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Discovering The Relationships between Fractional Order Derivatives and Complex Numbers

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Abstract. The derivative concept was defined by Newton and Leipzig. After these scientists, there are many approaches about the order of derivative, since derivative defined by Newton and Leipzig considered as order of 1. So, imaginary axis vanishes and the result of derivation is a real number / function. However, in case of other orders of derivations, the obtained results have real and imaginary axises, since complex numbers and derivative have directions and magnitudes. This paper includes these relationships by using fractional order derivative $_{\alpha}^{\partial}Kf(t)$ defined by Karcı.

Keywords: Fractional order derivatives, Complex numbers

Classification Code: 34L30, 32V05

1. Introduction

The fractional calculus (variational calculus) is a three centuries old concept and one of the branch of the fractional calculus is fractional order derivatives. The fractional order derivative concept was defined by many scientists such as Euler, Caputo, Riemman-Lioville, etc. (Das,2011). There is an idea such that the fractional calculus may depict the behaviours of nature almost in real behaviours of nature (Das,2011).

The classical approach for derivative (Newtonian Derivative) is a derivative whose result finds out the real axis and vanishes the imaginary axis, since derivative has direction and magnitude and complex number has direction and magnitude. Due to this case, the results of derivative should be complex numbers.

The Fractional Order Derivative (FOD) methods in the literature can be considered as different approaches instead of classical derivatives. In this point of view, Karcı defined fractional order derivative in a different manner and gave some properties of fractional order derivative (Karcı, 2013a; Karcı, 2013b; Karcı, 2015a; Karcı, 2015b; Karcı, 2015c; Karcı, 2015d; Karcı, 2015e; Karcı, 2016a; Karcı, 2016b; Karcı, 2017).

The definition developed by Karcı can be given as in Definition 1.

Definition1: Assume that f(t):R→R is a function, α∈R and L(.) be a L'Hospital process. The $_{\alpha}^{\partial}K$ of f(t) is (taken from Karcı, 2013a; Karcı, 2013b)

$$ {}^{\partial}_{\alpha}K = \lim_{\Delta t \to 0} L\left(\frac{f^{\alpha}(t+\Delta t) - f^{\alpha}(t)}{(t+\Delta t)^{\alpha} - t^{\alpha}}\right) = \lim_{\Delta t \to 0} \frac{\frac{d\left(f^{\alpha}(t+\Delta t) - f^{\alpha}(t)\right)}{\Delta t}}{\frac{d\left((t+\Delta t)^{\alpha} - t^{\alpha}\right)}{\Delta t}} = \left(\frac{f(t)}{t}\right)^{\alpha-1}\frac{df(t)}{dt} \quad (10) $$

This new definition for fractional order derivative was derived due to the many deficiencies of definitions for fractional order derivatives in literature. The obtained derivative is equal to Newtonian derivative in case of α=1, and in other cases, a non-linear operator exists.

2. Relationship between ${}^{\partial}_{\alpha}K$ and Complex Numbers

It is known that derivative has direction and magnitude. The complex numbers also have directions and magnitudes. Due to this case, the following theorem can be derived.

Theorem 1: Assume that f(t) : R→R is a positive real function and $f(t)=t^{2n}$, then ${}^{\partial}_{\alpha}Kf(t)$: R→R is a real function for even denominator in case of $\alpha = \frac{\mu}{\tau}$.

Proof: The FOD defined by Karcı, ${}^{\partial}_{\alpha}K$ has an exponent whose values can be any numbers. First of all, this value can be considered as a rational number. In order to verify this case, assume that $\alpha = \frac{\mu}{\tau}$ and $\tau \neq 0$. Assume that $f(t) \geq 0$ is a non-negative function and the denominator τ is an even number: In this case,

$$ {}^{\partial}_{\alpha}Kf(t) = \left(\frac{f(t)}{t}\right)^{\frac{\mu}{\tau}-1}\frac{df(t)}{dt} = \left(\frac{f(t)}{t}\right)^{\frac{\mu-\tau}{\tau}}\frac{df(t)}{dt} = \sqrt[\tau]{\left(\frac{f(t)}{t}\right)^{\mu-\tau}}\frac{df(t)}{dt}. $$

where $\frac{df(t)}{dt} \in R$. Any complex number can be illustrated as x+iy where $x,y \in R$. There are two the resultant number $\left(\frac{f(t)}{t}\right)^{\frac{\mu-\tau}{\tau}}\frac{df(t)}{dt}$ as follow:

a) $t \geq 0$: $\left(\frac{f(t)}{t}\right)^{\frac{\mu-\tau}{\tau}} \in \{0\} \cup R^{+}$, so, $\left(\frac{f(t)}{t}\right)^{\frac{\mu-\tau}{\tau}}\frac{df(t)}{dt} \in R$ and y=0.

