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Research Article

Solvable graphs of finite groups

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Abstract

Let G be a finite non-solvable group with solvable radical $\mathrm{Sol}(G)$. The solvable graph $\Gamma_s(G)$ of G is a graph with vertex set $G \setminus \mathrm{Sol}(G)$ and two distinct vertices u and v are adjacent if and only if $\langle u, v\rangle$ is solvable. We show that $\Gamma_s(G)$ is not a star graph, a tree, an n-partite graph for any positive integer $n \geq 2$ and not a regular graph for any non-solvable finite group G. We compute the girth of $\Gamma_s(G)$ and derive a lower bound of the clique number of $\Gamma_s(G)$. We prove the non-existence of finite non-solvable groups whose solvable graphs are planar, toroidal, double-toroidal, triple-toroidal or projective. We conclude the paper by obtaining a relation between $\Gamma_s(G)$ and the solvability degree of G.

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1. Introduction

Let G be a finite group and $u \in G$. The solvabilizer of u, denoted by $\mathrm{Sol}_G(u)$, is the set given by $\{v \in G : \langle u, v\rangle \text{ is solvable}\}$. Note that the centralizer $C_G(u) := \{v \in G : uv = vu\}$ is a subset of $\mathrm{Sol}_G(u)$ and hence the center $Z(G) \subseteq \mathrm{Sol}_G(u)$ for all $u \in G$. By [21, Proposition 2.13], $|C_G(u)|$ divides $|\mathrm{Sol}_G(u)|$ for all $u \in G$ though $\mathrm{Sol}_G(u)$ is not a subgroup of G in general. A group G is called a S-group if $\mathrm{Sol}_G(u)$ is a subgroup of G for all $u \in G$. A finite group G is a S-group if and only if it is solvable (see [21, Proposition 2.22]). Many other properties of $\mathrm{Sol}_G(u)$ can be found in [21]. We write $\mathrm{Sol}(G) = \{u \in G : \langle u, v\rangle \text{ is solvable for all } v \in G\}$. It is easy to see that $\mathrm{Sol}(G) = \bigcap_{u\in G} \mathrm{Sol}_G(u)$. Also, $\mathrm{Sol}(G)$ is the solvable radical of G (see [18]). The solvable graph of a finite non-solvable group G is a simple undirected graph whose vertex set is $G \setminus \mathrm{Sol}(G)$, and two vertices u and v are adjacent if $\langle u, v\rangle$ is a solvable. We write $\Gamma_s(G)$ to denote this graph. It is worth mentioning that $\Gamma_s(G)$ is the complement of the non-solvable graph of G considered in [4, 21] and extension of commuting and nilpotent graphs of finite groups that are studied extensively in [1–3, 5, 6, 9–11, 13–16, 25, 26]. It is worth mentioning that the study of commuting graphs of finite groups is originated from a question posed by Erdös [23].

In this paper, we show that $\Gamma_s(G)$ is not a star graph, a tree, an n-partite graph for any positive integer $n \geq 2$ and not a regular graph for any non-solvable finite group G. In Section 2, we also show that the girth of $\Gamma_s(G)$ is 3 and the clique number of $\Gamma_s(G)$ is

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greater than or equal to 4. In Section 3, we first show that for a given non-negative integer k, there are at the most finitely many finite non-solvable groups whose solvable graph have genus k. We also show that there is no finite non-solvable group, whose solvable graph is planar, toroidal, double-toroidal, triple-toroidal or projective. We conclude the paper by obtaining a relation between $\Gamma_s(G)$ and $P_s(G)$ in Section 4, where $P_s(G)$ is the probability that a randomly chosen pair of elements of G generate a solvable group (see [20]).

The reader may refer to [27] and [28] for various standard graph theoretic terminologies. For any subset X of the vertex set of a graph Γ, we write $\Gamma[X]$ to denote the induced subgraph of Γ on X. The girth of Γ is the minimum of the lengths of all cycles in Γ, and is denoted by girth(Γ). We write $\omega(\Gamma)$ to denote the clique number of Γ which is the least upper bound of the sizes of all the cliques of Γ. The smallest non-negative integer k is called the genus of a graph Γ if Γ can be embedded on the surface obtained by attaching k handles to a sphere. Let $\gamma(\Gamma)$ be the genus of Γ. Then, it is clear that $\gamma(\Gamma) \geq \gamma(\Gamma_0)$ for any subgraph Γ_0 of Γ. Let K_n be the complete graph on n vertices and mK_n the disjoint union of m copies of K_n. It was proved in [7, Corollary 1] that $\gamma(\Gamma) \geq \gamma(K_m) + \gamma(K_n)$ if Γ has two disjoint subgraphs isomorphic to K_m and K_n. Also, by [28, Theorem 6-38] we have

$$\gamma(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil \text{ if } n \geq 3. \tag{1.1}$$

A graph Γ is called planar, toroidal, double-toroidal and triple-toroidal if $\gamma(\Gamma) = 0, 1, 2$ and 3 respectively.

