Some Characterizations of Curves in n-dimensional Euclidean Space IE^n

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ABSTRACT: In this work, we deal with a curve whose position vector can be expressed with the help of Frenet Frame in $n-$dimensional Euclidean space IE^n. We classify this type of curve with regards to curvature functions and get certain consequences for $T-$constant, $N-$constant and constant ratio curves in IE^n.

Keywords: Constant ratio curves, $W-$curves, position vector

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INTRODUCTION

A unit speed curve is described by most geometers, as a curve $x = x(s)$ whose parametrization is written by arclength parameter s. So as to define curvatures of a curve, the well-known Frenet formulas can be explained as follows:

A regular curve $x = x(s) : I \subset IR \to IE^3$, given with the arclength parameter, has at least four continuous derivatives. Denote by the tangent vector $T = x'(s)$; we suppose $T'(s)$ is nonzero for all s. Then, it is possible to define the principle normal vector field N_1 and the curvature κ_1 by $T'(s) = \kappa_1 N_1$. Since the vectors T and N_1 are orthogonal, the binormal vector field is yield by the relation $N_2' = -\kappa_2 N_1$ defines the torsion κ_2. κ_1 and κ_2 are also called the first and the second curvatures of the curve.

The Frenet formulas are

$$T' = \kappa_1 N_1,$$
$$N_1' = -\kappa_1 T + \kappa_2 N_2, \quad (1)$$
$$N_2' = -\kappa_2 N_1.$$

The significant planes in IE^3 are osculating, rectifying and normal planes which are spanned by $\{T, N_1\}$, $\{T, N_2\}$, and $\{N_1, N_2\}$, respectively. If a curve's position vector is orthogonal to its normal vector field, then it is called a rectifying curve in IE^3. In $n-$ dimensional Euclidean space, we have a similar definition as $3-$ dimensional Euclidean space. In IE^n, rectifying curve's position vector is spanned by $\{T, N_2, N_3, ..., N_{n-1}\}$.

B.Y. Chen introduces the notion of the rectifying curves in his study(Chen, 2003). In that paper, he gives a basic classification of rectifying curves. Especially, in (Chen et al., 2005), the author gives the connection between these curves and centrodes that have a place in kinematics and mechanics. Also in the study (İlarslan et al., 2008), the authors give some characterizations about rectifying curves. Recently in (Cambie et al., 2016), rectifying curves have been studied in arbitrary Euclidean space.

In (Chen, 2001), B.Y. Chen gives a new concept for Euclidean spaces: constant ratio curves. The author presents in this study which curves satisfy the condition of being a constant ratio curve. Also in (Chen, 2002), B.Y. Chen announces the notion of $N-$ constant and $T-$ constant types of curves. In studies (Büyükkütük et al., 2015, 2016, 2017; Gürpınar et al., 2015; Kişi et al., 2015, 2017, 2018; Öztürk et al., 2017, 2018), by using Frenet and Bishop frame, the necessary and sufficient conditions are given for curves to become $T-$ constant and $N-$ constant, respectively.

In this study, we deal with any curve in $n-$ dimensional Euclidean space IE^n with regard to its Frenet frame $\{T, N_1, N_2, ..., N_{n-1}\}$. So, we can write the parametrization of the curve as:

$$x(s) = m_0(s)T(s) + \sum_{i=1}^{n-1} m_i(s)N_i(s) \quad (2)$$

where the functions $m_i(s)$, $i = 0, 1, 2,.., n-1$ are differentiable. We classify this curve with regards to curvature functions and obtain the sufficient and necessary conditions for this curve to become $T-$ constant, $N-$ constant and constant-ratio.

