Bayesian Confidence Interval Estimation of Weibull Modulus Under Increasing Failure Rate

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Highlights
• Two Bayesian models were proposed to estimate the confidence interval of the Weibull modulus.
• The Weibull modulus is an important problem in the fracture strength modeling.
• The Bayesian models were based on the prior of increasing failure rates of all most materials.
• This prior argument requires the Weibull modulus to be more than 1 due to wear-out failure.

Abstract

Estimating the confidence interval of the Weibull modulus is an important problem in the fracture strength modeling of ceramic and composite materials. It is particularly important in cases where the sample size is small due to high experimental costs. For this purpose, several classical methods, including the popular maximum likelihood method, and Bayesian methods have been developed in the literature. However, studies on Bayesian inference have remained very limited in the materials science literature. Recently a Bayesian Weibull model has been proposed for estimating confidence lower bounds for Weibull percentiles using the prior knowledge that the failure rates are increasing. This prior argument requires the Weibull modulus to be more than 1 due to wear-out failure. In this study, under the same prior information, two Bayesian Weibull models, one using the same prior argument and the other a relaxed version of it, have been developed for confidence interval estimation of the Weibull modulus. Their estimation performances have been compared against the maximum likelihood method with Monte Carlo simulations. The results show that the Bayesian Weibull models significantly outperform the maximum likelihood method for almost all Weibull modulus and sample size values.

1. INTRODUCTION

Weibull distribution [1] is a well-established characterization tool in the field of modeling failure times and analyzing the mechanical properties of ceramic and composite materials, such as fracture strength of ceramics, metallic matrix composites and ceramic matrix composites [2], flexural strength of brittle materials [3], fracture toughness behavior of steels in ductile-brittle transition region [4].

The two-parameter Weibull distribution is generally recommended for characterizing ceramic and composite material properties [4]. Let T denote a Weibull random variable modeling the strength of a material. Then T has the following probability density function with parameters σ_0 and m:

$$f(t) = \frac{m}{\sigma_0}\left(\frac{t}{\sigma_0}\right)^{m-1} e^{-\left(\frac{t}{\sigma_0}\right)^m} \qquad (1)$$

where $\sigma_0 > 0$ and $m > 0$ are the scale and shape parameters, respectively. The shape parameter is alternatively referred to as the Weibull modulus. Weibull modulus is used as a measure of the variability of the strength of materials [4] or as a measure of component reliability and system reliability [5].

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The cumulative distribution function, $F(t)$, giving the probability of fracture at a stress level t is expressed as Equation (2), where reliability function $R(t) = e^{-\left(\frac{t}{\sigma_0}\right)^m}$.

$$P(T \leq t) = F(t) = 1 - R(t) = 1 - e^{-\left(\frac{t}{\sigma_0}\right)^m} \tag{2}$$

In practice, high experimental costs limit the number of samples to be tested; and in general, as the sample size decreases, the statistical inferences for Weibull modulus (m) becomes worse, because the precision and the biasing (the coefficient of variation) of the Weibull estimators increase as the sample size decreases for any method [6-8], In addition, Khalili and Kromp [9] showed the coefficient of variation as a function of the inverse square root $\left(1/n^{1/2}\right)$. Therefore, for small sample sizes, new estimation methods or models with better statistical inferences are needed to be investigated.

In general, Bayesian methods have superior performance over classical ones for small samples. There are many studies on the estimation of the Weibull modulus (m) using classical and Bayesian inference methods; however Bayesian studies in the materials science literature are very limited. A previous study [6] proposed a Bayesian estimation model for confidence intervals of Weibull lower percentiles specifically designed for material science applications, and showed that it outperforms alternative classical methods for small sample sizes. The Bayesian model defines a prior distribution suitable for material science applications and generally applicable to any brittle material without any specific knowledge. The basic prior argument in the study is that ceramic and composite materials have increasing failure rates, which requires m to be greater than 1.

In reliability analysis increasing failure rates typically refer to the wear-out period in a bathtub curve, indicating late life of a product. Usually decreasing and constant failure rates are also observed in earlier periods in a product's life cycle. The random variable T is the failure time in such product-life cycle applications. In materials science applications, however, T denotes strength measurements. For increasing failure rates, the fracture probability increases by increasing pressure levels of t, which is normally expected for all brittle materials. Theoretical studies in materials science show that strength distribution of carbon fibers and fibrous composite materials exhibit increasing failure rates [10-12], and Yalcinkaya and Birgoren [13] discusses why increasing failure rates should also apply to ceramics: Increasing failure rates require $m > 1$, which explains the lower limit of the uniform prior. As for the upper limit, a value of 100 indicates an amount of variability that is negligibly small for most practical purposes. Estimated values for existing materials also support this upper limit: For metals such as aluminum and steel, and metal alloys, m is around 100 [14, 15]. For traditional ceramics (e.g. brick, pottery, chalk etc), m is between 1 and 3 [14, 15]; for engineered ceramics (e.g. SiC, Al2O3, Si3N4) it is between 5 and 10 [14, 15].

Motivated by this recent result [13], the two Bayesian models have been proposed, one using the same prior argument and the other a relaxed version of it, for confidence interval estimation of m and their performances have been assessed against the maximum likelihood method, which is the most frequently used one among classical estimation methods. The first Bayesian model uses a Uniform prior distribution with limits (1, 100) for m and the non-informative Jeffrey's prior for the Weibull scale parameter (σ_0). In theory, m values can range from 0 to ∞, but experimental and theoretical studies in materials science support these uniform prior limits. The first model strictly constrains m in the interval (1, 100), which may be argued to be valid for most materials, but certainly not all materials. The second model assigns a nonzero probability $1 - \alpha$ to the event that m is in (1, 100). A prior distribution can be elicited for m by following the steps of a maximal entropy hash (MEH) prior elicitation method, which will be explained in later. MEH priors encompass the standard maximal entropy (ME) priors and the maximal data information (MDI) priors in order to increase their chances of being proper [16]. For σ_0, the non-informative Jeffrey's prior has been used, as in the first Bayesian model.

In the first step of this study, credible interval (for Bayesian inference) and confidence interval (for classical inference) estimation algorithms have been developed based on the maximum likelihood (MLE) method and the two Bayesian Weibull (BW) models (using Uniform (1, 100) & Jeffrey's priors and MEH &

Jeffrey's priors). In the second step, Monte Carlo simulations have been designed and run in the C++ language for the comparisons with large simulation run numbers. Small sample sizes between 3 and 20 have been used in the simulations.

In the literature, many authors have proposed various estimation methods for the Weibull parameters and their confidence intervals for complete and censored data. The handbooks [17-22] provide a recent detailed discussion of classical and Bayesian methods. MLE [20], Linear Least Squares (LLS) [21], and Weighted Linear Least Squares (WLLS) [22-24] are among the popular classical estimation methods. The most common estimation method is the MLE; see Nelson [25], Johnson et al. [26], Murthy et al. [27], and Dodson [19] for a detailed discussion.

In the materials science literature, statistical inferences of the estimators and confidence intervals for m have been studied extensively using classical methods [2-6, 9, 20-24, 28-37]. Among these studies, Gong [31] introduced a new probability index for point and interval estimation of m for ceramics with an LLS method. In addition, McCool and Phan [38] proposed a simple method based on Menon's original work [39] for estimating exact confidence intervals of m and characteristic strength. McCool [40] developed a program for creating pivotal tables according to user-entered percentage and sample volume information for point and interval estimation of m and percentiles. Bütikofera et al. [41] compared point estimates and 95% confidence intervals for m by two LLS methods. They called conventional LLS methods as YonX methods, and proposed the XonY method by interchanging the axes in the least squares analysis.

While there are major differences among classical methods, Bayesian methods basically use the same formulation, as will be discussed later; they differ only by the choice of the prior distribution of the parameters [17,18]. In the general scientific literature, there are many Bayesian point estimation studies on m, see for example [42-47]; however Bayesian interval estimation studies are limited, see Aron et al. [48] as an example. The details of Bayesian Weibull analysis can be found in [49]. In the materials science literature, there is only one study on Bayesian interval estimation for the Weibull modulus: Simoa et al. [50] proposed a new Bayesian method for determination of strength and crack growth using fatigue data from previous tests as prior information. The remaining studies in materials science are generally based on point estimation of m for multiple purposes, such as investigating size effect of materials under different operational conditions [51] or understanding the statistical behavior of the strength of materials [52].