Let N_k be the connected sum of k projective planes. A simple graph which can be embedded in N_k but not in N_{k-1}, is called a graph of crosscap k. The notation $\bar{\gamma}(\Gamma)$ stand for the crosscap of a graph Γ. It is easy to see that $\bar{\gamma}(\Gamma) \geq \bar{\gamma}(\Gamma_0)$ for any subgraph Γ_0 of Γ. It was shown in [8] that

$$\bar{\gamma}(K_n) = \begin{cases} \lceil \frac{1}{6}(n-3)(n-4) \rceil & \text{if } n \geq 3 \text{ and } n \neq 7, \\ 3 & \text{if } n = 7. \end{cases} \tag{1.2}$$

A graph Γ is called a projective graph if $\bar{\gamma}(\Gamma) = 1$. It is worth mentioning that $2K_5$ is not projective graph (see [17]).

2. Graph realization

We begin with the following lemma.

Lemma 2.1. *For every $u \in G \setminus \mathrm{Sol}(G)$ we have*

$$\deg(u) = |\mathrm{Sol}_G(u)| - |\mathrm{Sol}(G)| - 1.$$

Proof. Note that $\deg(u)$ represents the number of vertices from $G \setminus \mathrm{Sol}(G)$ which are adjacent to u. Since $u \in \mathrm{Sol}_G(u)$, therefore $|\mathrm{Sol}_G(u)| - 1$ represents the number of vertices which are adjacent to u. Since we are excluding $\mathrm{Sol}(G)$ from the vertex set therefore $\deg(u) = |\mathrm{Sol}_G(u)| - |\mathrm{Sol}(G)| - 1$. □

Proposition 2.2. *$\Gamma_s(G)$ is not a star.*

Proof. Suppose for a contradiction $\Gamma_s(G)$ is a star. Let $|G| - |\mathrm{Sol}(G)| = n$. Then there exists $u \in G \setminus \mathrm{Sol}(G)$ such that $\deg(u) = n-1$. Therefore, by Lemma 2.1, $|\mathrm{Sol}_G(u)| = |G|$. This gives $u \in \mathrm{Sol}(G)$, a contradiction. Hence, the result follows. □

Proposition 2.3. *$\Gamma_s(G)$ is not complete bipartite.*

Proof. Let $\Gamma_s(G)$ be complete bipartite. Suppose that A_1 and A_2 are parts of the bipartition. Then, by Proposition 2.2, $|A_1| \geq 2$ and $|A_2| \geq 2$. Let $u \in A_1, v \in A_2$. If $|\langle u, v\rangle \mathrm{Sol}(G) \setminus \mathrm{Sol}(G)| > 2$, then there exists $y \in \langle u, v\rangle \mathrm{Sol}(G) \setminus \mathrm{Sol}(G)$ with $u \neq y \neq v$ such that $\langle u, y\rangle$ and $\langle v, y\rangle$ are both solvable. But then $y \notin A_1$ and $y \notin A_2$, a contradiction.

It follows that $|\langle u, v\rangle \operatorname{Sol}(G) \setminus \operatorname{Sol}(G)| = 2$. In particular, $\operatorname{Sol}(G) = 1$ and $\langle u, v\rangle$ is cyclic of order 3 or $|\operatorname{Sol}(G)| = 2$ and $v = uz$ for z an involution in $\operatorname{Sol}(G)$. Now the neighbours of $u \in A_1$ is just $u^2 \in A_2$ or uz in the respective cases. Hence $|A_2| = |A_1| = 1$, a contradiction. Hence, the result follows. □

Following similar arguments as in the proof of Proposition 2.3 we get the following result.

Proposition 2.4. *$\Gamma_s(G)$ is not complete n-partite.*

Proposition 2.5. *For any finite non-solvable group G, $\Gamma_s(G)$ has no isolated vertex.*

Proof. Suppose x is an isolated vertex of $\Gamma_s(G)$. Then $|\operatorname{Sol}(G)| = 1$; otherwise x is adjacent to xz for any $z \in \operatorname{Sol}(G) \setminus \{1\}$. Thus it follows that $o(x) = 2$; otherwise x is adjacent to x^2. Let $y \in G$. Then $\langle x, x^y\rangle$ is dihedral and so $x = x^y$ as x is isolated. Hence $x \in Z(G)$ and so $x \in Z(G) \leq \operatorname{Sol}(G)$, a contradiction. Hence, $\Gamma_s(G)$ has no isolated vertex. □

The following lemma is useful in proving the next two results as well as some results in subsequent sections.

