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Authors: Kerim BEKAR

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Left-sided Hermite-Hadamard Type Inequalities for Trigonometrically P-Functions

Kerim BEKAR*[1]

Abstract

In this paper, we obtain refinements of the left-sided Hermite-Hadamard inequality for functions whose first derivatives in absolute value are trigonometrically P-function.

Keywords: Convex function, trigonometrically convex function, trigonometrically P-functions, Hermite-Hadamard inequality

1. INTRODUCTION

Convexity theory provides powerful principles and techniques to study a wide class of problems in both pure and applied mathematics. See articles [2, 4, 7, 9, 11, 12] and the references therein.

Throughout the paper I is a non-empty interval in $\mathbb{R}$. Let $f: I \to \mathbb{R}$ be a convex function. Then the following inequality hold

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}$$

for all $a, b \in I$ with $a < b$. This double inequality is well known as the Hermite-Hadamard inequality (for more information, see [5]). Since then, some refinements of the Hermite-Hadamard inequality for convex functions have been obtained [3, 14].

Definition 1. [4] *A non-negative function $f: I \to \mathbb{R}$ is said to be a P-function if the inequality*

$$f(tx + (1-t)y) \leq f(x) + f(y)$$

holds for all $x, y \in I$ and $t \in [0,1]$. The set of P-functions on the interval I is denoted by $P(I)$.

Definition 2. [13] *Let $h: J \to \mathbb{R}$ be a non-negative function, $h \neq 0$. We say that $f: I \to \mathbb{R}$ is an h-convex function, or that f belongs to the class $SX(h, I)$, if f is non-negative and for all $x, y \in I$, $\alpha \in (0,1)$ we have*

$$f(\alpha x + (1-\alpha)y) \leq h(\alpha)f(x) + h(1-\alpha)f(y).$$

If this inequality is reversed, then f is said to be h-concave, i.e. $f \in SV(h, I)$.

*Corresponding Author: kebekar@gmail.com
[1] Department of Mathematics, Faculty of Arts and Sciences, Giresun University, 28200, Giresun-Turkey
ORCID ID: 0000-0002-7531-9345

In [8], Kadakal gave the concept of trigonometrically convex function as follows:

Definition 3. [8] *A non-negative function $f: I \to \mathbb{R}$ is called trigonometrically convex if for every $x, y \in I$ and $t \in [0,1]$,*

$$f(tx+(1-t)y) \leq \left(sin\frac{\pi t}{2}\right) f(x) + \left(cos\frac{\pi t}{2}\right) f(y).$$

The class of all trigonometrically convex functions is denoted by $TC(I)$ on interval I.

In [1], Bekar gave the concept of trigonometrically P-function as follows:

Definition 4. [1] *A non-negative function $f: I \to \mathbb{R}$ is called trigonometrically P-functions if for every $x, y \in I$ and $t \in [0,1]$,*

$$f(tx+(1-t)y) \leq \left(\sin\frac{\pi t}{2} + \cos\frac{\pi t}{2}\right)[f(x)+f(y)].$$

We will denote by $TP(I)$ the class of all trigonometrically P-functions on interval I. The range of the trigonometrically P-functions is greater than or equal to 0. Every non-negative trigonometrically convex function is trigonometrically P-functions. We note that, every trigonometrically convex function is a h-convex function for $h(t) = \sin\frac{\pi t}{2}$. Morever, if $f(x)$ is a nonnegative function, then every trigonometric convex function is a P-function.

We will denote by $L[a,b]$ the space of (Lebesgue) integrable functions on the interval $[a,b]$.

In [1], Bekar also obtained the following Hermite-Hadamard type inequalities for the trigonometrically P-function as follows:

Theorem 1. *Let the function $f: [a,b] \to \mathbb{R}$ be a trigonometrically P-function. If $a < b$ and $f \in L[a,b]$, then the following inequality holds:*

$$\frac{1}{2\sqrt{2}} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{4}{\pi}[f(a)+f(b)].$$

In [6], İşcan gave a refinement of the Hölder integral inequality as follows:

Theorem 2. [6] *Let $p > 1$ and $\frac{1}{p}+\frac{1}{q} = 1$. If f and g are real functions defined on interval $[a,b]$ and if $|f|^p$, $|g|^q$ are integrable functions on $[a,b]$ then*

$$\int_a^b |f(x)g(x)|dx \leq \frac{1}{b-a}\left\{\left(\int_a^b (b-x)|f(x)|^p dx\right)^{\frac{1}{p}} \times \left(\int_a^b (b-x)|g(x)|^q dx\right)^{\frac{1}{q}} + \left(\int_a^b (x-a)|f(x)|^p dx\right)^{\frac{1}{p}} \times \left(\int_a^b (x-a)|g(x)|^q dx\right)^{\frac{1}{q}}\right\}.$$

2. SOME NEW INEQUALITIES FOR TRIGONOMETRICALLY *P*-FUNCTION

The main purpose of this section is to establish new estimates that refine left-sided Hermite-Hadamard inequality for functions whose first derivative in absolute value, raised to a certain power which is greater than one, respectively at least one, is trigonometrically P-function. Kırmacı [10] used the following lemma:

Lemma 1. *Let $f: I^* \subset \mathbb{R} \to \mathbb{R}$ be differentiable mapping on I^*, $a, b \in I^\circ$ (I^* is the interior of I) with $a < b$. If $f' \in L[a,b]$, then we have*

$$\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) = (b-a)\left[\begin{array}{c}\int_0^{\frac{1}{2}} tf'(ta+(1-t)b)dt \\ +\int_{\frac{1}{2}}^1 (t-1)f'(ta+(1-t)b)dt\end{array}\right]$$

for $t \in [0,1]$.

Theorem 3. *Let $f: I \to \mathbb{R}$ be a continuously differentiable function, let $a < b$ in I and assume that $f' \in L[a,b]$. If $|f'|$ is trigonometrically P-function on interval $[a,b]$, then the following inequality*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right|$$

$$\leq 16(b-a)\left(\frac{\sqrt{2}-1}{\pi^2}\right) A(|f'(a)|, |f'(b)|)$$

holds for $t \in [0,1]$, where A is the arithmetic mean.

Proof. Using Lemma 1 and the inequality

$$|f'(ta+(1-t)b)|$$

$$\leq \left(\sin\frac{\pi t}{2}+\cos\frac{\pi t}{2}\right)[|f'(a)|+|f'(b)|],$$

we get

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right|$$

$$\leq (b-a)\begin{bmatrix} \int_0^{\frac{1}{2}} |t||f'(ta+(1-t)b)|dt \\ +\int_{\frac{1}{2}}^1 |t-1||f'(ta+(1-t)b)|dt \end{bmatrix}$$

$$\leq (b-a)\Big[\int_0^{\frac{1}{2}} |t|\left(\sin\frac{\pi t}{2}+\cos\frac{\pi t}{2}\right)[|f'(a)|+ |f'(b)|]dt$$

$$+\int_{\frac{1}{2}}^1 |t-1|\left(\sin\frac{\pi t}{2}+\cos\frac{\pi t}{2}\right)[|f'(a)|+ |f'(b)|]dt\Big]$$

$$= (b-a)[|f'(a)|+|f'(b)|]$$

$$\times\Big[\int_0^{\frac{1}{2}} |t|\left(\sin\frac{\pi t}{2}+\cos\frac{\pi t}{2}\right)dt$$

$$+\int_{\frac{1}{2}}^1 |t-1|\left(\sin\frac{\pi t}{2}+\cos\frac{\pi t}{2}\right)dt\Big]$$

$$= 2(b-a)[|f'(a)|+|f'(b)|]\left(\frac{4(\sqrt{2}-1)}{\pi^2}\right)$$

$$= 16(b-a)\left(\frac{\sqrt{2}-1}{\pi^2}\right)A(|f'(a)|,|f'(b)|),$$

where

$$\int_0^{\frac{1}{2}} |t|\left(\sin\frac{\pi t}{2}+\cos\frac{\pi t}{2}\right)dt = \frac{4(\sqrt{2}-1)}{\pi^2}$$

$$\int_{\frac{1}{2}}^1 |t-1|\left(\sin\frac{\pi t}{2}+\cos\frac{\pi t}{2}\right)dt = \frac{4(\sqrt{2}-1)}{\pi^2}.$$

This completes the proof of the theorem.