In this study, the two proposed BW models do not require any past data, and make use of the information that the Weibull modulus is more than 1 and less than 100, which is considered a proper upper value for brittle materials. MEH priors developed by Bousquet [16] encompass the ME and the MDI priors for eliciting a proper prior. In Bayesian inference, the ME and the MDI priors have been studied by Zellner [53-55], Berger [56] and Soofi [57], among many others. The interested reader is also referred to Skilling [58], Zellner [55], Le Besnerais et al. [59], Soofi [60], Miller and Yan [61].

2. MATERIAL METHOD

2.1. Estimation of Confidence Bounds for Weibull Modulus

The Weibull modulus, m, has been recently used as a measure of the variability of the strength, the mechanical reliability or the probability of failure of materials [62-64]. A higher Weibull modulus ($m > 20$) indicates a narrower distribution of failure strengths, and, therefore, a more reliable material such that failure occurs in a more predictable and controlled manner. A material characterized by a low Weibull modulus has a broad distribution of failure strengths and exhibits low predictable failure behavior. Since Weibull modulus characterizes the shape of the failure probability curve, it is used as a strength reliability parameter in materials science and has to be estimated accurately for performing a successful reliability analysis.

Weibull modulus can be estimated using classical and Bayesian inferences. For both inferences, the estimates of m, $\hat{m}$, can be quite unreliable, especially in small samples. Therefore, instead of $\hat{m}$ values, confidence bounds for m have been used for the characterization of mechanical properties [62-64].

Confidence intervals are called credible intervals in the parlance of Bayes' theory. Credible intervals of parameters and functions of parameters are calculated from posterior distributions in Bayesian inference [65].

2.2. Maximum Likelihood Method

The maximum likelihood estimation (MLE) is a method of estimating the parameters of a probability distribution by maximizing a likelihood function. Using the density of $f(t)$, as defined in Equation (1), the likelihood function for the Weibull distributed independent data observations $T_1, T_2, \ldots, T_n$ can be written as

$$L(m, \sigma_0) = L(t_1, t_2, \ldots, t_n, m, \sigma_0) = \prod_{i=1}^{n} f(t_i; m, \sigma_0) = m^n \, \sigma_0^n \, t_1^{\sigma_0 - 1} t_2^{\sigma_0 - 1} \ldots t_n^{\sigma_0 - 1} \, e^{-m \sum_{i=1}^{n} \sigma_i^{\sigma_0}}. \quad (3)$$

The likelihood function in Equation (3) can be maximized by taking logarithms of it (Equation (4)), differentiating with respect to m and σ_0 and equating to zero, $dl(m, \sigma_0)/dm = 0$ and $dl(m, \sigma_0)/d\sigma_0 = 0$:

$$l(m, \sigma_0) = InL(m, \sigma_0) = nInm - nIn\sigma_0 + (n - 1) \sum_{i=1}^{n} In\frac{t_i}{\sigma_0} - \sum_{i=1}^{n} In\left(\frac{t_i}{\sigma_0}\right)^m. \quad (4)$$

The estimating equations maximizing $l(m, \sigma_0)$ are shown in the set of Equation (5) and (6) and the solution set of these equations is called the maximum likelihood estimator of $\hat{m}$ ve $\hat{\sigma}_0$ [66]

$$\frac{n}{\hat{m}} - nIn\hat{\sigma}_0 + \sum_{i=1}^{n} In(t_i) - \sum_{i=1}^{n} \left(\frac{t_i}{\hat{\sigma}_0}\right)^{\hat{m}} In\left(\frac{t_i}{\hat{\sigma}_0}\right) \quad (5)$$

$$\hat{\sigma}_0 = \left(\frac{\sum_{i=1}^{n} (t_i)^{\hat{m}}}{n}\right)^{1/\hat{m}} \quad (6)$$

where $t_1, t_2, \ldots, t_n$ are an observed sample of size n. The Newton-Raphson method is usually employed for solving Equation (5) for $\hat{m}$. Then, $\hat{\sigma}_0$ is found by substituting $\hat{m}$ into Equation (6). Thoman [67] showed that $\hat{m}/m$ is a pivotal variable, that is, independent of σ_0 and m. The distribution of $\hat{m}/m$ can be found for a given n and α value by Monte Carlo simulation. Since the pivotal statistic $\hat{m}/m$ has the same distribution with m, $\hat{m}$ can be used instead of $\hat{m}/m$ in Equation (7) and then Equation (8) is obtained

$$P\left(d_{1-\alpha/2} < \frac{\hat{m}}{m} < d_{\alpha/2}\right) = 1 - \alpha \quad (7)$$

$$P\left(\frac{\hat{m}}{d_{\alpha/2}} < m < \frac{\hat{m}}{d_{1-\alpha/2}}\right) = 1 - \alpha \ . \quad (8)$$

The upper and lower bounds, respectively $d_{1-\alpha/2}$ and $d_{\alpha/2}$, are obtained by simulation of $\hat{m}$.

2.3. The Bayesian Weibull (BW) Models

Bayesian estimation inference makes use of one's prior knowledge about the parameters as well as the useable data. When any prior knowledge about the parameter is not available, it is possible to make use of the non-informative priors in Bayesian analysis [45]. In this article, Bayesian inference is applied to estimating the parameters of Weibull distribution assuming a prior knowledge about m exists.

Applying Bayes' rule [68] on the 2-parameter Weibull distribution and assuming the prior distributions of m and σ_0 are independent, the following posterior distribution is expressed as

$$P(m, \sigma_0 | \boldsymbol{T}) = \frac{L(m, \sigma_0)\, P(m)\, P(\sigma_0)}{\int_0^\infty \int_0^\infty L(m, \sigma_0)\, P(m)\, P(\sigma_0)\, d\sigma_0\, dm} \tag{9}$$

where $P(m, \sigma_0 | \boldsymbol{T})$ is the posterior distribution of m and σ_0, $L(m, \sigma_0) = \mathrm{P}(\boldsymbol{T}|\, m, \sigma_0) = \prod_{i=1}^n f(t_i; m; \sigma_0)$ is the likelihood function of the data given m and σ_0, $P(m)$ and $P(\sigma_0)$ are the prior distributions of m and σ_0, respectively. The double integrals in Equation (9) have no closed forms, since Weibull shape and scale parameters are treated as random variables having prior distributions. Thus, to get the estimators analytically based on the formulations is impossible, and some numerical integration methods, such as trapezoid rule and Simpson's rule, are needed to calculate the estimators [49]. For more information about numerical integration methods, see "Applied Mathematics for Science and Engineering", Chapter 4 "Numerical Quadrature" [69]. In this study, trapezoid integration rule was used for solving the double integrals.

After deriving the posterior distribution $P(m, \sigma_0 | \boldsymbol{T})$, the marginal posterior probability $P(m | \boldsymbol{T})$ of m is determined by summing the posterior probabilities across the alternative values for σ_0 and the credible interval of m is calculated based on $P(m | \boldsymbol{T})$. At a given confidence level a, the two-sided credible intervals can be calculated by solving Equation (10) for m with trapezoid integration rule:

$$\int_{m_{lower}}^{m_{upper}} P(m | \boldsymbol{T})\, dm = \int_{m_{lower}}^{m_{upper}} \int_0^\infty P(m, \sigma_0 | \boldsymbol{T})\, d\sigma_0\, dm = 1 - \alpha. \tag{10}$$

If the marginal posterior density of m, $P(m | \boldsymbol{T}) = \int_0^\infty P(m, \sigma_0 | \boldsymbol{T}) d\sigma_0$, is unimodal, then for a given value of α, the shortest credible interval for m is given by $CI = \{m: P(m | \boldsymbol{T}) \geq k\}$ where k is the largest number so that ;

$$\int_{\{m: P(m|\boldsymbol{T}) \geq k\}} P(m | \boldsymbol{T}) dm = \int_{\{m: P(m|\boldsymbol{T}) \geq k\}} \int_0^\infty P(m, \sigma_0 | \boldsymbol{T})\, d\sigma_0\, dm = 1 - \alpha. \tag{11}$$

The credible set described in Equation (11) is called a highest posterior density interval (HPD), as it consists of the values of the parameter for which the posterior density is the highest. An HPD interval has the shortest width among all other credible intervals. When the marginal distribution of the interest of parameter is not symmetric, a $100(1 - \alpha)\%$ HPD interval is more desirable. In this context, the HPD interval is more appropriate for m of which the marginal posterior distribution has been proven to be not symmetric by the simulations in this study. For a recent detailed discussion of HPD credible intervals, see [70].