BASIC NOTATION

In $n-$dimensional Euclidean space, a unit speed curve $x : I \rightarrow IE^n$ can be described by the Frenet frame, which is associated with the vectors $T, N_1, N_2, ..., N_{n-1}$ and n-1 functions $\kappa_1, \kappa_2, ..., \kappa_{n-1} : I \rightarrow IR$. Thus, for $n-$dimensional Euclidean space, Frenet formulas can be given by

$$
\begin{bmatrix} T' \\ N_1' \\ N_2' \\ . \\ . \\ . \\ N_{n-2}' \\ N_{n-1}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & 0. & . & . & . & 0 & 0 \\ -\kappa_1 & 0 & \kappa_2 & . & . & . & 0 & 0 \\ 0 & -\kappa_2 & 0 & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 0 & 0 & . & . & . & . & . & \kappa_{n-1} \\ 0 & 0 & . & . & . & . & -\kappa_{n-1} & 0 \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \\ . \\ . \\ . \\ N_{n-2} \\ N_{n-1} \end{bmatrix}. \quad (3)
$$

If any curve has constant Frenet curvatures, then it is known as $W-$curve. A parametrization of a unit speed $W-$curve in IE^{2r} is represented by

$$ x(s) = a_0 + \sum_{i=1}^{r} (a_i \cos \mu_i s + b_i \sin \mu_i s), \quad (4) $$

and a unit speed $W-$curve in IE^{2r+1} is represented by

$$ x(s) = a_0 + b_0 s + \sum_{i=1}^{r} (a_i \cos \mu_i s + b_i \sin \mu_i s), \quad (5) $$

where $a_0, b_0, a_1, ..., a_r, b_1, ..., b_r$ are constant vectors in IE^n and $\mu_1 < \mu_2 < ... < \mu_r$ are real numbers(Gluck, 1966; Gray, 1993; Klein et al., 1871).

CHARACTERIZATIONS OF CURVES IN IE^n

Now, we consider curves given with Equality 2. Differentiating Eq. 2. with regard to arclength parameter, we get

$$ x'(s) = m_0'(s)T(s) + m_0(s)T'(s) + \sum_{i=1}^{n-1} (m_i'(s)N_i(s) + m_i(s)N_i'(s)). \quad (6) $$

Then, by the use of Frenet formulas 3., we yield

$$ T(s) = m_0'(s)T(s) + m_0(\kappa_1(s)N_1(s)) $$

$$ + m_1'(s)N_1(s) + m_1(s)(-\kappa_1(s)T(s) + \kappa_2(s)N_2(s)) + \sum_{i=2}^{n-1} (m_i'(s)N_i(s)) $$

$$ + \sum_{i=2}^{n-2} m_i(s)(-\kappa_i(s)N_{i-1}(s) + \kappa_{i+1}(s)N_{i+1}(s)) - \kappa_{n-1}(s)N_{n-2}(s). \quad (7) $$

Equivalently,

$$T(s) = \left(m_0'(s) - \kappa_1(s)m_1(s)\right)T(s)$$
$$+ \sum_{i=2}^{n-2} (m_i'(s) + \kappa_i(s)m_{i-1}(s) - \kappa_{i+1}(s)m_{i+1}(s))N_i(s)$$
$$+ \left(m_{n-1}'(s) + \kappa_{n-1}(s)m_{n-2}(s)\right)N_{n-1}(s). \qquad (8)$$

It follows that,

$$m_0'(s) - \kappa_1(s)m_1(s) = 1$$
$$m_1'(s) + \kappa_1(s)m_0(s) - \kappa_2(s)m_2(s) = 0, \qquad (9)$$
$$m_i'(s) + \kappa_i(s)m_{i-1}(s) - \kappa_{i+1}(s)m_{i+1}(s) = 0 \quad with \quad i \in \{2,3,...,n-2\},$$
$$m_{n-1}'(s) + \kappa_{n-1}(s)m_{n-2}(s) = 0.$$

The functions $m_i(s)$, $1 \le i \le n-1$, can be given with the Frenet curvatures, the curvature function $m_0(s)$, and their derivatives. From the Equation system 9.,

$$m_1(s) = \frac{m_0'(s) - 1}{\kappa_1(s)} \qquad (10)$$

and

$$m_2(s) = \frac{\kappa_1^2(s)m_0(s)}{\kappa_2(s)\left(m_0'(s) - 1\right)} \left(\frac{m_0'(s) - 1}{\kappa_1(s)} \right) + \frac{1}{\kappa_2(s)} \left(\frac{m_0'(s) - 1}{\kappa_1(s)} \right)'. \qquad (11)$$