b) $t<0$: $\left(\frac{f(t)}{t}\right)^{\mu-\tau} \in R^{-}$, so, $\left(\frac{f(t)}{t}\right)^{\frac{\mu-\tau}{\tau}}\frac{df(t)}{dt} \in C$ ■

Example 1: Assume that $f(t)=t^2$, $t \in R$. The result is shown in Fig.1 and the denominator of order of FOD is even.

$$ {}^{\partial}_{\alpha}Kf(t) = \left(\frac{t^2}{t}\right)^{\alpha-1} 2t = \left(\frac{t^2}{t}\right)^{\frac{\mu}{\tau}-1} 2t = t^{\frac{\mu}{\tau}-1}2t = 2t^{\frac{\mu}{\tau}} $$

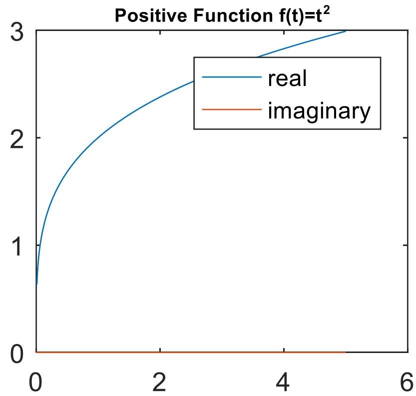


Figure 1. The results of FOD for a positive function in case of even denominator.

Example 2: $\forall x,y,\in R$, $f(x,y)=x^2y^2+x^2+y^2$ where $x,y\geq 0$. f(x,y) is a positive function for all values of positive x and y. Fig.2(a) shows f(x,y) for positive values of x and y. Fig.3(a) and Fig.3(b) show that ${}_{\alpha}^{\partial}Kf(x,y) = \frac{{}_{\alpha}^{\partial}Kf(x,y)}{{}_{\alpha}^{\partial}K(x)} = 2xy^2 + 2x$. Fig.3(a) and Fig.3(b) show that real parts are different from zero and imaginary parts are zero $\alpha = \frac{\mu}{\tau} = \frac{1}{2}$, $\alpha = \frac{\mu}{\tau} = -\frac{1}{2}$, respectively. Fig.2(b) shows f(x,y) for x,y<0. Fig.4(a) shows that real part of derivative is zero and imaginary part is different from zero where $\alpha = \frac{1}{2}$. Fig.4(b) shows the imaginary part of derivative for $\alpha = -\frac{1}{2}$ where real part is zero. Fig.4(c) shows that real part of derivative is zero and imaginary part is seen in figure where $\alpha = \frac{5}{2}$. Fig.4(d) shows that real part is zero and imaginary part is not zero for $\alpha = -\frac{5}{2}$. This example reveals the relationships between derivative and complex numbers.