2.4. Priors and Algorithms for the BW Models

As mentioned before, using the prior argument based on increasing failure rates of materials, two Bayesian Weibull models have been proposed for credible intervals of m in this study.

BW1 Model: $m \sim$ Uniform (1,100) & $\sigma_0 \sim$ Jeffrey's prior

The first Bayesian model (BW1) in Equation (12) assumes that the Weibull modulus has a uniform prior on the interval of (1, 100) as discussed before. It is also assumed that there is no prior knowledge about the scale parameter, so the Jeffrey's prior is chosen for σ_0 as it is non-informative, using only the information in a sample. The uniform distribution implies that any Weibull modulus value is equally likely in the interval (1, 100). An expert may specify a narrower interval, and even change the distribution. However, in materials testing, whereabouts of the Weibull modulus can hardly be known before actually conducting the tests. Because, in cases of where a Weibull material has such characteristic properties as relative flaw

density, fracture origins, internal stress fields, gradients, toughness etc., a deviation from Weibull behavior is identified and so m becomes stress-dependent [71]. An interval of (1, 100), on the other hand, will be valid for most, if not all, ceramics and composite materials, and hence eliminate any need for an expert opinion for a specific material.

$$f(m, \sigma_0 | T) = = \frac{\prod_{i=1}^{n} f(t_i; m; \sigma_0) \ Unf(m; 1,100) \ 1/\sigma_0}{\int_0^{\infty} \int_0^{\infty} \prod_{i=1}^{n} f(t_i; m; \sigma_0) \ Unf(m; 1,100) \ 1/\sigma_0 \ d\sigma_0 \, dm} \tag{12}$$

BW2 Model: $m \sim$ MEH ($\gamma_0 \cong 0.70$) & $\sigma_0 \sim$ Jeffrey's prior

A uniform distribution on the (1, 100) interval imposes strong constraints for the prior probability of m: the probability that $m < 1$ and $m > 100$ is zero. The second Bayesian model (BW2) assumes $P(1 < m < 100) = 1 - \alpha$, which provides a relaxation. Under this assumption, an MEH prior distribution is elicitied by the MEH elicitation method. MEH prior encompasses the ME priors and the MDI priors when there is no prior information or when little expert knowledge is considered only. The MEH prior distribution of m is given in Equation (13);

$$P_{MEH}(m) = m^{-1} \ exp(-\gamma_0 \gamma/m) \ / \ \Gamma^{\gamma_0}(1/m) \qquad , \qquad \gamma_0 > 0, m > 0 \tag{13}$$

where γ_0 is the shape parameter, γ is the Euler constant ($\gamma \cong 0.57721$) and $\Gamma(.)$ is the gamma function.

In the first step for the MEH elicitation method, an expert gives a range for m under a probability $1 - \alpha$ and the value of γ_0 is searched to satisfy Equation (14). In the second step, the MEH prior distribution is elicited by replacing Equation (13) with the result value of γ_0. In addition, by combining descent methods (such as Newton–Raphson) and Monte Carlo methods, numerous MEH prior distributions for the Weibull modulus can be derived, assessing a range of values for γ_0. See [16] for a detailed discussion of MEH prior and elicitation method:

$$P\big(m_{low} < m < m_{up}\big) = \frac{\int_{m_{low}}^{m_{up}} m^{-1} \ exp(-\gamma_0 \gamma/m) \ / \ \Gamma^{\gamma_0}(1/m) dm}{\int_0^{\infty} m^{-1} \ exp(-\gamma_0 \gamma/m) \ / \ \Gamma^{\gamma_0}(1/m) dm} = 1 - \alpha \ . \tag{14}$$

In this study, $\big[m_{low}, m_{up}\big] = [1,100].$ However, no parameter value for γ_0, can be found for high probabilities such 0.80 and 0.90 for this interval. This is a disadvantage of this prior distribution, while it is very practical for its simple, one-parameter construct. In this study, a high value for $1 - \alpha$ is desirable, thus, a value of γ_0 is searched to maximize $1 - \alpha$ in Equation (14). The Figure 1 is obtained by plotting Equation (14) for different γ_0 values for the (1, 100) interval. A maximum probability of $1 - \alpha = 0.66$ is obtained for $\gamma_0 \cong 0.70$. Therefore, the desired MEH prior is as follows: As in BW2, the Jeffrey's prior is chosen for σ_0:

$$P_{MEH}(m) = m^{-1} \ exp(-0.7 \ \gamma/m) \ / \ \Gamma^{\gamma_0}(1/m) \ . \tag{15}$$

After obtaining $P_{MEH}(m)$, the BW2 model has the following posterior distribution function:

$$f(m, \sigma_0 | T) = \frac{\prod_{i=1}^{n} f(t_i; m; \sigma_0) \ m^{-1} \ exp(-0.7 \ \gamma/m) \ / \ \Gamma^{\gamma_0}(1/m) \ 1/\sigma_0}{\int_0^{\infty} \int_0^{\infty} \prod_{i=1}^{n} f(t_i; m; \sigma_0) \ m^{-1} \ exp(-0.7 \ \gamma/m) \ / \ \Gamma^{\gamma_0}(1/m) \ 1/\sigma_0 \ d\sigma_0 \, dm} \ . \tag{16}$$
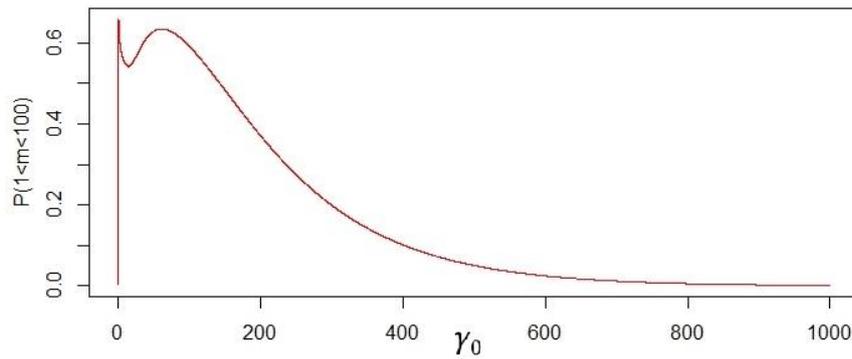
Figure 1. $P(1 < m < 100)$ *for different* γ_0 *values*