Introducing functions $m_{1,0}$, $m_{2,0}$, *and* $m_{2,1}$, we rewrite these equations as

$$m_1(s) = m_{1,0}(s) \frac{m_0'(s) - 1}{\kappa_1(s)}, \qquad (12)$$

$$m_2(s) = m_{2,0}(s) \left(\frac{m_0'(s) - 1}{\kappa_1(s)} \right) + m_{2,1}(s) \left(\frac{m_0'(s) - 1}{\kappa_1(s)} \right)'. \qquad (13)$$

By induction method for the Equation system 9., we get

$$m_i(s) = \sum_{k=0}^{i-1} m_{i,k}(s) \frac{\partial^k}{\partial s^k} \left(\frac{m_0'(s) - 1}{\kappa_1(s)} \right), \qquad (14)$$

for $i \in \{3,4,...,n-1,\}$. Inductively, we can give the functions $m_{i,k}$ by the following system

$$\begin{cases} m_{1,0} = 1 \\ m_{2,0} = \dfrac{\kappa_1^2(s)m_0(s)}{\kappa_2(s)\left(m_0'(s) - 1\right)}, \quad m_{2,1} = \dfrac{1}{\kappa_2(s)} \end{cases} \qquad (15)$$

and for $i \in \{3,4,...,n-1,\}$,

$$
\begin{cases}
m_{i,0}(s) = \dfrac{m'_{i-1,0}(s) + \kappa_{i-1}(s)m_{i-2,0}(s)}{\kappa_i(s)} \\[2ex]
m_{i,k}(s) = \dfrac{m'_{i-1,k}(s) + m_{i-1,k-1}(s) + \kappa_{i-1}(s)m_{i-2,k}(s)}{\kappa_i(s)}, \quad for\ \ k \in \{1,2,...,i-3\} \\[2ex]
m_{i,i-2}(s) = \dfrac{m'_{i-1,i-2}(s) + m_{i-1,i-3}(s)}{\kappa_i(s)}, \\[2ex]
m_{i,i-1}(s) = \dfrac{m_{i-1,i-2}(s)}{\kappa_i(s)}.
\end{cases}
\tag{16}
$$

Theorem 1: Let a unit speed curve $x : I \subset IR \to IE^n$ has nonzero curvatures. Thus, we have

$$
\kappa_{n-1}(s)\sum_{k=0}^{n-3} m_{n-2,k}(s)\frac{\partial^k}{\partial s^k}\left(\frac{m'_0(s)-1}{\kappa_1(s)}\right) + \left(\sum_{k=0}^{n-2} m_{n-1,k}(s)\frac{\partial^k}{\partial s^k}\left(\frac{m'_0(s)-1}{\kappa_1(s)}\right)\right)' = 0, \quad (17)
$$

where $m_{i,k}$ are inductively defined by the Systems 15. and 16.

Theorem 2: Let a unit speed curve $x : I \subset IR \to IE^n$ has the parametrization 2.

    (i) If the curve is unit speed $W-$ curve in IE^n (n=2r), then it satisfies the vectorial equation

$$
m_0^{(n)} + \sum_{k=1}^{\frac{n}{2}} a_k m_0^{(n-2k)} = 0. \quad (18)
$$

    (ii) If the curve is unit speed $W-$ curve in IE^n (n=2r+1), then it satisfies the vectorial equation

$$
m_0^{(n)} + \sum_{k=1}^{\frac{n-1}{2}} a_k m_0^{(n-2k)} - a_{\frac{n+1}{2}} = 0, \quad (19)
$$

where

$$
a_1 = \sum_{i=1}^{n-1} \kappa_i^2,
$$

$$
a_2 = \sum_{i=3}^{n-1} \kappa_i^2\left(\sum_{j=1}^{i-2} \kappa_j^2\right),
$$

$$
a_3 = \sum_{i=5}^{n-1} \kappa_i^2\left(\sum_{j=3}^{i-2} \kappa_j^2\left(\sum_{t=1}^{j-2} \kappa_t^2\right)\right),
$$