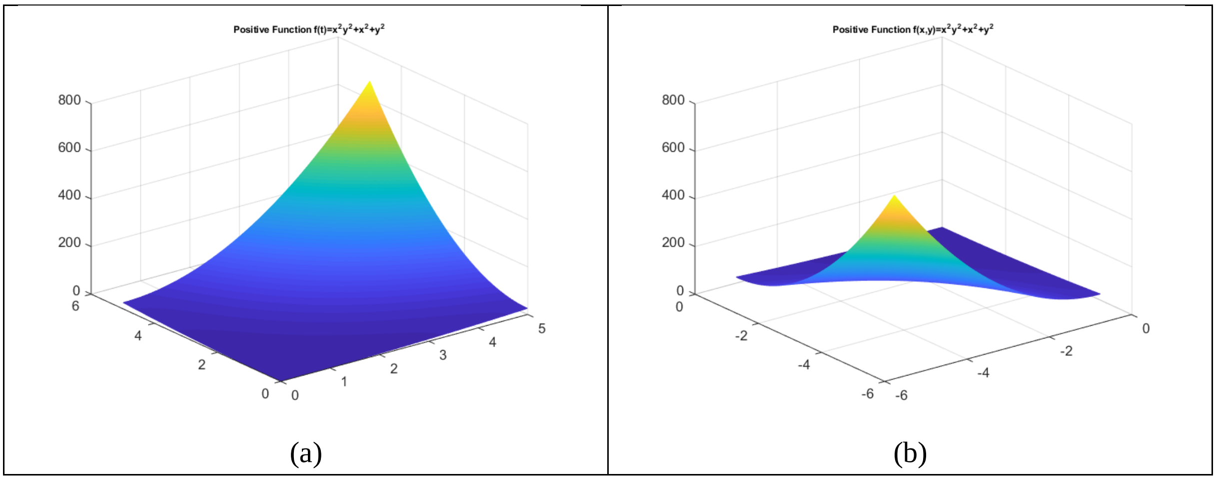


Figure 2. , $f(x,y)=x^2y^2+x^2+y^2$ (a) f(x,y) for $x,y\geq 0$, (b) f(x,y) for x,y<0.

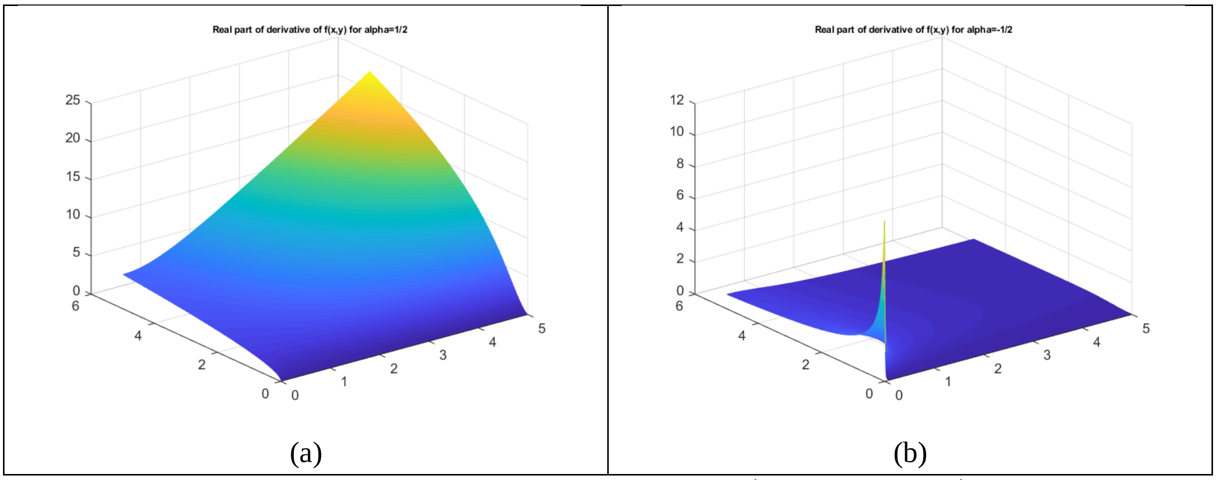


Figure 3. Derivatives of $f(x,y)=x^2y^2+x^2+y^2$, $x,y\geq 0$ for $\alpha = \frac{\mu}{\tau} = \frac{1}{2}$ and $\alpha = \frac{\mu}{\tau} = -\frac{1}{2}$.