3. MONTE CARLO SIMULATION STUDY

3.1. Simulation Inputs

An extensive Monte Carlo simulation has been performed to compare the performances of the MLE, BW1 and BW2 models. Figure 2 shows the flowchart of the simulation procedure implemented in C++.
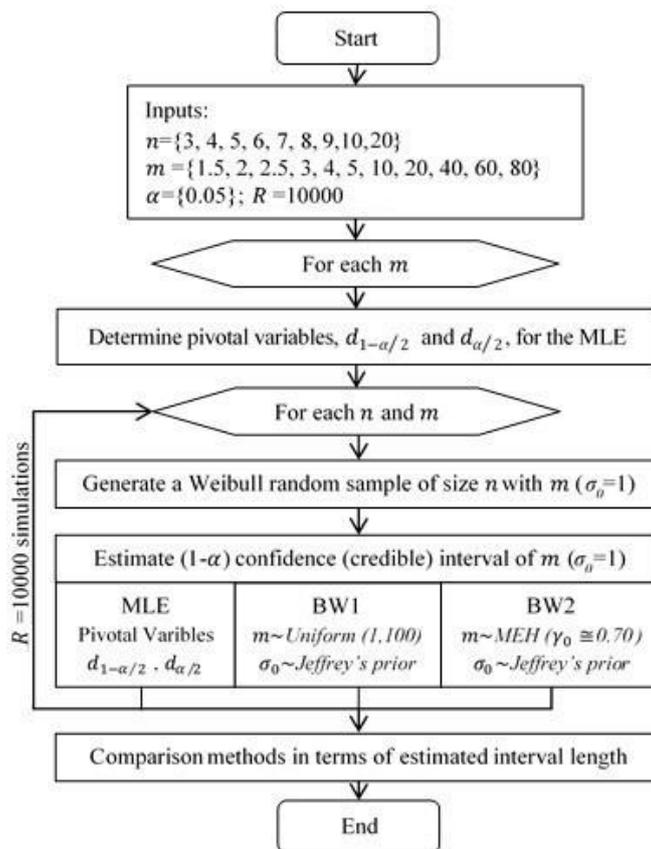
Figure 2. *Flow chart of the simulation procedure*

In the simulation, σ_0 is a fixed at $\sigma_0 = 1$, since it has no effect on the results. For each set of given sample size and the prescribed Weibull modulus, $R = 10000$ intervals are estimated using the methods described in the experimental section. The reason for selecting the small m values in Figure 2 is to examine the effect of the shape of the Weibull probability density function, which is positively skewed for $m < 2.6$, approximates the normal pdf for $2.6 < m < 3.7$ and is negatively skewed for $m > 3.7$ [49].

3.2. Comparison Criterion: Median Interval Length

In this study, median interval length is chosen for comparing the performances of MLE and BW methods. The median statistic is more representative of the central tendency due to the asymmetry of the distribution of $\hat{m}$ for most classical methods [9]. For Bayesian methods, the obtained $\hat{m}$ distributions in this study have verified the existence of asymmetry. However, median interval lengths cannot be compared without considering simulation errors. The replication number $R = 10000$ is quite a large number considering computationally expensive Bayesian methods; yet, it still produces a significant simulation error. Simulation errors will be quantified by constructing confidence intervals for the median interval length. Let $Z_1, \dots, Z_R$ be estimated interval lengths from a simulation of R replications for a particular n and m. Also let $Z_{(1)} \leq \cdots \leq Z_{(R)}$ denote the associated order statistics, and $\tilde{Z}$ denote the associated sample median. A 95% confidence interval for the true median $\tilde{\mu}_Z$ is $(Z_{(l_1)}, Z_{(u_1)})$, where [72]

$$l_1 = \left[(0.5)R - 1.96\sqrt{(0.5)^2 R} + 0.5\right] \& u_1 = \left[(0.5)R + 1.96\sqrt{(0.5)^2 R} + 1.5\right]. \tag{17}$$

3.3. Simulation Results

A detailed summary of the simulation results is presented in Tables 1-2 for each n and m according to the 95% confidence intervals of $\tilde{\mu}_Z$ in Equation (17). The results are also shown graphically in Figures 3 and 4. In Table 3, for each n and m, the result of the best method, that is the shortest interval length, is shown in bold and the second best method is shown underlined and italic. In some cases, two methods are shown to be best methods at the same time due to intersecting confidence intervals, which indicates that there is no statistically significant difference between the true medians $(\tilde{\mu}_Z)$.

Table 1. *Estimated Interval Lengths for Weibull Modulus (m) for n=3, 4, 5, 6 and 7*

		MLE		BW1		BW2	
n	*m*	$\tilde{Z}$	95% CI of $\tilde{\mu}_Z$	$\tilde{Z}$	95% CI of $\tilde{\mu}_Z$	$\tilde{Z}$	95% CI of $\tilde{\mu}_Z$
3	1.5	3.720	(3.701, 3.738)	3.557	(3.512, 3.622)	2.910	(2.867, 2.951)
3	2	4.978	(4.952, 5.001)	4.852	(4.776, 4.946)	3.838	(3.789, 3.894)
3	2.5	6.185	(6.156, 6.218)	6.261	(6.149, 6.372)	4.799	(4.735, 4.868)
3	3	7.462	(7.426, 7.498)	7.646	(7.511, 7.736)	5.715	(5.624, 5.795)
3	4	9.913	(9.857, 9.963)	10.459	(10.286, 10.652)	7.612	(7.511, 7.735)
3	5	12.398	(12.34, 12.459)	13.062	(12.87, 13.27)	9.355	(9.211, 9.486)
3	10	24.850	(24.737, 24.97)	26.462	(26.04, 26.859)	18.381	(18.123, 18.639)
3	20	49.499	(49.258, 49.737)	53.943	(53.013, 54.783)	36.991	(36.384, 37.632)
3	40	99.411	(98.936, 99.863)	72.256	(72.029, 72.441)	71.898	(70.902, 72.886)
3	60	148.569	(147.845, 149.261)	72.549	(72.378, 72.747)	108.077	(106.386, 111.164)
3	80	199.439	(198.412, 200.454)	70.342	(70.17, 70.499)	145.418	(143.286, 146.866)
4	1.5	2.915	(2.903, 2.926)	2.612	(2.566, 2.65)	2.446	(2.417, 2.476)
4	2	3.895	(3.879, 3.909)	3.695	(3.642, 3.75)	3.297	(3.254, 3.338)
4	2.5	4.878	(4.86, 4.897)	4.790	(4.712, 4.865)	4.119	(4.072, 4.166)
4	3	5.861	(5.839, 5.881)	5.976	(5.911, 6.06)	5.014	(4.953, 5.068)
4	4	7.772	(7.742, 7.801)	8.073	(7.972, 8.177)	6.608	(6.546, 6.688)
4	5	9.734	(9.695, 9.771)	10.186	(10.093, 10.303)	8.305	(8.207, 8.397)
4	10	19.510	(19.439, 19.581)	20.296	(20.059, 20.568)	16.541	(16.355, 16.749)
4	20	38.835	(38.674, 38.989)	40.905	(40.506, 41.385)	33.349	(32.971, 33.792)
4	40	78.164	(77.899, 78.47)	66.403	(66.166, 66.663)	66.475	(65.705, 67.296)
4	60	116.602	(116.169, 117.013)	67.472	(67.29, 67.68)	100.313	(98.003, 102.123)
4	80	155.602	(154.992, 156.125)	64.329	(64.089, 64.517)	131.289	(129.999, 132.689)
5	1.5	2.470	(2.462, 2.478)	2.173	(2.152, 2.202)	2.182	(2.158, 2.206)
5	2	3.289	(3.279, 3.299)	3.085	(3.049, 3.124)	2.909	(2.877, 2.944)
5	2.5	4.126	(4.112, 4.139)	4.099	(4.054, 4.148)	3.701	(3.655, 3.737)
5	3	4.920	(4.904, 4.937)	5.019	(4.974, 5.07)	4.429	(4.395, 4.469)
5	4	6.592	(6.57, 6.613)	6.743	(6.682, 6.798)	5.894	(5.838, 5.945)
5	5	8.213	(8.185, 8.237)	8.458	(8.366, 8.551)	7.365	(7.287, 7.439)
5	10	16.441	(16.386, 16.491)	17.066	(16.93, 17.271)	14.982	(14.833, 15.113)
5	20	32.963	(32.866, 33.076)	33.929	(33.648, 34.293)	29.666	(29.45, 29.956)
5	40	65.823	(65.611, 66.059)	61.285	(60.915, 61.583)	59.558	(58.967, 60.172)
5	60	98.623	(98.303, 98.943)	63.706	(63.499, 63.889)	90.950	(90.45, 91.367)
5	80	131.510	(131.068, 131.953)	60.164	(59.903, 60.389)	118.539	(117.608, 119.479)
6	1.5	2.185	(2.18, 2.191)	1.893	(1.869, 1.912)	1.975	(1.958, 1.993)
6	2	2.911	(2.903, 2.919)	2.762	(2.731, 2.795)	2.664	(2.639, 2.686)
6	2.5	3.647	(3.636, 3.658)	3.667	(3.628, 3.705)	3.378	(3.348, 3.407)
6	3	4.355	(4.342, 4.367)	4.433	(4.386, 4.48)	4.030	(3.986, 4.065)
6	4	5.812	(5.797, 5.83)	5.955	(5.897, 6.021)	5.390	(5.338, 5.455)
6	5	7.260	(7.241, 7.28)	7.444	(7.375, 7.526)	6.733	(6.662, 6.801)
6	10	14.548	(14.51, 14.592)	15.065	(14.945, 15.201)	13.647	(13.51, 13.75)
6	20	29.135	(29.054, 29.221)	29.849	(29.618, 30.096)	27.115	(26.854, 27.369)
6	40	58.194	(58.027, 58.359)	56.623	(56.325, 56.895)	54.436	(53.926, 54.922)
6	60	87.460	(87.203, 87.705)	60.890	(60.684, 61.101)	81.105	(80.678, 81.44)
6	80	116.202	(115.874, 116.533)	56.752	(56.536, 56.974)	107.413	(106.462, 108.299)
7	1.5	1.984	(1.979, 1.989)	1.731	(1.712, 1.747)	1.838	(1.827, 1.854)
7	2	2.639	(2.633, 2.646)	2.523	(2.493, 2.557)	2.453	(2.431, 2.476)
7	2.5	3.310	(3.302, 3.319)	3.321	(3.295, 3.35)	3.098	(3.076, 3.124)
7	3	3.970	(3.96, 3.981)	3.986	(3.954, 4.021)	3.687	(3.663, 3.719)
7	4	5.294	(5.281, 5.308)	5.354	(5.311, 5.402)	4.943	(4.904, 4.983)
7	5	6.613	(6.595, 6.631)	6.644	(6.599, 6.699)	6.134	(6.086, 6.192)
7	10	13.215	(13.181, 13.249)	13.419	(13.305, 13.53)	12.427	(12.334, 12.546)
7	20	26.456	(26.398, 26.522)	26.754	(26.511, 26.989)	24.804	(24.614, 24.998)
7	40	52.946	(52.807, 53.088)	52.323	(51.964, 52.697)	49.760	(49.403, 50.212)
7	60	79.275	(79.081, 79.488)	58.316	(58.108, 58.518)	74.215	(73.268, 75.16)
7	80	105.890	(105.604, 106.183)	53.973	(53.752, 54.189)	98.197	(97.521, 98.852)