Lemma 2.6. *Let G be a finite non-solvable group. Then there exist $x \in G$ such that $x, x^2 \notin \operatorname{Sol}(G)$.*

Proof. Suppose that for all $x \in G$, we have $x^2 \in \operatorname{Sol}(G)$. Therefore, $G/\operatorname{Sol}(G)$ is elementary abelian and hence solvable. Also, $\operatorname{Sol}(G)$ is solvable. It follows that G is solvable, a contradiction. Hence, the result follows. □

Theorem 2.7. *Let G be a finite non-solvable group. Then* $\operatorname{girth}(\Gamma_s(G)) = 3$.

Proof. Suppose for a contradiction that $\Gamma_s(G)$ has no 3-cycle. Let $x \in G$ such that $x, x^2 \notin \operatorname{Sol}(G)$ (by Lemma 2.6). Suppose $|\operatorname{Sol}(G)| \geq 2$. Let $z \in \operatorname{Sol}(G), z \neq 1$, then x, x^2 and xz form a 3-cycle, which is a contradiction. Thus $|\operatorname{Sol}(G)| = 1$. In this case, every element of G has order 2 or 3; otherwise, $\{x, x^2, x^3\}$ forms a 3-cycle in $\Gamma_s(G)$ for all $x \in G$ with $o(x) > 3$. Therefore, $|G| = 2^m 3^n$ for some non-negative integers m and n. By Burnside's Theorem, it follows that G is solvable; a contradiction. Hence, $\operatorname{girth}(\Gamma_s(G)) = 3$. □

Theorem 2.8. *Let G be a finite non-solvable group. Then $\omega(\Gamma_s(G)) \geq 4$.*

Proof. Suppose for a contradiction that G is a finite non-solvable group with $\omega(\Gamma_s(G)) \leq 3$. Let $x \in G \setminus \operatorname{Sol}(G)$ such that $x^2 \notin \operatorname{Sol}(G)$ according to Lemma 2.6. Suppose $|\operatorname{Sol}(G)| \geq 2$. Let $z \in \operatorname{Sol}(G), z \neq 1$, then $\{x, x^2, xz, x^2z\}$ is a clique which is a contradiction. Thus $|\operatorname{Sol}(G)| = 1$. In this case every element of $G \setminus \operatorname{Sol}(G)$ has order $2, 3$ or 4 otherwise $\{x, x^2, x^3, x^4\}$ is a clique with $o(x) > 4$, which is a contradiction. Therefore $|G| = 2^m 3^n$ where m, n are non-negative integers. Again, by Burnside's Theorem, it follows that G is solvable; a contradiction. This completes the proof. □

As a consequence of Theorem 2.7 and Theorem 2.8 we have the following corollary.

Corollary 2.9. *The solvable graph of a finite non-solvable group is not a tree.*

We conclude this section with the following result.

Proposition 2.10. *$\Gamma_s(G)$ is not regular.*

Proof. Follows from [21, Corollary 3.17], noting the fact that a graph is regular if and only if its complement is regular. □

3. Genus and diameter

We begin this section with the following useful lemma.

Lemma 3.1. *Let G be a finite group and H a solvable subgroup of G. Then $\langle H, \mathrm{Sol}(G)\rangle$ is a solvable subgroup of G.*

Proposition 3.2. *Let G be a finite non-solvable group such that $\gamma(\Gamma_s(G)) = m$.*

(a) *If S is a nonempty subset of $G \setminus \mathrm{Sol}(G)$ such that $\langle x, y\rangle$ is solvable for all $x, y \in S$, then $|S| \leq \left\lfloor \frac{7+\sqrt{1+48m}}{2} \right\rfloor$.*

(b) $|\mathrm{Sol}(G)| \leq \frac{1}{t-1} \left\lfloor \frac{7+\sqrt{1+48m}}{2} \right\rfloor$, *where $t = \max\{o(x\,\mathrm{Sol}(G)) \mid x\,\mathrm{Sol}(G) \in G/\mathrm{Sol}(G)\}$.*

(c) *If H is a solvable subgroup of G, then $|H| \leq \left\lfloor \frac{7+\sqrt{1+48m}}{2} \right\rfloor + |H \cap \mathrm{Sol}(G)|$.*

Proof. We have $\Gamma_s(G)[S] \cong K_{|S|}$ and $\gamma(K_{|S|}) = \gamma(\Gamma_s(G)[S]) \leq \gamma(\Gamma_s(G))$. Therefore, if $m = 0$ then $\gamma(K_{|S|}) = 0$. This gives $|S| \leq 4$, otherwise $K_{|S|}$ will have a subgraph K_5 having genus 1. If $m > 0$ then, by Heawood's formula [27, Theorem 6.3.25], we have

$$|S| = \omega(\Gamma_s(G)[S]) \leq \omega(\Gamma_s(G)) \leq \chi(\Gamma_s(G)) \leq \left\lfloor \frac{7+\sqrt{1+48m}}{2} \right\rfloor$$

where $\chi(\Gamma_s(G))$ is the chromatic number of $\Gamma_s(G)$. Hence part (a) follows.