Theorem 4. *Let $f: I \to \mathbb{R}$ be a continuously differentiable function, let $a < b$ in I and assume that $q > 1$. If $|f'|^q$ is a trigonometrically P-function on interval $[a,b]$, then the following inequality*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right|$$

$$\leq 2^{\frac{3}{q}-1}\left(\frac{1}{\pi}\right)^{\frac{1}{q}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}(b-a)A^{\frac{1}{q}}(|f'(a)|^q,|f'(b)|^q)$$

holds for $t \in [0,1]$, where $\frac{1}{p}+\frac{1}{q}=1$ and A is the arithmetic mean.

Proof. Using Lemma 1, Hölder's integral inequality and the following inequality

$$|f'(ta+(1-t)b)|^q$$

$$\leq \left(\sin\frac{\pi t}{2}+\cos\frac{\pi t}{2}\right)[|f'(a)|^q+|f'(b)|^q]$$

which comes from the definition of trigonometrically P-function for $|f'|^q$, we get

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right|$$

$$\leq \left|(b-a)\begin{bmatrix} \int_0^{\frac{1}{2}} tf'(ta+(1-t)b)dt \\ +\int_{\frac{1}{2}}^1 (t-1)f'(ta+(1-t)b)dt \end{bmatrix}\right|$$

$$\leq (b-a)\left(\int_0^{\frac{1}{2}} |t|^p dt\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^{\frac{1}{2}} |f'(ta+(1-t)b)|^q dt\right)^{\frac{1}{q}}$$

$$+(b-a)\left(\int_{\frac{1}{2}}^1 |t-1|^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1 |f'(ta+(1-t)b)|^q dt\right)^{\frac{1}{q}}$$

$$\leq (b-a)\left(\int_0^{\frac{1}{2}} |t|^p dt\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^{\frac{1}{2}}\left(\sin\frac{\pi t}{2}+\cos\frac{\pi t}{2}\right)[|f'(a)|^q+|f'(b)|^q]dt\right)^{\frac{1}{q}}$$

$$+(b-a)\left(\int_{\frac{1}{2}}^1 |t-1|^p dt\right)^{\frac{1}{p}}$$

$$\times\left(\int_{\frac{1}{2}}^{1}\left(\sin\frac{\pi t}{2}+\cos\frac{\pi t}{2}\right)[|f'(a)|^q+|f'(b)|^q]dt\right)^{\frac{1}{q}}$$

$$=(b-a)2^{\frac{1}{q}}A^{\frac{1}{q}}(|f'(a)|^q,|f'(b)|^q)$$

$$\times\left(\frac{1}{(p+1)2^{p+1}}\right)^{\frac{1}{p}}\left[\int_0^1\left(\sin\frac{\pi t}{2}+\cos\frac{\pi t}{2}\right)dt\right]^{\frac{1}{q}}$$

$$+(b-a)2^{\frac{1}{q}}A^{\frac{1}{q}}(|f'(a)|^q,|f'(b)|^q)$$

$$\times\left(\frac{1}{(p+1)2^{p+1}}\right)^{\frac{1}{p}}\left[\int_{\frac{1}{2}}^{1}\left(\sin\frac{\pi t}{2}+\cos\frac{\pi t}{2}\right)dt\right]^{\frac{1}{q}}$$

$$=2^{\frac{3}{q}-1}\left(\frac{1}{\pi}\right)^{\frac{1}{q}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}(b-a)A^{\frac{1}{q}}(|f'(a)|^q,|f'(b)|^q),$$

where

$$\int_0^{\frac{1}{2}}|t|^p dt=\int_{\frac{1}{2}}^{1}|t-1|^p dt=\frac{1}{(p+1)2^{p+1}}$$

$$\int_0^{\frac{1}{2}}\left(\sin\frac{\pi t}{2}+\cos\frac{\pi t}{2}\right)dt=\frac{2}{\pi}$$

$$\int_{\frac{1}{2}}^{1}\left(\sin\frac{\pi t}{2}+\cos\frac{\pi t}{2}\right)dt=\frac{2}{\pi}.$$

This completes the proof of the theorem.

Theorem 5. *Let $f: I\subseteq\mathbb{R}\to\mathbb{R}$ be a continuously differentiable function, let $a<b$ in I and assume that $q\geq 1$. If $|f'|^q$ is a trigonometrically P-function on the interval $[a,b]$, then the following inequality holds for $t\in[0,1]$*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-f\left(\frac{a+b}{2}\right)\right|$$

$$\leq(b-a)2^{\frac{6}{q}-2}A^{\frac{1}{q}}(|f'(a)|^q,|f'(b)|^q)\left(\frac{\sqrt{2}-1}{\pi^2}\right)^{\frac{1}{q}},$$

where A is the arithmetic mean.

