{ is solvable}\}.
\]

Therefore

\[
|S| = |\text{Sol}(G)|^2 + 2|\text{Sol}(G)|(|G| - |\text{Sol}(G)|) + 2|E(\Gamma_s(G))| + |G| - |\text{Sol}(G)|
\]

\[
\implies |G|^2 P_s(G) = |G|(2|\text{Sol}(G)| + 1) - |\text{Sol}(G)|^2 - |\text{Sol}(G)| + 2|E(\Gamma_s(G))|.
\]

Hence, the result follows.
We conclude this paper noting that lower bounds for $|E(\Gamma_s(G))|$ can be obtained from Theorem 4.5 using the lower bounds given in Theorem 4.3, Theorem 4.4 and the lower bounds for $\Pr(G)$ obtained in [22].

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References