Part (b) follows from Lemma 3.1 and part (a) considering $S = \bigsqcup_{i=1}^{t-1} y^i\,\mathrm{Sol}(G)$, where $y \in G \setminus \mathrm{Sol}(G)$ such that $o(y\,\mathrm{Sol}(G)) = t$.

Part (c) follows from part (a) noting that $H = (H \setminus \mathrm{Sol}(G)) \cup (H \cap \mathrm{Sol}(G))$. $\square$

Theorem 3.3. *Let G be a finite non-solvable group. Then $|G|$ is bounded above by a function of $\gamma(\Gamma_s(G))$.*

Proof. Let $\gamma(\Gamma_s(G)) = m$ and $h_m = \left\lfloor \frac{7+\sqrt{1+48m}}{2} \right\rfloor$. By Lemma 3.1, we have $\Gamma_s(G)[x\,\mathrm{Sol}(G)] \cong K_{|\mathrm{Sol}(G)|}$, where $x \in G \setminus \mathrm{Sol}(G)$. Therefore by Proposition 3.2(a), $|\mathrm{Sol}(G)| \leq h_m$.

Let P be a Sylow p-subgroup of G for any prime p dividing $|G|$ having order p^n for some positive integer n. Then P is a solvable. Therefore, by Proposition 3.2(c), we have $|P| \leq h_m + |\mathrm{Sol}(G)| \leq 2h_m$. Hence, $|G| < (2h_m)^{h_m}$ noting that the number of primes less than $2h_m$ is at most h_m. This completes the proof. $\square$

As an immediate consequence of Theorem 3.3 we have the following corollary.

Corollary 3.4. *Let n be a non-negative integer. Then there are at the most finitely many finite non-solvable groups G such that $\gamma(\Gamma_s(G)) = n$.*

The following two lemmas are essential in proving the main results of this section.

Lemma 3.5. [24, Lemma 3.4] *Let G be a finite group.*

(a) *If $|G| = 7m$ and the Sylow 7-subgroup is normal in G, then G has an abelian subgroup of order at least 14 or $|G| \leq 42$.*

(b) *If $|G| = 9m$, where $3 \nmid m$ and the Sylow 3-subgroup is normal in G, then G has an abelian subgroup of order at least 18 or $|G| \leq 72$.*

Lemma 3.6. *If G is a non-solvable group of order not exceeding 120 then $\Gamma_s(G)$ has a subgraph isomorphic to K_{11} and $\gamma(\Gamma_s(G)) \geq 5$.*

Proof. If G is a non-solvable group and $|G| \leq 120$ then G is isomorphic to A_5, $A_5 \times \mathbb{Z}_2$, S_5 or $SL(2,5)$. Note that $|\mathrm{Sol}(A_5)| = |\mathrm{Sol}(S_5)| = 1$ and $|\mathrm{Sol}(A_5 \times \mathbb{Z}_2)| = |\mathrm{Sol}(SL(2,5))| = 2$. Also, A_5 has a solvable subgroup of order 12 and S_5, $A_5 \times \mathbb{Z}_2$, $SL(2,5)$ have solvable subgroups of order 24. It follows that $\Gamma_s(G)$ has a subgraph isomorphic to K_{11}. Therefore, by (1.1), $\gamma(\Gamma_s(G)) \geq \gamma(K_{11}) = 5$. $\square$

Theorem 3.7. *The solvable graph of a finite non-solvable group is neither planar, toroidal, double-toroidal nor triple-toroidal.*

Proof. Let G be a finite non-solvable group. Note that it is enough to show $\gamma(\Gamma_s(G)) \geq 4$ to complete the proof. Suppose that $\gamma(\Gamma_s(G)) \leq 3$. Let $x \in G \setminus \mathrm{Sol}(G)$ such that $x^2 \notin \mathrm{Sol}(G)$. Such element exists by Lemma 2.6. Since any two elements of the set $A = x\,\mathrm{Sol}(G) \cup x^2\,\mathrm{Sol}(G)$ generate a solvable group, by Proposition 3.2(a), we have $2|\mathrm{Sol}(G)| = |A| \leq \left\lfloor \frac{7+\sqrt{1+48\cdot 3}}{2} \right\rfloor = 9$. Thus $|\mathrm{Sol}(G)| \leq 4$. Let p be a prime divisor of $|G|$ and P is a Sylow p-subgroup of G. Since P is solvable, by Proposition 3.2(c), we get $|P| \leq 9 + |P \cap \mathrm{Sol}(G)| \leq 13$. If $|P| = 11$ or 13 then $|P \cap \mathrm{Sol}(G)| = 1$. Therefore, $\Gamma_s(G)[P \setminus \mathrm{Sol}(G)] \cong K_{10}$ or K_{12}. Using (1.1), we get $\gamma(\Gamma_s(G)[P \setminus \mathrm{Sol}(G)]) = 4$ or 6. Therefore, $\gamma(\Gamma_s(G)) \geq \gamma(\Gamma_s(G)[P \setminus \mathrm{Sol}(G)]) \geq 4$, a contradiction. Thus $|P| \leq 9$ and hence $p \leq 7$. This shows that $|G|$ divides $2^3.3^2.5.7$.