$$a_k = \sum_{i=2k-1}^{n-1} \kappa_i^2 \left(\sum_{j=2k-3}^{i-2} \kappa_j^2 \left(\sum_{t=2k-5}^{j-2} \kappa_t^2 \left(\sum ... \right) \right) \right), \quad (20)$$

$$a_{\frac{n-1}{2}} = \prod_{i=1}^{\frac{n-1}{2}} \kappa_{2i-1}^2,$$

$$a_{\frac{n}{2}} = \prod_{i=1}^{\frac{n}{2}} \kappa_{2i-1}^2,$$

$$a_{\frac{n+1}{2}} = \prod_{i=1}^{\frac{n+1}{2}} \kappa_{2i}^2,$$

Proof. Let a unit speed curve $x : I \subset IR \to IE^n$ has the parametrization 2. By the use of the first equation in Eq. 9., we write

$$m_1(s) = \frac{m_0'(s) - 1}{\kappa_1(s)}. \quad (21)$$

Since all curvatures are real constants, we get

$$m_1' = \frac{m_0''}{\kappa_1}. \quad (22)$$

Putting this value into the second equation in Eq. 9., we write

$$m_2' = \frac{m_0''}{\kappa_1 \kappa_2} + \frac{\kappa_1}{\kappa_2} m_0. \quad (23)$$

In this way, by induction, for odd α and for even β, we obtain

$$m_\alpha = \frac{m_0^{(\alpha)}}{\prod_{i=1}^{\alpha} \kappa_i} + \frac{\sum_{i=1}^{\alpha-1} \kappa_i^2}{\prod_{i=1}^{\alpha} \kappa_i} m_0^{(\alpha-2)} + \frac{\sum_{i=3}^{\alpha-1} \kappa_i^2 \left(\sum_{j=1}^{i-2} \kappa_j^2 \right)}{\prod_{i=1}^{\alpha} \kappa_i} m_0^{(\alpha-4)} + ... - \frac{\prod_{i=1}^{\frac{\alpha-1}{2}} (\kappa_{2i})^2}{\prod_{i=1}^{\alpha} \kappa_i} \quad (24)$$

and

$$m_\beta = \frac{m_0^{(\beta)}}{\prod_{i=1}^{\beta} \kappa_i} + \frac{\sum_{i=1}^{\beta-1} \kappa_i^2}{\prod_{i=1}^{\beta} \kappa_i} m_0^{(\beta-2)} + \frac{\sum_{i=3}^{\beta-1} \kappa_i^2 \left(\sum_{j=1}^{i-2} \kappa_j^2 \right)}{\prod_{i=1}^{\beta} \kappa_i} m_0^{(\beta-4)} + ... + \frac{\prod_{i=1}^{\frac{\beta}{2}} (\kappa_{2i-1})^2}{\prod_{i=1}^{\beta} \kappa_i} m_0. \quad (25)$$

We substitute these results into the last equation of the System 9., so obtain the differential equations 18. and 19., which completes the proof of the theorem.

Using Theorem 2, we can get all curvature functions. Hence, we have the following result:

Corollary 1: Let a unit speed curve $x : I \subset IR \to IE^n$ has the parametrization 2.

(i) If the curve is unit speed $W-$curve in IE^n (n=2r), then the curvature functions of the curve are

$$m_0(s) = \sum_{i=1}^{\frac{n}{2}} c_i e^{-\lambda_i s} + \sum_{i=1}^{\frac{n}{2}} d_i e^{\lambda_i s},$$

$$m_1(s) = \frac{1}{\kappa_1}\left(\sum_{i=1}^{\frac{n}{2}} -c_i \lambda_i e^{-\lambda_i s} + \sum_{i=1}^{\frac{n}{2}} d_i \lambda_i e^{\lambda_i s} - 1\right),$$

$$m_j(s) = \sum_{i=1}^{\frac{n}{2}} c_{ij} e^{-\lambda_i s} + \sum_{i=1}^{\frac{n}{2}} d_{ij} e^{\lambda_i s}, \qquad j = 2,4,...,n-2 \quad (26)$$

$$m_k(s) = \sum_{i=1}^{\frac{n}{2}} c_{ik} e^{-\lambda_i s} + \sum_{i=1}^{\frac{n}{2}} d_{ik} e^{\lambda_i s} + a_k, \quad k = 3,5,...,n-1,$$