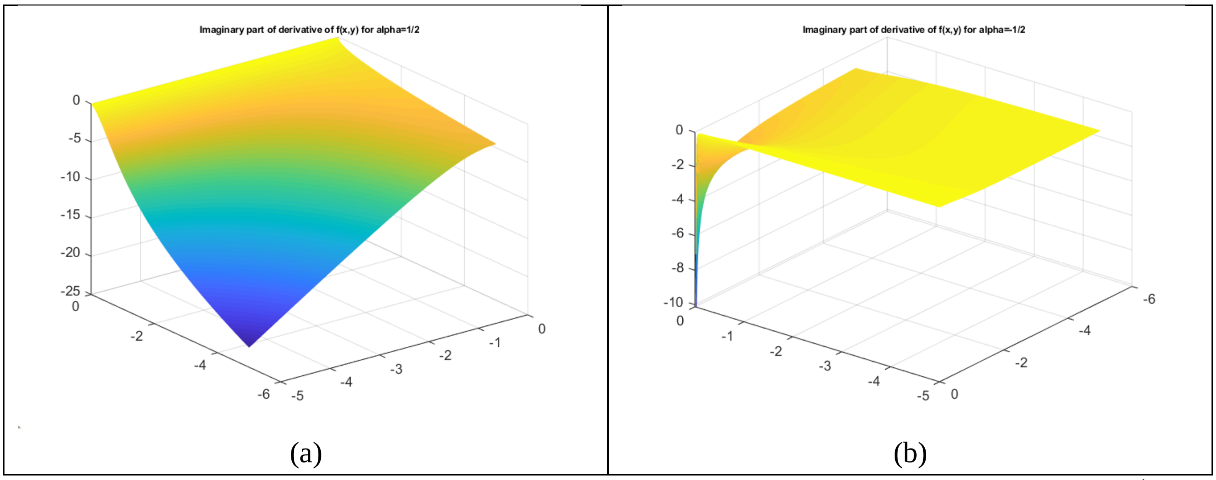


Figure 4. Derivatives of f(x,y)=$x^2y^2+x^2+y^2$, x, y<0. (a) imaginary part of derivative for $\alpha=\frac{1}{2}$, (b) imaginary part of derivative for $\alpha=-\frac{1}{2}$.

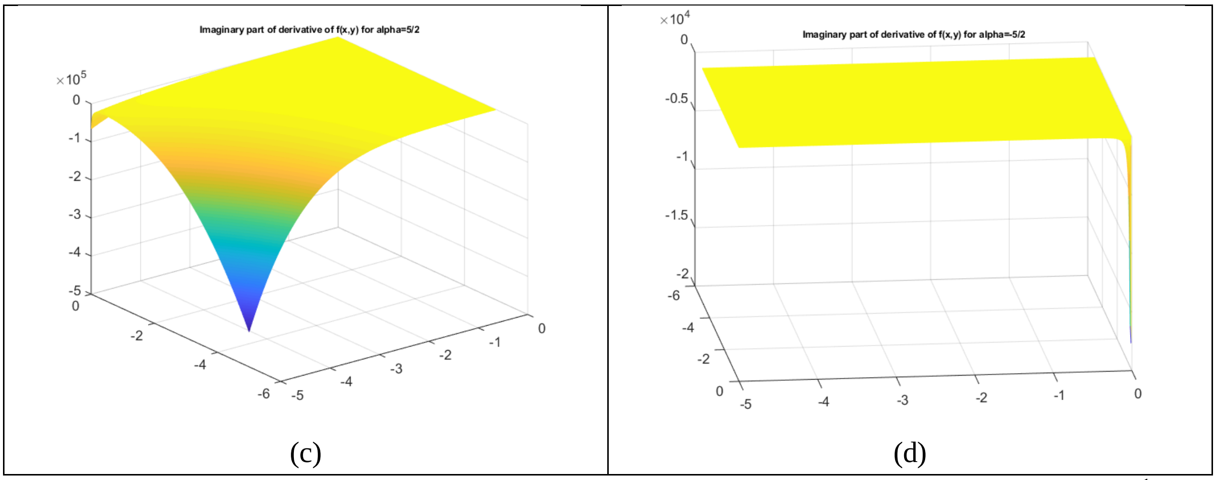


Figure 4. Derivatives of f(x,y)=$x^2y^2+x^2+y^2$, x, y<0. (c) imaginary part of derivative for $\alpha=-\frac{1}{2}$, (d) imaginary part of derivative for $\alpha=-\frac{5}{2}$.

Theorem 2: Assume that f(t) : R→R is a positive real function, then ${}_{\alpha}^{\partial}Kf(t)$: R→R is a real function for odd denominator in case of $\alpha=\frac{\mu}{\tau}$.