We consider the following cases.

Case 1. $|\mathrm{Sol}(G)| = 4$.

If H is a Sylow p-subgroup of G where $p = 5$ or 7 then $\langle H, \mathrm{Sol}(G)\rangle$ is solvable since H is solvable (by Lemma 3.1). We have $|H \cap \mathrm{Sol}(G)| = 1$ and $|\langle H, \mathrm{Sol}(G)\rangle| = 20, 28$ according as $p = 5, 7$ respectively. Therefore $\Gamma_s(G)[\langle H, \mathrm{Sol}(G)\rangle \setminus \mathrm{Sol}(G)] \cong K_{16}$ or K_{24}. By (1.1) we get $\gamma(\Gamma_s(G)) \geq \gamma(\Gamma_s(G)[\langle H, \mathrm{Sol}(G)\rangle \setminus \mathrm{Sol}(G)]) \geq 13$, which is a contradiction.

Thus $|G|$ is a divisor of 72. Therefore, by Lemma 3.6 we have $\gamma(\Gamma_s(G)) \geq 5$, a contradiction.

Case 2. $|\mathrm{Sol}(G)| = 3$.

If H is a Sylow p-subgroup of G where $p = 5$ or 7 then $\langle H, \mathrm{Sol}(G)\rangle$ is solvable. We have $|H \cap \mathrm{Sol}(G)| = 1$ and $|\langle H, \mathrm{Sol}(G)\rangle| = 15, 21$ according as $p = 5, 7$ respectively. Therefore $\Gamma_s(G)[\langle H, \mathrm{Sol}(G)\rangle \setminus \mathrm{Sol}(G)] \cong K_{12}$ or K_{18}. By (1.1) we get $\gamma(\Gamma_s(G)) \geq \gamma(\Gamma_s(G)[\langle H, \mathrm{Sol}(G)\rangle \setminus \mathrm{Sol}(G)]) \geq 6$, which is a contradiction.

Thus $|G|$ is a divisor of 72. Therefore, by Lemma 3.6 we have $\gamma(\Gamma_s(G)) \geq 5$, a contradiction.

Case 3. $|\mathrm{Sol}(G)| = 2$.

If H is a Sylow 7-subgroup of G then $\langle H, \mathrm{Sol}(G)\rangle$ is solvable. We have $|H \cap \mathrm{Sol}(G)| = 1$ and $|\langle H, \mathrm{Sol}(G)\rangle| = 14$. So, $\Gamma_s(G)[\langle H, \mathrm{Sol}(G)\rangle \setminus \mathrm{Sol}(G)] \cong K_{12}$. By (1.1) we get $\gamma(\Gamma_s(G)) \geq \gamma(\Gamma_s(G)[\langle H, \mathrm{Sol}(G)\rangle \setminus \mathrm{Sol}(G)]) \geq 6$, which is a contradiction. Let K be a Sylow 3-subgroup of G. If $|K| = 9$ then $\langle K, \mathrm{Sol}(G)\rangle$ is solvable since K is solvable (by Lemma 3.1). We have $|K \cap \mathrm{Sol}(G)| = 1$ and $|\langle K, \mathrm{Sol}(G)\rangle| = 18$. So, $\Gamma_s(G)[\langle K, \mathrm{Sol}(G)\rangle \setminus \mathrm{Sol}(G)] \cong K_{16}$. By (1.1) we get $\gamma(\Gamma_s(G)) \geq \gamma(\Gamma_s(G)[\langle K, \mathrm{Sol}(G)\rangle \setminus \mathrm{Sol}(G)]) = 13$, which is a contradiction.

Thus $|G|$ is a divisor of 120. Therefore, by Lemma 3.6 we have $\gamma(\Gamma_s(G)) \geq 5$, a contradiction.

Case 4. $|\mathrm{Sol}(G)| = 1$.