(ii) If the curve is unit speed $W-$ curve in IE^n (n=2r+1), then the curvature functions of the curve are

$$m_0(s) = \sum_{i=1}^{\frac{n-1}{2}} c_i e^{-\lambda_i s} + \sum_{i=1}^{\frac{n-1}{2}} d_i e^{\lambda_i s} + as + b,$$

$$m_1(s) = \frac{1}{\kappa_1}\left(\sum_{i=1}^{\frac{n-1}{2}} -c_i \lambda_i e^{-\lambda_i s} + \sum_{i=1}^{\frac{n-1}{2}} d_i \lambda_i e^{\lambda_i s} + a - 1\right),$$

$$m_j(s) = \sum_{i=1}^{\frac{n-1}{2}} c_{ij} e^{-\lambda_i s} + \sum_{i=1}^{\frac{n-1}{2}} d_{ij} e^{\lambda_i s} + a_j s + b_j, \qquad j = 2,4,...,n-1 \quad (27)$$

$$m_k(s) = \sum_{i=1}^{\frac{n-1}{2}} c_{ik} e^{-\lambda_i s} + \sum_{i=1}^{\frac{n-1}{2}} d_{ik} e^{\lambda_i s} + a_k, \qquad k = 3,5,...,n-1,$$

where $a, a_j, a_k, b, b_j, c_i, c_{ij}, c_{ik}, d_i, d_{ij}, d_{ik}$ and λ_i are real constants.

Curves of Constant-ratio

Definition 1: Let any curve $x: I \subset IR \rightarrow IE^n$ be unit speed in IE^n. The sum of the tangential and normal part of the curve gives its position vector as:
$$x = x^T + x^N. \quad (28)$$

In case of $\|x^T\| : \|x^N\|$ or $\|x^T\| : \|x\|$ is constant on all points of the curve, then x is called as a constant ratio curve (Chen, 2001).

Let $\rho = \|x(s)\|$ is the distance function and T is the tangent vector field, then the gradient is defined as

$$grad\rho = \frac{d\rho}{ds}T(s) = \frac{\langle x(s), T(s)\rangle}{\|x(s)\|}T(s). \quad (29)$$

Theorem 3: (Chen, 2003) Let any curve $x: I \subset IR \rightarrow IE^n$ be unit speed in IE^n. Then $\|grad\rho\|$ is constant if and only if there exist three possible cases:

(i) The curve is in the sphere of IE^n where the center is the origin.

(ii) The curve is a line which passes the origin

(iii) The curve is given by $x(s) = cs\alpha(s), \ c \in (0,1)$. Here, $\alpha = \alpha(t), \ t = \dfrac{\sqrt{1-c^2}}{c}\ln s$ is a unit speed curve and it lies on a sphere whose center is origin.

Corollary 2: (Chen, 2003) Let any curve $x : I \subset IR \to IE^n$ be unit speed in IE^n. Then

(i) $\|grad\rho\| = 0$ if and only if the curve is in a sphere of IE^n whose center is the origin.

(ii) $\|grad\rho\| = 1$ if and only if the curve is a line which passes the origin.

(iii) $\|grad\rho\| = c$ if and only if, for $c \in (0,1)$, $cs = \|x(s)\| = \rho$.

(iv) In case of $\|grad\rho\| = c, \ c \in (0,1)$ and two-dimensional Euclidean space, the curvature is given by

$$\kappa^2 = \frac{1-c^2}{c^2(s^2+a)} \quad (30)$$

where a is a real constant.

The following result characterizes constant ratio curves in IE^n.