Proof: Assume that f(t)≥0 is a non-negative function and the denominator τ is an odd number: In this case,

$$ {}_{\alpha}^{\partial}Kf(t)=\left(\frac{f(t)}{t}\right)^{\frac{\mu}{\tau}-1}\frac{df(t)}{dt}=\left(\frac{f(t)}{t}\right)^{\frac{\mu-\tau}{\tau}}\frac{df(t)}{dt}=\sqrt[\tau]{\left(\frac{f(t)}{t}\right)^{\mu-\tau}}\frac{df(t)}{dt} $$

where $\frac{df(t)}{dt}\in R$ and τ is an odd number so, $\sqrt[\mu]{\left(\frac{f(t)}{t}\right)^{\mu-\tau}}\in R$, since assume that f(t) is positive function and the result is also a positive. The complex number is two-tuple structure such as (x,y) where complex number C=x+iy where x,y∈R. In this case, y=0, so resultant number is equal to a=$\sqrt[\mu]{\left(\frac{f(t)}{t}\right)^{\mu-\tau}}\frac{df(t)}{dt}$, not a complex number and result is positive. There are two cases:

a) t≥0 and μ-τ is even: $\left(\frac{f(t)}{t}\right)^{\mu-\tau} \in R^{+} \cup \{0\}$, so, $\sqrt[\mu]{\left(\frac{f(t)}{t}\right)^{\mu-\tau}} \frac{df(t)}{dt} \in R^{+} \cup \{0\}$.

b) t<0 and μ-τ is even: $\left(\frac{f(t)}{t}\right)^{\mu-\tau} \geq 0$, so, $\sqrt[\mu]{\left(\frac{f(t)}{t}\right)^{\mu-\tau}} \frac{df(t)}{dt} \in R^{+} \cup \{0\}$ ■

Example 3: Assume that $f(t)=t^2$, t∈R. The result is shown in Fig.3. The denominator of order of FOD is odd.

$$\frac{\partial}{\alpha}Kf(t) = \left(\frac{t^2}{t}\right)^{\alpha-1} 2t = \left(\frac{t^2}{t}\right)^{\frac{\mu}{\tau}-1} 2t = t^{\frac{\mu}{\tau}-1} 2t = 2t^{\frac{\mu}{\tau}} = 2\sqrt[\tau]{t^{\mu}}$$

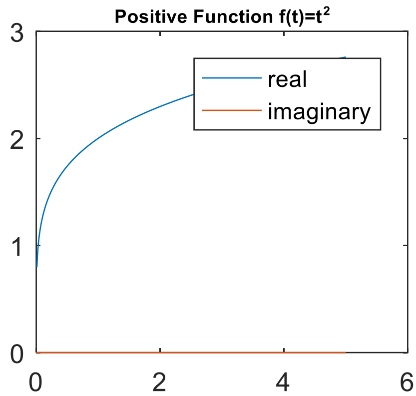
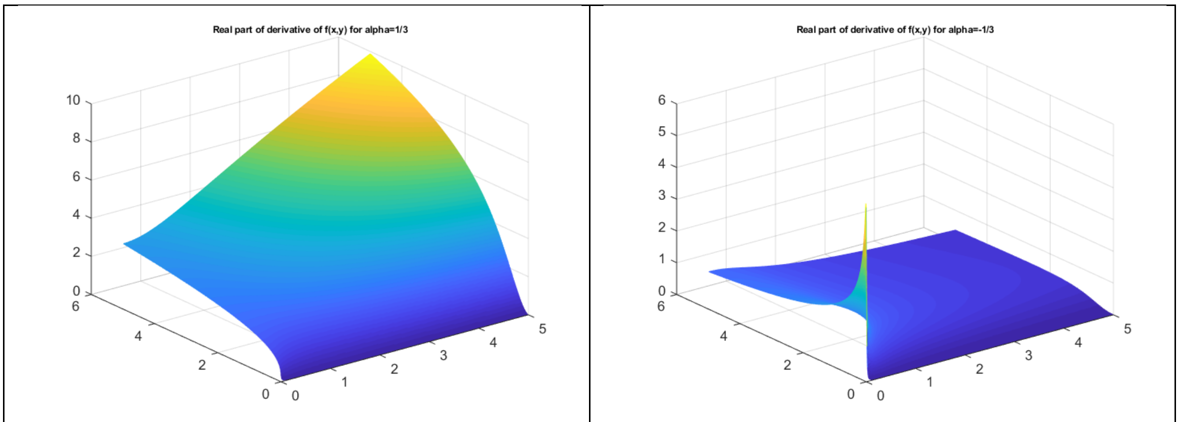


Figure 5. The results of FOD for a positive function in case of odd denominator.