Proof. Assume first that $q>1$. From Lemma 1, Hölder integral inequality and the property of $|f'|^q$ which is trigonometrically P-function, we obtain

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-f\left(\frac{a+b}{2}\right)\right|\leq(b-a)\left(\int_0^{\frac{1}{2}}|t|dt\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^{\frac{1}{2}}|t||f'(ta+(1-t)b)|^q dt\right)^{\frac{1}{q}}$$

$$+(b-a)\left(\int_{\frac{1}{2}}^{1}|t-1|dt\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_{\frac{1}{2}}^{1}|t-1||f'(ta+(1-t)b)|^q dt\right)^{\frac{1}{q}}$$

$$\leq(b-a)\left(\int_0^{\frac{1}{2}}|t|dt\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^{\frac{1}{2}}|t|\left(\sin\frac{\pi t}{2}+\cos\frac{\pi t}{2}\right)[|f'(a)|^q+|f'(b)|^q]dt\right)^{\frac{1}{q}}$$

$$+(b-a)\left(\int_{\frac{1}{2}}^{1}|t-1|dt\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_{\frac{1}{2}}^{1}|t-1|\left(\sin\frac{\pi t}{2}+\cos\frac{\pi t}{2}\right)[|f'(a)|^q+|f'(b)|^q]dt\right)^{\frac{1}{q}}$$

$$=2(b-a)\left(\frac{1}{8}\right)^{1-\frac{1}{q}}8^{\frac{1}{q}}A^{\frac{1}{q}}(|f'(a)|^q,|f'(b)|^q)\left(\frac{4(\sqrt{2}-1)}{\pi^2}\right)^{\frac{1}{q}}$$

$$=(b-a)2^{\frac{6}{q}-2}A^{\frac{1}{q}}(|f'(a)|^q,|f'(b)|^q)\left(\frac{\sqrt{2}-1}{\pi^2}\right)^{\frac{1}{q}}.$$

It can be seen that

$$\int_0^{\frac{1}{2}}|t|dt=\int_{\frac{1}{2}}^{1}|t-1|dt=\frac{1}{8}$$

$$\int_0^{\frac{1}{2}}|t|\left(\sin\frac{\pi t}{2}+\cos\frac{\pi t}{2}\right)dt=\frac{4(\sqrt{2}-1)}{\pi^2}$$

$$\int_{\frac{1}{2}}^{1}|t-1|\left(\sin\frac{\pi t}{2}+\cos\frac{\pi t}{2}\right)dt=\frac{4(\sqrt{2}-1)}{\pi^2}.$$

Therefore, the desired result is obtained.

For $q=1$ we use the estimates from the proof of the Theorem 3, which also follow step by step the above estimates.

This completes the proof of the theorem.

Corollary 1. *Under the assumption of the Theorem 5 with $q = 1$, we get the conclusion of the Theorem 3.*

Theorem 6. *Let $f: I \to \mathbb{R}$ be a continuously differentiable function, let $a < b$ in I and assume that $q > 1$. If $|f'|^q$ is a trigonometrically P-function on interval $[a, b]$, then the following inequality*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right|$$

$$\leq 2^{\frac{1}{q}+2}(b-a)A^{\frac{1}{q}}(|f'(a)|^q, |f'(b)|^q)\left(\frac{1}{p+2}\right)^{\frac{1}{p}}$$

$$\times\left[\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{\pi-4\sqrt{2}+4}{\pi^2}\right)^{\frac{1}{q}} + \left(\frac{4\sqrt{2}-4}{\pi^2}\right)^{\frac{1}{q}}\right]$$

holds for $t \in [0,1]$, where $\frac{1}{p} + \frac{1}{q} = 1$ and A is the arithmetic mean.

