Title: Degree Distance of Zero-Divisor Graph Γ[Z_n]

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Degree Distance of Zero-Divisor Graph $\Gamma[\mathbb{Z}_n]$

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Abstract

In this article, degree distance of zero-divisor graph $\Gamma[\mathbb{Z}_n]$ is computed for $n = p^2$, $n = pq$ and $n = p^3$, where p, q are distinct prime numbers.

Keywords: zero-divisor graph, degree distance, topological index, graph

1. INTRODUCTION

Let $G = (V(G), E(G))$ be a simple graph with $V(G)$, set of all vertices and $E(G)$, the set of all edges that join the vertices. If there is an edge joining a pair of vertices, those vertices are said to be adjacent. An edge is incident to a vertex if the edge is joined to the vertex. If $e = uv$ be an edge of the graph G, e is incident to the vertices u and v. Degree of a vertex u in a graph G is the number of edges incident to u and is denoted by $deg_G(u)$. If there is a path between any two distinct vertices of a graph, then the graph is called connected. Length of the shortest path connecting the vertices u and v in a graph G is called distance between u and v, denoted by $d_G(u, v)$. If any two vertices of a graph is adjacent then the graph is called complete. A complete graph has n vertices is denoted by K_n. Complement graph of the complete graph K_n is $\overline{K}_n$, which has n vertices and no edges, namely null graph of order n. A complete bipartite graph is a graph that the set of vertices $V(G)$ can be decomposed into two disjoint subsets such that none of two vertices in the same set is not adjacent and every pair of vertices in the two sets. If these two sets have n and m vertices, then the complete bipartite graph is denoted by $K_{n,m}$.

$Top(G)$, topological index of a graph G is a real number which is invariant under isomorphism of graphs. In other words, let $\mathcal{G}$ be the class of all finite graphs and if $G, H \in \mathcal{G}$ are isomorphic, then $Top(G) = Top(H)$. Topological indices are basically related to connectivity and distance in a graph. Let $G = (V(G), E(G))$ be a simple finite connected graph. In [4] degree distance of a graph G is defined as

$$D'(G) = \sum_{v \in V(G)} deg_G(v) \, D_G(v),$$

where $D_G(v) = \sum_{u \in V(G)} d_G(u, v)$, which is the distance of a vertex v in G. "Schultz index" name was proposed by Gutman in [5] for degree distance of a graph which can also be expressed as

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$$D'(G) =$$

$$\sum_{\{u,v\}\subseteq V(G)} [deg_G(u) + deg_G(v)]d_G(u,v).$$

Wiener index which is equal to the sum of distances between all pairs of vertices and introduced in [10] as

$$W(G) = \sum_{\{u,v\}\subseteq V(G)} d_G(u,v).$$

Degree distance of a graph can be also considered as a weighted version of the Wiener index.

Anderson and Livingston [2] introduced the zero-divisor graph of a commutative ring R (with 1) denoted by $\Gamma[R]$ with the vertex set $Z^*(R) = Z(R)\backslash\{0\}$ via this adjacency: for distinct $x,y \in Z^*(R)$, the vertices x and y are adjacent if and only if $xy = 0$. Actually, zero-divisor graph of a commutative ring is due to Beck [3] who considered the set of all elements of the ring as vertices and two distinct x, y elements of the ring are adjacent iff $xy = 0$. Beck is mainly dealt with colourings in [3].

Let $\mathbb{Z}_n$ be the commutative ring of all residue classes of integers modulo n. Ahmadi and Jahani-Nezhad [1] computed energy and Wiener index of $\Gamma[\mathbb{Z}_n]$ for $n = pq$ and $n = p^2$, where p, q are prime numbers. Mohammad and Authman [8] determined Hosoya polynomial of $\Gamma[\mathbb{Z}_n]$ and computed the Wiener index of $\Gamma[\mathbb{Z}_n]$ for $n = p^m$ and $n = p^m q$, where p, q are distinct prime numbers and $m \geq 2$ is an integer. Reddey et al. [9] consider $\Gamma[\mathbb{Z}_n]$ for $n = p^2q$ and $n = p^3$, where p, q are prime numbers and studied adjacency matrix of $\Gamma[\mathbb{Z}_n]$, its eigenvalues and computed the Wiener index of $\Gamma[\mathbb{Z}_n]$.

In this article $\Gamma[\mathbb{Z}_n]$, the zero-divisor graph of $\mathbb{Z}_n$ is considered for $n = p^2$, $n = pq$ and $n = p^3$, where p, q are distinct prime numbers and degree distance of $\Gamma[\mathbb{Z}_n]$ is computed which is a weighted version of the Wiener index.