In this case, first we shall show that $7 \nmid |G|$. On the contrary, assume that $7 \mid |G|$. Let n be the number of Sylow 7-subgroups of G. Then $n \mid 2^3.3^2.5$ and $n \equiv 1 \mod 7$. If $n \neq 1$ then $n \geq 8$. Let $H_1, \ldots, H_8$ be the eight distinct Sylow 7-subgroups of G. Then the induced subgraphs $\Gamma_S(G)[H_i \setminus \mathrm{Sol}(G)]$ for each $1 \leq i \leq 8$ contribute $\gamma(\Gamma_S(G)[H_i \setminus \mathrm{Sol}(G)]) = 1$ to the genus of $\Gamma_S(G)$. Thus

$$\gamma(\Gamma_S(G)) \geq \sum_{i=1}^{8} \gamma(\Gamma_S(G)[H_i \setminus \mathrm{Sol}(G)]) = 8,$$

a contradiction. Therefore, Sylow 7-subgroup of G is unique and hence normal. Since we have started with a non-solvable group, by Lemma 3.5, it follows that G has an abelian subgroup of order at least 14. Therefore, by (1.1) we have $\gamma(\Gamma_S(G)) \geq \gamma(K_{13}) = 8$, a contradiction. Hence, $|G|$ is a divisor of $2^3.3^2.5$.

Now, we shall show that $9 \nmid |G|$. Assume that, on the contrary, $9 \mid |G|$. If Sylow 3-subgroup of G is not normal in G, then the number of Sylow 3-subgroups is greater than or equal to 4. Let H_1, H_2, H_3 be the three Sylow 3-subgroups of G. Then the induced subgraph $\Gamma_S(G)[H_1 \setminus \mathrm{Sol}(G)] \cong K_8$ and so it contributes $\gamma(\Gamma_S(G)[H_1 \setminus \mathrm{Sol}(G)]) = 2$ to the genus of $\Gamma_S(G)$. If $|H_1 \cap H_2| = 1$, then the induced subgraph $\Gamma_S(G)[H_2 \setminus \mathrm{Sol}(G)] \cong K_8$ and so it contributes $+2$ to the genus $\Gamma_S(G)$. Thus

$$\gamma(\Gamma_S(G)) \geq \gamma(\Gamma_S(G)[(H_1 \cup H_2) \setminus \mathrm{Sol}(G)]) = 4$$

which is a contradiction. So assume that $|H_1 \cap H_2| = 3$. Similarly $|H_1 \cap H_3| = 3$ and $|H_2 \cap H_3| = 3$. Let $M = H_2 \setminus H_1$. Then $|M| = 6$. Also note that if $L = H_1 \cup H_2$ and $K = H_3 \setminus L$, then $|K| \geq 4$. Also $H_1 \cap M = H_1 \cap K = M \cap K = \emptyset$.

If $|K| \geq 5$ then H_1 contribute $+2$ to genus of $\Gamma_S(G)$, M and K each contribute $+1$ to genus of $\Gamma_S(G)$. Hence genus of $\Gamma_S(G)$ is greater than or equal to 4, a contradiction.

Assume that $|K| = 4$. In this case $|M \cap H_3| = 2$. Let $x \in M \cap H_3$. Then H_1 contribute $+2$ to genus of $\Gamma_S(G)$, $M \setminus \{x\}$ and $K \cup \{x\}$ each contribute $+1$ to genus of $\Gamma_S(G)$. Hence genus of $\Gamma_S(G)$ is greater than or equal to 4, a contradiction.

These show that Sylow 3-subgroup of G is unique and hence normal in G. Therefore, by Lemma 3.5 and Lemma 3.6, G has an abelian subgroup A of order at least 18. Hence,

$$\gamma(\Gamma_S(G)) \geq \gamma(\Gamma_S(G)[A \setminus \mathrm{Sol}(G)]) \geq \gamma(K_{17}) = 16$$

which is a contradiction.

It follows that $9 \nmid |G|$ and $|G|$ is a divisor of 120. Therefore, by Lemma 3.6 we get $\gamma(\Gamma_S(G)) \geq 5$, a contradiction. Hence, $\gamma(\Gamma_s(G)) \geq 4$ and the result follows. □

The above theorem gives that $\gamma(\Gamma_s(G)) \geq 4$. Usually, genera of solvable graphs of finite non-solvable groups are very large. For example, if G is the smallest non-solvable group A_5 then $\Gamma_s(G)$ has 59 vertices and 571 edges. Also $\gamma(\Gamma_s(G)) \geq 571/6 - 59/2 + 1 = 68$ (follows from [28, Corollary 6–14]). The following theorem shows that the crosscap number of the solvable graph of a finite non-solvable group is greater than 1.

Proposition 3.8. *The solvable graph of a finite non-solvable group is not projective.*

Proof. Suppose G is a finite non-solvable group whose solvable graph is projective. Note that if $\Gamma_s(G)$ has a subgraph isomorphic to K_n then, by (1.2), we must have $n \leq 6$. Let $x \in G$, such that $x, x^2 \notin \mathrm{Sol}(G)$. Then

$$\Gamma_s(G)[x\,\mathrm{Sol}(G) \cup x^2\,\mathrm{Sol}(G)] \cong K_{2|\mathrm{Sol}(G)|}.$$

Therefore, $2|\mathrm{Sol}(G)| \leq 6$ and hence $|\mathrm{Sol}(G)| \leq 3$.