Table 2. *Estimated Interval Lengths for Weibull Modulus (m) for n=8, 9, 10 and 20*

n	m	MLE $\tilde{Z}$	MLE 95% CI of $\tilde{\mu}_Z$	BW1 $\tilde{Z}$	BW1 95% CI of $\tilde{\mu}_Z$	BW2 $\tilde{Z}$	BW2 95% CI of $\tilde{\mu}_Z$
8	1.5	1.824	(1.82, 1.828)	**1.576**	(1.558, 1.595)	*1.699*	(1.687, 1.711)
8	2	2.438	(2.431, 2.444)	*2.325*	(2.305, 2.35)	**2.270**	(2.252, 2.287)
8	2.5	*3.041*	(3.034, 3.048)	*3.042*	(3.016, 3.064)	**2.872**	(2.848, 2.892)
8	3	*3.656*	(3.647, 3.665)	*3.674*	(3.645, 3.704)	**3.438**	(3.414, 3.464)
8	4	*4.868*	(4.856, 4.879)	4.953	(4.915, 4.994)	**4.637**	(4.599, 4.676)
8	5	*6.090*	(6.077, 6.106)	*6.127*	(6.079, 6.177)	**5.744**	(5.708, 5.791)
8	10	*12.188*	(12.156, 12.22)	*12.296*	(12.201, 12.392)	**11.544**	(11.446, 11.637)
8	20	*24.372*	(24.318, 24.423)	*24.461*	(24.269, 24.649)	**23.011**	(22.849, 23.17)
8	40	48.682	(48.565, 48.8)	*48.268*	(48.002, 48.563)	**46.179**	(45.813, 46.556)
8	60	73.083	(72.911, 73.258)	**56.143**	(55.991, 56.292)	*69.359*	(69.074, 69.659)
8	80	97.440	(97.204, 97.69)	**51.766**	(51.536, 51.982)	*92.057*	(91.249, 92.774)
9	1.5	1.693	(1.689, 1.697)	**1.477**	(1.462, 1.493)	*1.599*	(1.588, 1.612)
9	2	2.256	(2.251, 2.261)	*2.208*	(2.189, 2.227)	**2.151**	(2.132, 2.166)
9	2.5	*2.828*	(2.821, 2.834)	*2.829*	(2.81, 2.85)	**2.688**	(2.67, 2.707)
9	3	*3.389*	(3.382, 3.397)	*3.416*	(3.393, 3.437)	**3.227**	(3.203, 3.248)
9	4	*4.520*	(4.51, 4.531)	*4.555*	(4.52, 4.588)	**4.311**	(4.282, 4.34)
9	5	*5.657*	(5.645, 5.669)	*5.715*	(5.668, 5.75)	**5.396**	(5.361, 5.443)
9	10	*11.292*	(11.27, 11.319)	11.494	(11.4, 11.57)	**10.904**	(10.828, 10.984)
9	20	*22.601*	(22.552, 22.653)	22.842	(22.666, 23.002)	**21.688**	(21.521, 21.821)
9	40	*45.097*	(45, 45.194)	*45.254*	(44.975, 45.619)	**43.201**	(42.922, 43.507)
9	60	67.735	(67.583, 67.88)	**54.238**	(54.098, 54.371)	*66.198*	(65.733, 66.654)
9	80	90.484	(90.283, 90.697)	**50.060**	(49.81, 50.327)	*85.335*	(84.809, 85.924)
10	1.5	1.591	(1.587, 1.594)	**1.405**	(1.392, 1.417)	*1.520*	(1.508, 1.528)
10	2	2.125	(2.12, 2.129)	*2.079*	(2.063, 2.093)	**2.023**	(2.01, 2.037)
10	2.5	*2.651*	(2.645, 2.656)	*2.642*	(2.625, 2.659)	**2.522**	(2.507, 2.539)
10	3	*3.183*	(3.177, 3.19)	*3.191*	(3.169, 3.215)	**3.042**	(3.019, 3.062)
10	4	*4.250*	(4.241, 4.259)	*4.286*	(4.259, 4.311)	**4.085**	(4.06, 4.11)
10	5	*5.300*	(5.288, 5.311)	5.360	(5.325, 5.399)	**5.110**	(5.074, 5.14)
10	10	*10.597*	(10.575, 10.618)	10.757	(10.674, 10.826)	**10.267**	(10.195, 10.345)
10	20	*21.164*	(21.121, 21.212)	21.532	(21.387, 21.667)	**20.556**	(20.442, 20.705)
10	40	*42.485*	(42.394, 42.573)	*42.803*	(42.565, 43.049)	**40.831**	(40.554, 41.148)
10	60	63.513	(63.389, 63.642)	**52.596**	(52.472, 52.712)	61.186	(60.498, 61.869)
10	80	84.785	(84.596, 84.97)	**48.616**	(48.408, 48.875)	80.861	(80.362, 81.266)
20	1.5	1.073	(1.071, 1.074)	**1.021**	(1.013, 1.028)	*1.051*	(1.046, 1.055)
20	2	*1.430*	(1.428, 1.432)	*1.434*	(1.425, 1.44)	**1.405**	(1.399, 1.411)
20	2.5	*1.787*	(1.784, 1.789)	*1.796*	(1.788, 1.804)	**1.761**	(1.753, 1.769)
20	3	*2.145*	(2.142, 2.148)	*2.155*	(2.144, 2.164)	**2.112**	(2.101, 2.121)
20	4	*2.859*	(2.855, 2.863)	*2.865*	(2.852, 2.878)	**2.810**	(2.798, 2.821)
20	5	*3.573*	(3.568, 3.578)	*3.583*	(3.566, 3.601)	**3.517**	(3.499, 3.534)
20	10	*7.145*	(7.134, 7.156)	*7.180*	(7.146, 7.211)	**7.049**	(7.02, 7.079)
20	20	*14.304*	(14.285, 14.324)	*14.359*	(14.301, 14.429)	**14.097**	(14.033, 14.17)
20	40	28.609	(28.569, 28.649)	*28.251*	(28.133, 28.361)	**27.807**	(27.686, 27.907)
20	60	42.879	(42.818, 42.939)	**40.032**	(39.899, 40.178)	*42.039*	(41.721, 42.353)
20	80	57.195	(57.11, 57.276)	**40.194**	(40.075, 40.299)	*54.416*	(54.067, 54.775)

As shown in Tables 1 and 2 and Figures 3 and 4, the BW2 model generally outperforms the others by yielding the smallest values for the median interval length $(\tilde{Z})$. The exceptional cases are m=60, 80 for all n, and m=1.5 for $n >5$, where BW1 model is the best method. These are the cases where m is close to the uniform prior boundaries 1 and 100. The restriction of the prior that m cannot take values below 1 and above 100 is likely to induce an effect of narrowing of the interval length. Figures 3-4 show that performance of the BW1 model becomes significantly better as m approaches 1 or 100.