Proposition 1: Let any curve $x : I \subset IR \to IE^n$ be unit speed in IE^n. In case of x is a constant ratio, then it has the parametrization

$$x(s) = \left(c^2 s + bc\right)T(s) + (1-c^2)\left(\frac{-1}{\kappa_1(s)}\right)N_1(s)$$

$$+ \left(\frac{-\kappa_1^2(s)\left(c^2 s + cb\right)}{\kappa_2(s)}\left(\frac{-1}{\kappa_1(s)}\right) + \frac{1-c^2}{\kappa_2(s)}\left(\frac{-1}{\kappa_1(s)}\right)'\right)N_2(s) \quad (31)$$

$$+ \sum_{i=3}^{n-1}\left(\sum_{k=0}^{i-1} m_{i,k}(s)\frac{\partial^k}{\partial s^k}\left(\frac{-1}{\kappa_1(s)}\right)\right)N_i(s).$$

Here, $m_{i,k}$ are inductively defined by the system 16. where $m_{1,0}(s) = 1 - c^2$, $m_{2,0}(s) = \dfrac{-\kappa_1^2(s)\left(c^2 s + cb\right)}{\kappa_2(s)}$, and $m_{2,1}(s) = \dfrac{1-c^2}{\kappa_2(s)}$.

Proof. Let any curve $x : I \subset IR \to IE^n$ be unit speed and constant ratio in IE^n. Then $\rho = \|x(s)\| = cs$. By the use of Equation 29., we obtain

$$\|grad\rho\| = \frac{\langle x(s), x'(s)\rangle}{\|x(s)\|} = c. \quad (32)$$

So, from the Equation 2. and 32., we yield $m_0 = c^2 s + cb$. Then, if we substitute it into Eq. 9, we have

$$m_1(s) = \left(1 - c^2\right)\left(\frac{-1}{\kappa_1(s)}\right), \qquad\qquad\qquad (33)$$

$$m_2(s) = \frac{-\kappa_1^2(s)\left(c^2 s + cb\right)}{\kappa_2(s)}\left(\frac{-1}{\kappa_1(s)}\right) + \frac{1 - c^2}{\kappa_2(s)}\left(\frac{-1}{\kappa_1(s)}\right)'.$$

By using the functions $m_{1,0}, m_{2,0}, and\ m_{2,1}$, we rewrite these equations as

$$m_1(s) = m_{1,0}\left(\frac{-1}{\kappa_1(s)}\right) \quad \text{and} \quad m_2(s) = m_{2,0}\left(\frac{-1}{\kappa_1(s)}\right) + m_{2,1}\left(\frac{-1}{\kappa_1(s)}\right)'. \qquad (34)$$

By induction method for equation system,

$$m_i(s) = \sum_{k=0}^{i-1} m_{i,k}(s)\frac{\partial^k}{\partial s^k}\left(\frac{-1}{\kappa_1(s)}\right) \quad (35)$$

for $i \in \{3, 4, ..., n-1,\}$. Here

$$\begin{cases} m_{1,0} = 1 - c^2 \\ m_{2,0} = \dfrac{-\kappa_1^2(s)\left(c^2 s + cb\right)}{\kappa_2(s)}, \quad m_{2,1} = \dfrac{1 - c^2}{\kappa_2(s)} \end{cases} \quad (36)$$

and for $i \in \{3, 4, ..., n-1\}$, $m_{i,k}$ satisfy the Equation system 16. As a result of these, substituting the

values into Eq 2., we get the result.

T − Constant Curves

Definition 2: Let any curve $x : I \subset IR \to IE^n$ be unit speed in IE^n. In case of $\left\|x^T\right\|$ is constant or zero on all points of the curve, then x is called as a T − constant curve (Gürpınar et al., 2015). Besides that, if $\left\|x^T\right\| = 0$, then x is called T − constant curve of the first kind, if not second kind.

As a result of Eq. 9., we have the following theorem.