2. PRELIMINARIES

In this section we give some definitions and lemmas which are required for proofs of the results will be presented.

Definition 2.1. A sum (join) of graphs G and H with disjoint vertex sets $V(G)$ and $V(H)$ is a graph has vertex set $V(G) \cup V(H)$ and the edge set $E(G) \cup E(H) \cup \{uv: u \in V(G) \text{ and } v \in V(H)\}$. The sum of the graphs G and H is denoted by $G + H$.

Lemma 2.1. Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be graphs. Then the following assertions are hold.

i. $|E(G + H)| = |E(G)| + |E(H)| + |V(G)||V(H)|$,

ii. $d_{G+H}(u,v)=$
$$\begin{cases} 0, & u = v \\ 1, & uv \in E(G) \text{ or } uv \in E(H) \text{ or} \\ & (u \in V(G) \ \& \ v \in V(H)) \\ 2, & otherwise \end{cases},$$

iii. $deg_{G+H}(u) =$
$$\begin{cases} deg_G(u) + |V(H)|, & u \in V(G) \\ deg_H(u) + |V(G)|, & u \in V(H) \end{cases}.$$

Proof. The proof of the parts (i)-(ii) and (iii) can be seen from [6] and [7], respectively.

Lemma 2.2. ([1]) Let $G = \Gamma[\mathbb{Z}_n]$ and let $n = p^2$, where p is a prime number. Then $Z^*(\mathbb{Z}_{p^2}) = \{p, 2p, \dots, (p-1)p^2\}$. Thus for every $x, y \in Z^*(\mathbb{Z}_{p^2})$, $xy = 0$ and $\Gamma[\mathbb{Z}_{p^2}] = K_{p-1}$. If $n = pq$ such that p, q are distinct prime numbers, then

$$Z^*(\mathbb{Z}_{pq}) = A \cup B,$$

where $A = \{pk: k = 1, 2, \dots, q - 1\}$ and $B = \{qk: k = 1, 2, \dots, p - 1\}$. For any $x, y \in Z^*(\mathbb{Z}_{pq})$, $xy = 0$ if and only if $x \in A, y \in B$ or $x \in B, y \in A$. So, $\Gamma[\mathbb{Z}_{pq}] = K_{p-1,q-1}$.

Theorem 2.1. ([8]) $\Gamma[\mathbb{Z}_{p^3}] \cong K_{p-1} + \overline{K}_{p^2-p}$, where $\overline{K}_{p^2-p}$ is complement graph of complete graph K_{p^2-p}.

The zero-divisor graph of the ring $\mathbb{Z}_{27}$ is illustrated in Figure 1.
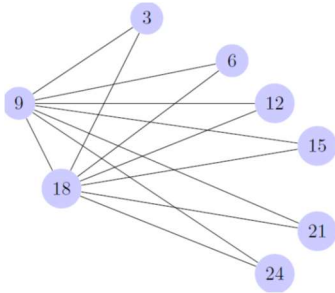

Figure 1. $\Gamma[\mathbb{Z}_{27}]$

3. MAIN RESULTS

In this section we compute the degree distance of the zero-divisor graph $\Gamma[\mathbb{Z}_n]$ for $n = p^2$, $n = pq$ and $n = p^3$, where p, q are distinct prime numbers.

Theorem 3.1. The degree distance of $G = \Gamma[\mathbb{Z}_{p^2}]$ is

$$D'(G) = (p-1)(p-2)^2.$$

Proof. From Lemma 2.2, we have $\Gamma[\mathbb{Z}_{p^2}] = K_{p-1}$. Since for any $x, y \in V(K_{p-1})$, $deg(x) = p - 2$ and $d_G(x,y) = 1$, we obtain

$$D'(G) =$$
$$\sum_{\{u,v\}\subseteq V(G)} [deg_G(u) + deg_G(v)]d_G(u,v)$$
$$= \sum_{u \neq v}(p - 2 + p - 2).1$$
$$= |E(K_{p-1})|.(2p - 4)$$
$$= \frac{(p-1)(p-2)}{2}.(2p - 4)$$
$$= (p-1)(p-2)^2,$$

which completes proof.