The MLE method is the worst method according to $\tilde{Z}$ values, however there is no statistical difference between the MLE method and the BW1 model for $2.5\leq m \leq 20$ and all n, since the corresponding confidence intervals in Tables 1 and 2 intersect with each other. Also, by increasing n, the performances of all the three methods become closer to each other.
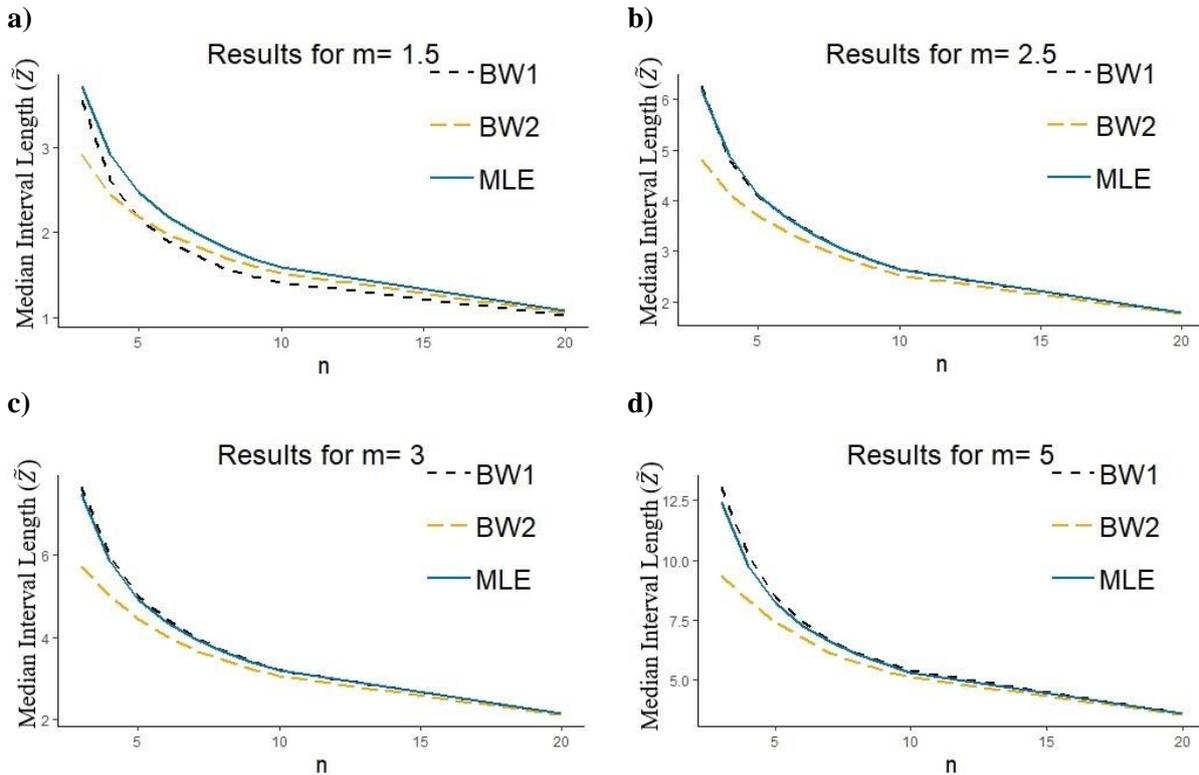
Figure 3. *Expected Median Interval Lengths for (a) $m=1.5$ (b) $m=2.5$, (c) $m=3$, (d) $m=5$*
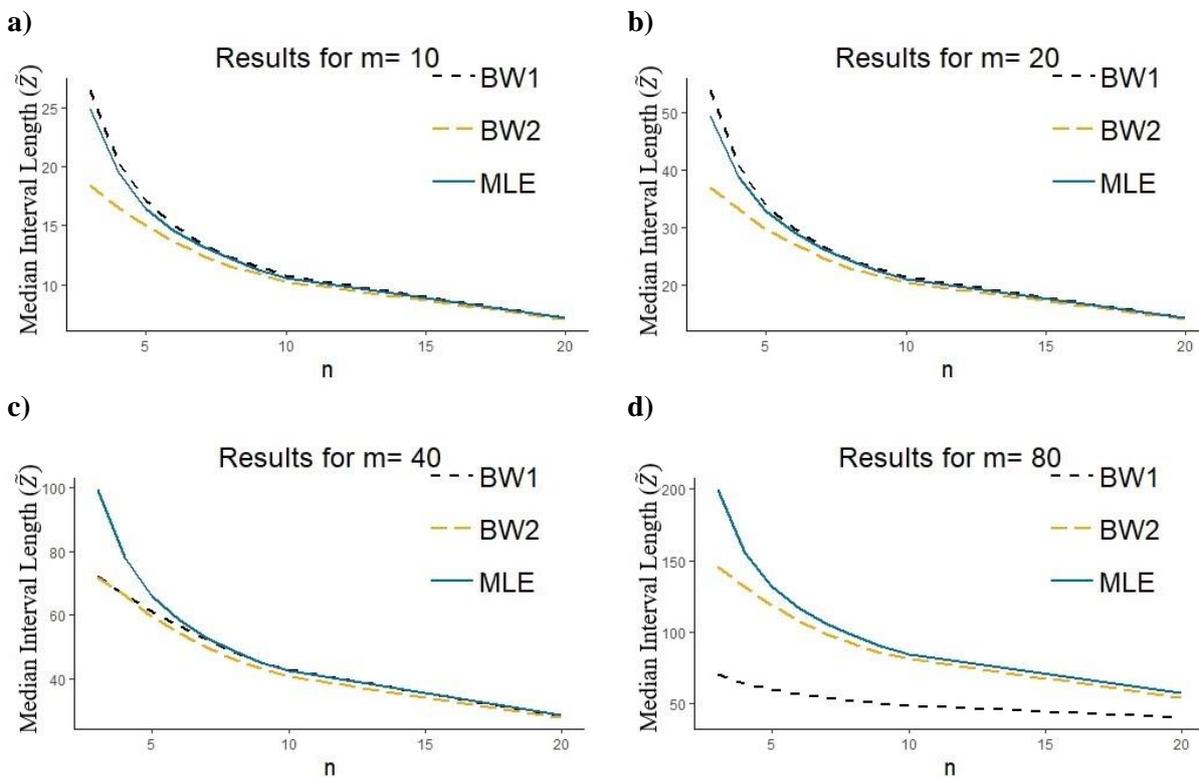

Figure 4. *Expected Median Interval Lengths for (a) $m=10$ (b) $m=20$, (c) $m=40$, (d) $m=80$*

Having established the superior performance of the Bayesian models, a more practical way to compare in terms of engineering costs is to calculate the relative sample sizes needed to obtain approximately the same $\tilde{Z}$ values. For this purpose, the BW2 model has been chosen as the reference method for $m < 60$ since it produces the smallest values for the median interval length $(\tilde{Z})$ in the majority of the cases. Similarly, the

BW1 model has the smallest lengths, and hence is the reference method for $m \geq 60$. Tables 3 and 4 show the results for a comparison of the three methods used where the reference method, as described above, is the method that produces the minimum sample size in each row. The sample sizes required by the other two methods that produce the same or very similar interval lengths as the reference method have been calculated and listed in the same row.