Let $p \mid |G|$ be a prime and P be a Sylow p-subgroup of G. Then $\Gamma_s(G)[P \setminus \mathrm{Sol}(G)] \cong K_{|P \setminus \mathrm{Sol}(G)|}$ since P is solvable. Therefore, $|P \setminus \mathrm{Sol}(G)| = |P| - |P \cap \mathrm{Sol}(G)| \leq 6$ and hence $|P| \leq 9$. This shows that $|G|$ is a divisor of $2^3.3^2.5.7$.

If $7 \mid |G|$ then Sylow 7-subgroup of G is unique and hence normal in G; otherwise, let H and K be two Sylow 7-subgroups of G. Then $|H \cap K| = |H \cap \mathrm{Sol}(G)| = |K \cap \mathrm{Sol}(G)| = 1$. Therefore, $\Gamma_s(G)[(H \cup K) \setminus \mathrm{Sol}(G)]$ has a subgraph isomorphic to $2K_6$. Hence, $\Gamma_s(G)$ has a subgraph isomorphic to $2K_5$, which is a contradiction. Similarly, if $9 \mid |G|$, then the Sylow 3-subgroup of G is normal in G. Therefore, by Lemma 3.5, it follows that $|G| \leq 72$ or $|G|$ is a divisor of $2^3.3.5$. In the both cases, by Lemma 3.6, $\Gamma_s(G)$ has complete subgraphs isomorphic to K_{11}, which is a contradiction. This completes the proof. □

We conclude this section, by an observation and a couple of problems regarding the diameter and connectedness of $\Gamma_s(G)$. Using the following programme in GAP[29], we see that the solvable graph of the groups $A_5, S_5, A_5 \times \mathbb{Z}_2, SL(2,5), PSL(3,2)$ and $GL(2,4)$ are connected with diameter 2. The solvable graphs of S_6 and A_6 are connected with diameters greater than 2.

```
g:=PSL(3,2);
sol:=RadicalGroup(g);
L:=[];
gsol:=Difference(g,sol);
for x in gsol do
 AddSet(L,[x]);
 for y in Difference(gsol,L) do
  if IsSolvable(Subgroup(g,[x,y]))=true then
   break;
  fi;
  i:=0;
  for z in gsol do
   if IsSolvable(Subgroup(g,[x,z]))=true and

   IsSolvable(Subgroup(g,[z,y]))=true
   then
    i:=1;
    break;
   fi;
  od;
  if i=0 then
   Print("Diameter>2");
   Print(x,"   ",y);
  fi;
 od;
od;
```

In this connection, we have the following problems.

Problem 3.1. Is $\Gamma_s(G)$ connected for any finite non-solvable group G?

Problem 3.2. Is there any finite bound for the diameter of $\Gamma_s(G)$ when $\Gamma_s(G)$ is connected?

4. Relations with solvability degree

The solvability degree of a finite group G is defined by the following ratio

$$P_s(G) := \frac{|\{(u,v) \in G \times G : \langle u,v\rangle \text{ is solvable}\}|}{|G|^2}.$$

Using the solvability criterion (see [12, Section 1]),

"A finite group is solvable if and only if every pair of its elements generates a solvable group"

for finite groups we have G is solvable if and only if its solvability degree is 1. It was shown in [20, Theorem A] that $P_s(G) \leq \frac{11}{30}$ for any finite non-solvable group G. In this section, we study a few properties of $P_s(G)$ and derive a connection between $P_s(G)$ and $\Gamma_s(G)$ for finite non-solvable groups G. We begin with the following lemma.

Lemma 4.1. *Let G be a finite group. Then $P_s(G) = \frac{1}{|G|^2}\sum\limits_{u\in G}|\operatorname{Sol}_G(u)|$.*

Proof. Let $\mathcal{S} = \{(u,v) \in G \times G : \langle u,v\rangle \text{ is solvable}\}$. Then

$$\mathcal{S} = \bigcup_{u\in G}(\{u\} \times \{v \in G : \langle u,v\rangle \text{ is solvable}\}) = \bigcup_{u\in G}(\{u\} \times \operatorname{Sol}_G(u)).$$

Therefore, $|\mathcal{S}| = \sum\limits_{u\in G}|\operatorname{Sol}_G(u)|$. Hence, the result follows. $\square$

Corollary 4.2. *$|G|P_s(G)$ is an integer for any finite group G.*

Proof. By Proposition 2.16 of [21] we have that $|G|$ divides $\sum_{u\in G}|\operatorname{Sol}_G(u)|$. Hence, the result follows from Lemma 4.1. □

We have the following lower bound for $P_s(G)$.