Theorem 3.2. The degree distance of $G = \Gamma[\mathbb{Z}_{pq}]$ is

$$D'(G) = 5(p^2q + pq^2 - (p^2 + q^2)) - 28pq + 23(p + q) - 18.$$

Proof. From Lemma 2.2, we have $Z^*(\mathbb{Z}_{pq}) = A \cup B$, where $A = \{pk: k = 1,2,\ldots,q-1\}$ and $B = \{qk: k = 1,2,\ldots,p-1\}$ and $\Gamma[\mathbb{Z}_{pq}] = K_{p-1,q-1}$. For $u \in A, v \in B$, we have $deg_G(u) = p - 1$, $deg_G(v) = q - 1$. The number of paths of length 2 for $u, v \in A$ is $(q-1)(q-2)$ and for $u, v \in B$ is $(p-1)(p-2)$. Thus we get

$$D'(G) =$$
$$\sum_{\{u,v\}\subseteq V(G)} [deg_G(u) + deg_G(v)]d_G(u,v)$$
$$= \sum_{u \in A, v \in B} [deg_G(u) + deg_G(v)].1$$
$$+ \sum_{u,v \in A, u \neq v} [deg_G(u) + deg_G(v)].2$$
$$+ \sum_{u,v \in B, u \neq v} [deg_G(u) + deg_G(v)].2$$

$$= (p + q - 2)(p - 1)(q - 1)$$
$$+2(2p - 2)(q - 1)(q - 2)$$
$$+2(2q - 2)(p - 1)(p - 2)$$
$$= 5(p^2q + pq^2 - (p^2 + q^2)) - 28pq$$
$$+23(p + q) - 18.$$

Thus proof is complete.

Theorem 3.3. The degree distance of $G = \Gamma[\mathbb{Z}_{p^3}]$ is

$$D'(G) = 3p^5 - 6p^4 - 3p^3 + 9p^2 + p - 4.$$

Proof. From proof of the Theorem 2.1 in [8], we have

$$Z^*(\mathbb{Z}_{p^3}) = A \cup B,$$

where $A = \{p^2k: k = 1,2,\ldots,p-1\}$ and $B = \{pk: k = 1,2,\ldots,p^2-1; p \nmid k\}$. From the features of these sets and adjacency relation of zero-divisor graph, three cases occur:

If $u, v \in A$, then u is adjacent to v; if $u \in A, v \in B$, then u is adjacent to v; if $u, v \in B$, then u is not adjacent to v, thus $d_G(u,v) = 2$. Since by Theorem 2.1 we have $\Gamma[\mathbb{Z}_{p^3}] \cong K_{p-1} + \bar{K}_{p^2-p}$,

we will consider the graph G as $K_{p-1} + \overline{K}_{p^2-p}$ with $|V(G)| = |V(\Gamma[\mathbb{Z}_{p^3}])| = p^2 - 1$ vertices. From Lemma 2.1, we have

$$deg_G(a) = \begin{cases} p^2 - 2, & a \in V(K_{p-1}) \\ p - 1, & a \in V(\overline{K}_{p^2-p}) \end{cases} \quad (1)$$

and

$$|E(G)| = |E(\Gamma[\mathbb{Z}_{p^3}])|$$

$$= |E(K_{p-1})| + |E(\overline{K}_{p^2-p})|$$
$$\quad + |V(K_{p-1})| \cdot |V(\overline{K}_{p^2-p})|$$
$$= \frac{(p-1)(p-2)}{2} + (p-1)(p^2-p)$$

$$= \frac{(p-1)(2p^2-p-2)}{2}. \quad (2)$$

We can also write

$$D'(G) =$$

$$\sum_{\{u,v\}\subseteq V(G)} [deg_G(u) + deg_G(v)]d_G(u,v)$$

$$= \sum_{u \in A, v \in B} [deg_G(u) + deg_G(v)].1$$

$$+ \sum_{u,v \in A, u \neq v} [deg_G(u) + deg_G(v)].1$$

$$+ \sum_{u,v \in B, u \neq v} [deg_G(u) + deg_G(v)].2. \quad (3)$$

If we consider the distance matrix of the zero-divisor graph $\Gamma[\mathbb{Z}_{p^3}]$ with $p^2 - 1$ vertices, the number of entries above the main diagonal is $\frac{(p^2-1)(p^2-2)}{2}$. The total number of paths of length 1 is the number of the edges, that is $\frac{(p-1)(2p^2-p-2)}{2}$. Therefore the number of paths of length 2 is

$$\frac{(p^2-1)(p^2-2)}{2} - \frac{(p-1)(2p^2-p-2)}{2}$$
$$= \frac{p^4 - 2p^3 + p}{2}. \quad (4)$$

By using (1), (2), (4) in (3), we obtain

$$D'(G) = (p-1)(p^2-p)(p^2+p-3)$$

$$+ \frac{(p-1)(p-2)}{2}.2(p^2-2)$$

$$+ 2\left(\frac{p^4-2p^3+p}{2}\right).2(p-1)$$

$$= 3p^5 - 6p^4 - 3p^3 + 9p^2 + p - 4.$$

So, proof is complete.

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