First, let's examine the graph of $m = 40$ in Figure 4(c). For $n = 3$, the $\tilde{Z}$ value of the BW2 model is approximately 72; and this value is achieved by the MLE method at $n = 4$ and by the BW1 at $n = 3$. This means that the BW1 and BW2 models show approximately the same performance, but as compared to the MLE method, a saving of (4-3)/4 = 25% is achieved by the BW2 model in testing costs for reliability analysis. This is reflected at the row with $m = 40$ and the BW2 column's value at $n = 3$ in Table 3 with a 0% saving as compared to the BW1 model and 25% saving as compared to the MLE method. Similarly, as shown in Table 4, the $\tilde{Z}$ value of the BW1 model is 67 for $n = 4$ and $m = 60$, and this value is approximately achieved by the MLE method at $n=10$ indicating a saving of (10-4)/10 = 60% if the BW1 model were used, and is achieved by the BW2 model at $n=9$ indicating a saving of (9-4)/9 = 56% if the BW1 model were used.

According to Table 3, the percent savings achieved by using the BW2 model is at least 9% for of $3 \leq m \leq 40$ and $n \leq 10$. For $m = 1.5$, there is almost no difference between the BW1 and BW2 models for $n \leq 6$, and the BW1 model starts to perform better for $n > 6$. Similarly, Table 4 indicates a minimal saving of 9% for $n \leq 20$ when the BW1 model is used.

Table 3. *Comparison of median expected lengths in terms of sample size (Ref: BW2)*

Reference: BW2			Equivalent n to produce $\tilde{Z}$				Reference: BW2			Equivalent n to produce $\tilde{Z}$			
			BW1		MLE					BW1		MLE	
m	n	$\tilde{Z}$	n	% Saving	n	% Saving	m	n	$\tilde{Z}$	n	% Saving	n	% Saving
1.5	3	2.91	4	0.25	4	0.25	5	3	9.355	4	0.25	4	0.25
1.5	4	2.446	4	0.00	5	0.20	5	4	8.305	5	0.20	5	0.20
1.5	5	2.182	5	0.00	6	0.17	5	5	7.365	6	0.17	6	0.17
1.5	6	1.975	6	0.00	7	0.14	5	6	6.733	7	0.14	7	0.14
1.5	7	1.838	6	-0.17	8	0.13	5	7	6.134	8	0.13	8	0.13
1.5	8	1.699	7	-0.14	9	0.11	5	8	5.744	9	0.11	9	0.11
1.5	9	1.599	8	-0.13	10	0.10	5	9	5.396	10	0.10	10	0.10
1.5	10	1.52	9	-0.11	11	0.09	5	10	5.11	11	0.09	11	0.09
1.5	20	1.051	19	-0.05	20	0.00	5	20	3.517	20	0.00	20	0.00
2	3	3.838	4	0.25	4	0.25	10	3	18.381	5	0.40	4	0.25
2	4	3.297	5	0.20	5	0.20	10	4	16.541	5	0.20	5	0.20
2	5	2.909	5	0.00	6	0.17	10	5	14.982	6	0.17	6	0.17
2	6	2.664	6	0.00	7	0.14	10	6	13.647	7	0.14	7	0.14
2	7	2.453	7	0.00	8	0.13	10	7	12.427	8	0.13	8	0.13
2	8	2.27	8	0.00	9	0.11	10	8	11.544	9	0.11	9	0.11
2	9	2.151	9	0.00	10	0.10	10	9	10.904	10	0.10	10	0.10
2	10	2.023	11	0.09	11	0.09	10	10	10.267	11	0.09	11	0.09
2	20	1.405	20	0.00	20	0.00	10	20	7.049	20	0.00	20	0.00
3	3	5.715	4	0.25	4	0.25	20	3	36.991	4	0.25	4	0.25
3	4	5.014	5	0.20	5	0.20	20	4	33.349	5	0.20	5	0.20
3	5	4.429	6	0.17	6	0.17	20	5	29.666	6	0.17	6	0.17
3	6	4.03	7	0.14	7	0.14	20	6	27.115	7	0.14	7	0.14
3	7	3.687	8	0.13	8	0.13	20	7	24.804	8	0.13	8	0.13
3	8	3.438	9	0.11	9	0.11	20	8	23.011	9	0.11	9	0.11
3	9	3.227	10	0.10	10	0.10	20	9	21.688	10	0.10	10	0.10
3	10	3.042	11	0.09	11	0.09	20	10	20.556	11	0.09	11	0.09
3	20	2.112	20	0.00	20	0.00	20	20	14.097	20	0.00	20	0.00
4	3	7.612	4	0.25	4	0.25	40	3	71.898	3	0.00	4	0.25
4	4	6.608	5	0.20	5	0.20	40	4	66.475	4	0.00	5	0.20
4	5	5.894	6	0.17	6	0.17	40	5	59.558	5	0.00	6	0.17
4	6	5.39	7	0.14	7	0.14	40	6	54.436	6	0.00	7	0.14
4	7	4.943	8	0.13	8	0.13	40	7	49.76	8	0.13	8	0.13
4	8	4.637	9	0.11	9	0.11	40	8	46.179	9	0.11	9	0.11
4	9	4.311	10	0.10	10	0.10	40	9	43.201	10	0.10	10	0.10
4	10	4.085	11	0.09	11	0.09	40	10	40.831	11	0.09	11	0.09
4	20	2.81	20	0.00	20	0.00	40	20	27.807	20	0.00	21	0.05

Table 4. *Comparison of median expected lengths in terms of sample size (Ref: BW1)*

Reference: BW1			Equivalent n to produce $\tilde{Z}$				Reference: BW1			Equivalent n to produce $\tilde{Z}$			
			BW2		MLE					BW2		MLE	
m	n	$\tilde{Z}$	n	% Saving	n	% Saving	m	n	$\tilde{Z}$	n	% Saving	n	% Saving
60	3	72.549	8	0.63	9	0.67	80	3	70.342	14	0.79	15	0.80
60	4	67.472	9	0.56	10	0.60	80	4	64.329	16	0.75	17	0.76
60	5	63.706	10	0.50	11	0.55	80	5	60.164	18	0.72	19	0.74
60	6	60.890	11	0.45	11	0.45	80	6	56.752	19	0.68	21	0.71
60	7	58.316	12	0.42	12	0.42	80	7	53.973	21	0.67	22	0.68
60	8	56.143	12	0.33	13	0.38	80	8	51.766	22	0.64	23	0.65
60	9	54.238	13	0.31	14	0.36	80	9	50.060	22	0.59	24	0.63
60	10	52.596	14	0.29	15	0.33	80	10	48.616	23	0.57	24	0.58
60	20	40.032	22	0.09	22	0.09	80	20	40.194	27	0.26	28	0.29