Proof. Using Lemma 1, Hölder-İşcan integral inequality and the following inequality

$$|f'(ta + (1-t)b)|^q$$

$$\leq \left(sin\frac{\pi t}{2} + cos\frac{\pi t}{2}\right)[|f'(a)|^q + |f'(b)|^q]$$

which comes from the definition of trigonometrically P-function for $|f'|^q$, we get

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right|$$

$$\leq (b-a)\left[\begin{array}{c}\int_0^{\frac{1}{2}} |t||f'(ta+(1-t)b)|dt \\ +\int_{\frac{1}{2}}^1 |t-1||f'(ta+(1-t)b)|dt\end{array}\right]$$

$$\leq 2(b-a)\left[\left(\int_0^{\frac{1}{2}}\left|\frac{1}{2}-t\right||t|^p dt\right)^{\frac{1}{p}}\right.$$

$$\times\left(\int_0^{\frac{1}{2}}\left|\frac{1}{2}-t\right||f'(ta+(1-t)b)|^q dt\right)^{\frac{1}{q}}$$

$$+\left(\int_0^{\frac{1}{2}}|t||t|^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}|t||f'(ta+(1-t)b)|^q dt\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{1}{2}}^1 |1-t||t-1|^p dt\right)^{\frac{1}{p}}$$

$$\times\left(\int_{\frac{1}{2}}^1 |1-t||f'(ta+(1-t)b)|^q dt\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{1}{2}}^1\left|t-\frac{1}{2}\right||t-1|^p dt\right)^{\frac{1}{p}}$$

$$\left.\left(\int_{\frac{1}{2}}^1\left|t-\frac{1}{2}\right||f'(ta+(1-t)b)|^q dt\right)^{\frac{1}{q}}\right]$$

$$\leq 2(b-a)\left[\left(\int_0^{\frac{1}{2}}\left|\frac{1}{2}-t\right||t|^p dt\right)^{\frac{1}{p}}\right.$$

$$\times\left(\int_0^{\frac{1}{2}}\left|\frac{1}{2}-t\right|\left(\sin\frac{\pi t}{2}+\cos\frac{\pi t}{2}\right)[|f'(a)|^q+|f'(b)|^q]dt\right)^{\frac{1}{q}}$$

$$+\left(\int_0^{\frac{1}{2}}|t||t|^p dt\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^{\frac{1}{2}}|t|\left(\sin\frac{\pi t}{2}+\cos\frac{\pi t}{2}\right)[|f'(a)|^q+|f'(b)|^q]dt\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{1}{2}}^1 |1-t||t-1|^p dt\right)^{\frac{1}{p}}$$

$$\times\left(\int_{\frac{1}{2}}^1 |1-t|\left(\sin\frac{\pi t}{2}+\cos\frac{\pi t}{2}\right)[|f'(a)|^q+|f'(b)|^q]dt\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{1}{2}}^1\left|t-\frac{1}{2}\right||t-1|^p dt\right)^{\frac{1}{p}}$$

$$\times\left(\int_{\frac{1}{2}}^1\left|t-\frac{1}{2}\right|\left(\sin\frac{\pi t}{2}+\cos\frac{\pi t}{2}\right)[|f'(a)|^q+\right.$$

$$\left.\left.|f'(b)|^q]dt\right)^{\frac{1}{q}}\right]$$

$$= 2^{1+\frac{1}{q}}(b-a)A^{\frac{1}{q}}(|f'(a)|^q, |f'(b)|^q)$$

$$\times\left[\left(\frac{2^{-(p+2)}}{(p+1)(p+2)}\right)^{\frac{1}{p}}\left(\frac{\pi-4\sqrt{2}+4}{\pi^2}\right)^{\frac{1}{q}}\right.$$

$$+\left(\frac{2^{-(p+2)}}{p+2}\right)^{\frac{1}{p}}\left(\frac{4\sqrt{2}-4}{\pi^2}\right)^{\frac{1}{q}}+\left(\frac{2^{-(p+2)}}{p+2}\right)^{\frac{1}{p}}\left(\frac{4\sqrt{2}-4}{\pi^2}\right)^{\frac{1}{q}}$$

$$+\left(\frac{2^{-(p+2)}}{(p+1)(p+2)}\right)^{\frac{1}{p}}\left(\frac{\pi-4\sqrt{2}+4}{\pi^2}\right)^{\frac{1}{q}}\Bigg]$$

$$=2^{\frac{1}{q}+2}(b-a)A^{\frac{1}{q}}(|f'(a)|^q,|f'(b)|^q)\left(\frac{1}{p+2}\right)^{\frac{1}{p}}$$