Theorem 4.3. *For any finite group G,*

$$P_s(G) \geq \frac{|\operatorname{Sol}(G)|}{|G|} + \frac{2(|G| - |\operatorname{Sol}(G)|)}{|G|^2}.$$

Proof. By Lemma 4.1, we have

$$\begin{aligned} |G|^2 P_s(G) &= \sum_{u\in \operatorname{Sol}(G)} |\operatorname{Sol}_G(u)| + \sum_{u\in G\setminus \operatorname{Sol}(G)} |\operatorname{Sol}_G(u)| \\ &= |G||\operatorname{Sol}(G)| + \sum_{u\in G\setminus \operatorname{Sol}(G)} |\operatorname{Sol}_G(u)|. \end{aligned} \tag{4.1}$$

By Proposition 2.13 of [21], $|C_G(u)|$ is a divisor of $|\operatorname{Sol}_G(u)|$ for all $u \in G$ where $C_G(u) = \{v \in G : uv = vu\}$, the centralizer of $u \in G$. Since $|C_G(u)| \geq 2$ for all $u \in G$ we have $|\operatorname{Sol}_G(u)| \geq 2$ for all $u \in G$. Therefore

$$\sum_{u\in G\setminus \operatorname{Sol}(G)} |\operatorname{Sol}_G(u)| \geq 2(|G| - |\operatorname{Sol}(G)|).$$

Hence, the result follows from (4.1). □

The following theorem shows that $P_s(G) > \Pr(G)$ for any finite non-solvable group where $\Pr(G)$ is the commuting probability of G (see [19]).

Theorem 4.4. *Let G be a finite group. Then $P_s(G) \geq \Pr(G)$ with equality if and only if G is a solvable group.*

Proof. The result follows from Lemma 4.1 and the fact that $\Pr(G) = \frac{1}{|G|^2}\sum_{u\in G}|C_G(u)|$ noting that $C_G(u) \subseteq \operatorname{Sol}_G(u)$ and so $|\operatorname{Sol}_G(u)| \geq |C_G(u)|$ for all $u \in G$.

The equality holds if and only if $C_G(u) = \operatorname{Sol}_G(u)$ for all $u \in G$, that is $\operatorname{Sol}_G(u)$ is a subgroup of G for all $u \in G$. Hence, by Proposition 2.22 of [21], the equality holds if and only if G is solvable. □

Let $|E(\Gamma_s(G))|$ be the number of edges of the non-solvable graph $\Gamma_s(G)$ of G. The following theorem gives a relation between $P_s(G)$ and $|E(\Gamma_s(G))|$.

Theorem 4.5. *Let G be a finite non-solvable group. Then*

$$2|E(\Gamma_s(G))| = |G|^2 P_s(G) + |\operatorname{Sol}(G)|^2 + |\operatorname{Sol}(G)| - |G|(2|\operatorname{Sol}(G)| + 1).$$

Proof. We have

$$2|E(\Gamma_s(G))| = |\{(x, y) \in (G \setminus \operatorname{Sol}(G)) \times (G \setminus \operatorname{Sol}(G)) : \langle x, y\rangle \text{ is solvable}\}| - |G| + |\operatorname{Sol}(G)|.$$

Also

$$\begin{aligned} \mathcal{S} &= \{(x, y) \in G \times G : \langle x, y\rangle \text{ is solvable}\} \\ &= \operatorname{Sol}(G) \times \operatorname{Sol}(G) \quad \sqcup \quad \operatorname{Sol}(G) \times (G \setminus \operatorname{Sol}(G)) \quad \sqcup \quad (G \setminus \operatorname{Sol}(G)) \times \operatorname{Sol}(G) \\ &\quad \sqcup \quad \{(x, y) \in (G \setminus \operatorname{Sol}(G)) \times (G \setminus \operatorname{Sol}(G)) : \langle x, y\rangle \text{ is solvable}\}. \end{aligned}$$

Therefore

$$\begin{aligned} &|\mathcal{S}| = |\operatorname{Sol}(G)|^2 + 2|\operatorname{Sol}(G)|(|G| - |\operatorname{Sol}(G)|) + 2|E(\Gamma_s(G))| + |G| - |\operatorname{Sol}(G)| \\ \Longrightarrow\ &|G|^2 P_s(G) = |G|(2|\operatorname{Sol}(G)| + 1) - |\operatorname{Sol}(G)|^2 - |\operatorname{Sol}(G)| + 2|E(\Gamma_s(G))|. \end{aligned}$$

Hence, the result follows. □

We conclude this paper noting that lower bounds for $|E(\Gamma_s(G))|$ can be obtained from Theorem 4.5 using the lower bounds given in Theorem 4.3, Theorem 4.4 and the lower bounds for $\Pr(G)$ obtained in [22].

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