Figure 5 plots median confidence intervals of the methods for m= 1.5, 10 and 80, that is, any lower or upper bound in the plots is the median of the lower or upper bounds of 10,000 confidence intervals estimated by Monte Carlo simulations. They show that the BW2 model generates confidence intervals that are more

centered around m as compared to the MLE and the BW1. This phenomenon is particularly noticeable for small samples.
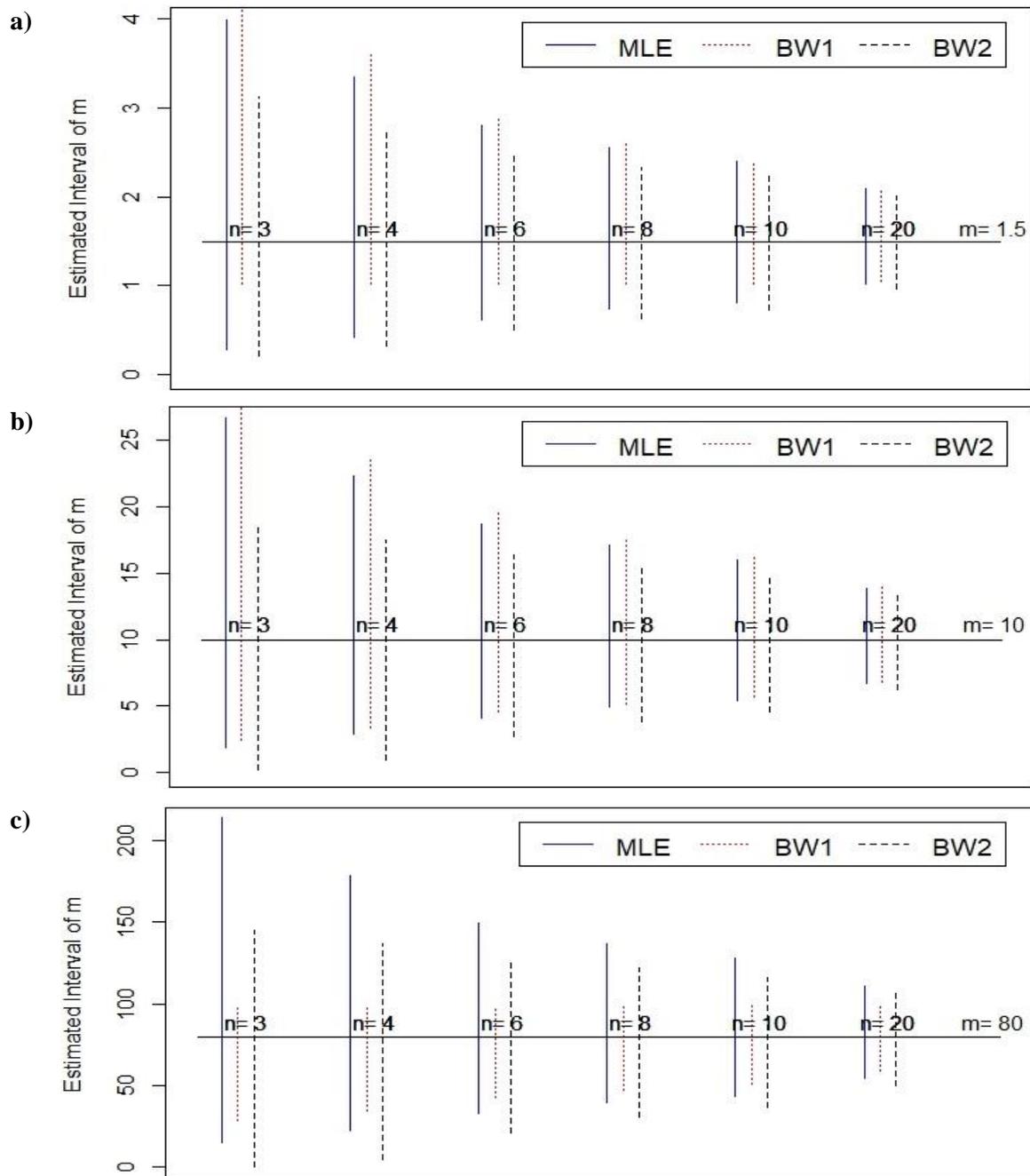
Figure 5. *The position of estimated confidence intervals according to a) m=1.5 b) m=10 and c) m=80*

4. VERIFICATION OF THE SIMULATION

Verification of the simulation results was carried out by an experimental study cited in the literature [73]. The experimental study was on the fracture strengths of 19 identical composite specimens. The composite specimens were prepared from quasi-isotropic carbon– epoxy sheets with $(0°)_3$ configuration, 0.89 mm thickness, and 295 g/m² weight. The tension experiments were carried out on an Instron 8516+universal testing machine according to ASTM D3039 standard [73]. A crosshead speed of 1.33 mm/min was used and room temperature conditions existed during the tests. The fracture strength values measured are presented in Table 5.

Table 5. *Fracture strength values from tension experiments*

Test No	1	2	3	4	5	6	7	8	9	10
Fracture strength (MPa)	532.7	502.5	442	473	519	502.7	477	510	522	552
Test No	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	
Fracture strength (MPa)	522	439	513.6	497.5	521.6	450.9	476.5	507.3	463.5	

In order to diversify the application study in terms of sample size, besides the entire sample, the random subsets of size of 3, 5 and 10 were drawn from the sample of 19 observations. The data set consists of {513.6, 442, 519} for n=3, {510, 521.6, 442, 439, 519} for n=5 and {439, 521.6, 513.6, 450.9, 442, 463.5, 473, 519, 476.5, 477} for n=10. By the three methods, the confidence intervals of m were estimated and the result of the best method for each n was shown in bold in Table 6.

Table 6. *The experimental results for n=3,5,10 and 19*

n	Method	Confidence Interval of m ($\alpha = 0.05$)		
		Lower Bound	**Upper Bound**	**Interval Length**
3	MB	2.463	35.720	33.257
	BW1	2.919	41.927	39.008
	BW2	**0.409**	**25.774**	**25.365**
5	MB	4.890	27.463	22.573
	BW1	5.014	30.329	25.315
	BW2	**2.256**	**23.490**	**21.234**
10	MB	8.536	25.191	16.655
	BW1	9.091	25.595	16.504
	BW2	**7.442**	**23.318**	**15.876**
19	MB	11.980	25.132	13.152
	BW1	12.162	25.205	13.043
	BW2	**11.171**	**24.021**	**12.85**

As can be seen in Table 6, the experimental results are quite similar to simulation results and the BW2 model outperforms the other methods. Especially for $n = 3$, it yields a much narrower interval than those of the MLE and BW1. As n increases, performances of the methods become closer to each other, but the BW2 model maintains its superiority by a small difference in estimated interval length.

5. RESULTS

This study compares the performances of the two proposed Bayesian models against the Maximum Likelihood Method (MLE) for estimating confidence intervals of the Weibull modulus in small samples. The first Bayesian model (BW1) assumes that the Weibull modulus has a uniform prior on the interval of (1, 100). The second Bayesian model (BW2) assumes that the Weibull modulus is between 1 and 100 with a probability, 1-α. Non-informative priors are used for the scale parameter.

In the BW1 model, use of uniform prior on the interval of (1, 100) was recently proposed in the materials science literature, and the lower and uppers limits, 1 and 100, was supported by practical and theoretical studies in the literature. The BW2 model is a relaxed version of the BW1 model, assigning a high probability (1-α) to the event that Weibull modulus is in this interval, instead of requiring it to be in the interval with a probability of 1, which is the case in the BW1 model.

The Weibull modulus values between 1.5 and 80, and sample size values between 3 and 20 are considered in this study, and the simulations showed that the Bayesian models always have better performance than the MLE method. In general, the BW1 model is the best method for the Weibull modulus values close to 1

and 100, and the BW2 model is the best for the other values between 1 and 100, particularly for $2 \leq m \leq 40$. The performances of the Bayesian models become better as the sample size decreases, particularly for $n \leq 10$, which is an expected result from the general Bayesian theory. In addition, the BW2 model generates confidence intervals that are more centered around m. This result is important for engineers, because they usually expect the unknown parameter to lie at or near the center of the estimated interval.

In summary, this study proposes the BW2 model for estimating the confidence (credible) interval of Weibull modulus, especially for small sample sizes since the BW2 model has more relaxed assumptions than the BW1 model, and performs better unless the Weibull modulus values close to 1 and 100.

There are many alternative Bayesian methods that can be used for Weibull applications in materials science such as recursive methods. This study demonstrates that Bayesian methods have a promising future in reliability studies in materials science, because, they attempt to capture the existing domain knowledge, such as increasing failure rates, and reflect it in the estimation process. This results in higher estimation precision.

A final word of caution is that while using priors based on unfounded assumptions and non-existing knowledge may result in Bayesian methods with superior performances, the obtained results would likely be non-representative of the domain and misleading. In Bayesian analysis, priors should reflect the true domain knowledge. In this study, care has been taken to ensure compliance to this general rule.

CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

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