Theorem 4: Let any curve $x : I \subset IR \to IE^n$ be unit speed in IE^n with the parametrization 2. In case of x is a T − constant curve of the first kind, then it has the position vector as

$$x(s) = \left(\frac{-1}{\kappa_1(s)}\right)N_1(s) + \frac{1}{\kappa_2(s)}\left(\frac{-1}{\kappa_1(s)}\right)'N_2(s) + \sum_{i=3}^{n-1}\left(\sum_{k=0}^{i-1} m_{i,k}(s)\frac{\partial^k}{\partial s^k}\left(\frac{-1}{\kappa_1(s)}\right)\right)N_i(s). \quad (37)$$

Here, $m_{i,k}$ are inductively defined by the system 16. where $m_{1,0}(s) = 1$, $m_{2,0}(s) = 0$, and

$$m_{2,1}(s) = \frac{1}{\kappa_2(s)}.$$

Proof. Let any curve $x : I \subset IR \to IE^n$ be $T-$constant first kind in IE^n. Then, from the Frenet equations 9., we obtain $m_1(s) = \dfrac{-1}{\kappa_1(s)}$ and $m_2(s) = \dfrac{1}{\kappa_2(s)}\left(\dfrac{-1}{\kappa_1(s)}\right)'$. By using the functions $m_{1,0}$, $m_{2,0}$, and $m_{2,1}$, we rewrite these equations as

$$m_1(s) = m_{1,0}\left(\frac{-1}{\kappa_1(s)}\right) \quad \text{and} \quad m_2(s) = m_{2,0}\left(\frac{-1}{\kappa_1(s)}\right) + m_{2,1}\left(\frac{-1}{\kappa_1(s)}\right)'. \quad (38)$$

By induction method for equation system,

$$m_i(s) = \sum_{k=0}^{i-1} m_{i,k}(s)\frac{\partial^k}{\partial s^k}\left(\frac{-1}{\kappa_1(s)}\right) \quad (39)$$

for $i \in \{3,4,...,n-1,\}$. Here

$$\begin{cases} m_{1,0} = 1 - c^2 \\ m_{2,0} = \dfrac{-\kappa_1^2(s)(c^2 s + cb)}{\kappa_2(s)}, \quad m_{2,1} = \dfrac{1-c^2}{\kappa_2(s)} \end{cases} \quad (40)$$

and for $i \in \{3,4,...,n-1\}$, $m_{i,k}$ satisfy the equation system 16. As a result of these, substituting the values into Eq. 2., we get the result.

Theorem 5: Let any curve $x : I \subset IR \to IE^n$ be unit speed in IE^n. In case of x is $T-$constant curve of second kind, then it has the position vector as

$$x(s) = cT(s) + \left(\frac{-1}{\kappa_1(s)}\right)N_1(s) + \left(\frac{-\kappa_1^2(s)c}{\kappa_2(s)}\left(\frac{-1}{\kappa_1(s)}\right) + \frac{1}{\kappa_2(s)}\left(\frac{-1}{\kappa_1(s)}\right)'\right)N_2(s) + \sum_{i=3}^{n-1}\left(\sum_{k=0}^{i-1} m_{i,k}(s)\frac{\partial^k}{\partial s^k}\left(\frac{-1}{\kappa_1(s)}\right)\right)N_i(s).$$

where c is a constant function.

Proof. Assume that x is $T-$constant curve ofthe second kind $(m_0 = c)$. Then, from by using Eq. 9., we obtain $m_1(s) = \dfrac{-1}{\kappa_1(s)}$ and $m_2(s) = \dfrac{-\kappa_1^2(s)c}{\kappa_2(s)}\left(\dfrac{-1}{\kappa_1(s)}\right) + \dfrac{1}{\kappa_2(s)}\left(\dfrac{-1}{\kappa_1(s)}\right)'$. By using the functions $m_{1,0}$, $m_{2,0}$, and $m_{2,1}$, we rewrite these equations as

$$m_1(s) = m_{1,0}\left(\frac{-1}{\kappa_1(s)}\right) \quad \text{and} \quad m_2(s) = m_{2,0}\left(\frac{-1}{\kappa_1(s)}\right) + m_{2,1}\left(\frac{-1}{\kappa_1(s)}\right)'. \quad (41)$$