$$\times\left[\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{\pi-4\sqrt{2}+4}{\pi^2}\right)^{\frac{1}{q}}+\left(\frac{4\sqrt{2}-4}{\pi^2}\right)^{\frac{1}{q}}\right]$$

where

$$\int_0^{\frac{1}{2}}\left|\frac{1}{2}-t\right||t|^p dt=\frac{2^{-(p+2)}}{(p+1)(p+2)}$$

$$\int_{\frac{1}{2}}^{1}\left|t-\frac{1}{2}\right||t-1|^p dt=\frac{2^{-(p+2)}}{(p+1)(p+2)}$$

$$\int_0^{\frac{1}{2}}|t||t|^p dt=\int_{\frac{1}{2}}^{1}|1-t||t-1|^p dt=\frac{2^{-(p+2)}}{p+2}$$

$$\int_0^{\frac{1}{2}}\left|\frac{1}{2}-t\right|\left(\sin\frac{\pi t}{2}+\cos\frac{\pi t}{2}\right)dt=\frac{\pi-4\sqrt{2}+4}{\pi^2}$$

$$\int_{\frac{1}{2}}^{1}\left|t-\frac{1}{2}\right|\left(\sin\frac{\pi t}{2}+\cos\frac{\pi t}{2}\right)dt=\frac{\pi-4\sqrt{2}+4}{\pi^2}$$

$$\int_0^{\frac{1}{2}}|t|\left(\sin\frac{\pi t}{2}+\cos\frac{\pi t}{2}\right)dt=\frac{4\sqrt{2}-4}{\pi^2}$$

$$\int_{\frac{1}{2}}^{1}|1-t|\left(\sin\frac{\pi t}{2}+\cos\frac{\pi t}{2}\right)=\frac{4\sqrt{2}-4}{\pi^2}.$$

This completes the proof of the theorem.

REFERENCES

[1] K. Bekar, "Hermite-Hadamard type inequalities for trigonometrically P-functions", C. R. Acad. Bulgare Sci., 72(11), 2019.

[2] S.S. Dragomir and R.P. "Agarwal, Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula", Appl. Math. Lett. 11 (1998), 91-95.

[3] S.S. Dragomir and C.E.M. Pearce, "Selected Topics on Hermite-Hadamard Inequalities and Its Applications", RGMIA Monograph 2002.

[4] S.S. Dragomir, J. Pečarić and LE.Persson, "Some inequalities of Hadamard Type", Soochow J. Math., 21 (3)(2001), pp. 335-341.

[5] J. Hadamard, "Étude sur les propriétés des fonctions entières en particulier d'une fonction considérée par Riemann", J. Math. Pures Appl. 58(1893), 171-215.

[6] İ. İşcan, "New refinements for integral and sum forms of Hölder inequality", J. Inequal. Appl., (2019) 2019:304, 11 pages.

[7] İ. İşcan and M. Kunt, "Hermite-Hadamard-Fejer type inequalities for quasi geometrically convex functions via fractional integrals", J. Math., Volume 2016, Article ID 6523041, 7 pages.

[8] H. Kadakal, "Hermite-Hadamard type inequalities for trigonometrically convex functions", Sci. Stud. Res. Ser. Math. Inform., Vol. 28(2018), No. 2, 19-28.

[9] M. Kadakal, H. Kadakal and İ. İşcan, "Some new integral inequalities for n-times differentiable s-convex functions in the first sense", TJANT, Vol. 5, No. 2 (2017), 63-68.

[10] U.S., Kırmacı, "Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula", Appl. Math. Comput. 147 (1) (2004), 137-146.

[11] S. Maden, H. Kadakal, M. Kadakal and İ, İşcan, "Some new integral inequalities for n-times differentiable convex and concave functions", J. Nonlinear Sci. Appl., 10, 12(2017), 6141-6148.

[12] S. Özcan, and İ. İşcan, "Some new Hermite-Hadamard type inequalities for s-convex functions and their applications", J. Inequal. Appl., Article number: 2019:201 (2019).

[13] S. Varošanec, "On h-convexity", J. Math. Anal. Appl. 326 (2007) 303-311.

[14] G. Zabandan, "A new refinement of the Hermite-Hadamard inequality for convex functions", J. Inequal. Pure Appl. Math. 10 2(2009), Article ID 45.