By induction method for equation system,

$$m_i(s) = \sum_{k=0}^{i-1} m_{i,k}(s)\frac{\partial^k}{\partial s^k}\left(\frac{-1}{\kappa_1(s)}\right) \quad (42)$$

for $i \in \{3,4,...,n-1,\}$. Here

$$\begin{cases} m_{1,0} = 1 \\ m_{2,0} = \dfrac{-\kappa_1^2(s)c}{\kappa_2(s)}, \quad m_{2,1} = \dfrac{1}{\kappa_2(s)} \end{cases} \quad (43)$$

and for $i \in \{3,4,...,n-1\}$, $m_{i,k}$ satisfy the equation system 16. As a result of these, substituting the

values into Eq. 2., we get the result.

N – Constant Curves

Definition 3: Let any curve $x : I \subset IR \rightarrow IE^n$ be unit speed in IE^n. In case of $\|x^N\|$ is constant or zero on all points of the curve, then x is called as N – constant curve (Chen, 2002). In addition that, if $\|x^N\| = 0$, then x is called a N – constant curve of the first kind, if not second kind (Gürpınar et al., 2015).

Thus, for the curve of N – constant

$$\|x^N(s)\|^2 = \sum_{i=1}^{n-1} m_i^2(s) \quad (44)$$

becomes a constant function.

As a result of Eq. 2., 9. and 44., we have the following:

Lemma 1: Let any curve $x : I \subset IR \rightarrow IE^n$ be unit speed in IE^n. The necessary and sufficient condition for x to become N – constant is

$$m_0'(s) = 1 + \kappa_1(s)m_1(s)$$
$$m_1'(s) = \kappa_2(s)m_2(s) - \kappa_1(s)m_0(s),$$
$$m_i'(s) = \kappa_{i+1}(s)m_{i+1}(s) - \kappa_i(s)m_{i-1}(s) \quad with \ i \in \{2,3,...,n-2\} \quad (45)$$
$$m_{n-1}'(s) = -\kappa_{n-1}(s)m_{n-2}(s)$$
$$0 = \sum_{i=1}^{n-1} m_i^2(s)$$

hold, where $m_i(s)$, $1 \le i \le n-1$ are differentiable functions.

Theorem 6: Let any curve $x : I \subset IR \rightarrow IE^n$ be unit speed in IE^n. The necessary and sufficient condition for the curve to become N – constant first kind is that x is congruent to a line which passes the origin.

Proof. Assume that x is N – constant in IE^n, then the equation system 45. holds. Further, x is a N – constant curve of the first kind then from Eq. 45., $m_i = 0$ for $i \in \{1,2,...,n-1\}$. Namely, the parametrization is $x(s) = (s + b)T(s)$. Consequently, x is congruent to a line.

Theorem 7: Let a unit speed curve $x : I \subset IR \rightarrow IE^n$ has nonzero curvatures in IE^n. In case of x is a N – constant curve of the second kind, then the following cases are validated:

(i) x is $T-$constant curve of first kind,

(ii) x is congruent to a rectifying curve in IE^n.

Proof. Let x is a $N-$constant curve of the second kind, then the equation system 45. holds. By multiplying every $k-$th equations by $-m_{k-1}$ $(1 \leq k \leq n)$ in Eq. 45., and combining them with the last equation in Eq. 45., we get $\kappa_1(s)m_1(s)m_0(s) = 0$.

Since all curvatures are nonzero, there are two possible cases; $m_0 = 0$, $m_1 = 0$. If $m_0 = 0$, then x is of a $T-$constant first kind. Finally, if $m_1 = 0$, then x is congruent to a rectifying curve in IE^n.

CONCLUSION

$T-$constant, $N-$constant and constant ratio curves are first defined by B.Y. Chen. In this study, with regards to these definitions, we focus on these types of curves with Frenet frame in $n-$dimensional Euclidean space and we present some results about $T-$constant, $N-$constant and constant ratio curves.

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