Example 4: ∀x,y,∈R, $f(x,y)=x^2y^2+x^2+y^2$ where x,y≥0. f(x,y) is a positive function for all values of x and y. Fig.6(a) and Fig.6(b) show that the imaginary parts of derivatives are zero and real parts are different from zero where x,y≥0, $\alpha = \frac{1}{3}$, and $\alpha = -\frac{1}{3}$, respectively. Fig.6(c) shows that real part of derivative is different from zero for $\alpha = \frac{1}{3}$, x,y<0. Fig.d(d) shows that real part of derivative is different from zero for $\alpha = -\frac{1}{3}$, x,y<0. All results are real.



(a) (b)

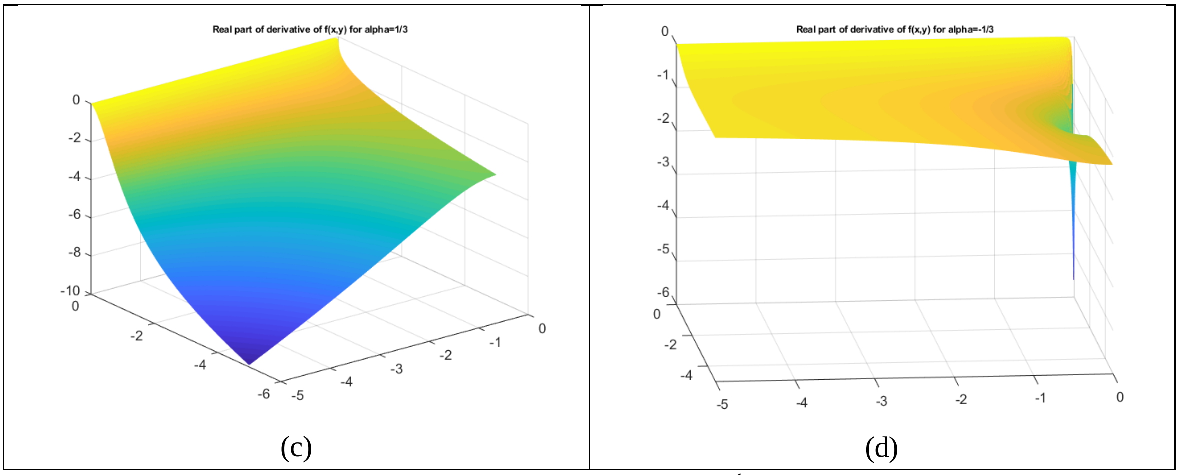


Figure 6. $f(x,y)=x^2y^2+x^2+y^2$. (a) real part of derivative for $\alpha=\frac{1}{3}$, and x,y≥0, (b) real part of derivative for $\alpha=-\frac{1}{3}$, and x,y≥0, (c) real part of derivative for $\alpha=\frac{1}{3}$, and x,y<0, (d) real part of derivative for $\alpha=-\frac{1}{3}$, and x,y<0.

Theorem 3: Assume that f(t) : R→R is a negative real function, then ${}_{\alpha}^{\partial}Kf(t)$: R→R is a complex function for even denominator in case of $\alpha=\frac{\mu}{\tau}$.

Proof: Assume that f(t) is a negative function and τ is an even number:

$$ {}_{\alpha}^{\partial}Kf(t)=\left(\frac{f(t)}{t}\right)^{\frac{\mu}{\tau}-1}\frac{df(t)}{dt}=\left(\frac{f(t)}{t}\right)^{\frac{\mu-\tau}{\tau}}\frac{df(t)}{dt}=\sqrt[\mu]{\left(\frac{f(t)}{t}\right)^{\mu-\tau}}\frac{df(t)}{dt} $$

The even exponent of any negative number is positive.

a) t≥0 and μ-τ is odd: $\left(\frac{f(t)}{t}\right)^{\mu-\tau}\in R^{-}$, so, $\sqrt[\mu]{\left(\frac{f(t)}{t}\right)^{\mu-\tau}}\frac{df(t)}{dt}\in C$.

b) t<0 and μ-τ is odd: $\left(\frac{f(t)}{t}\right)^{\mu-\tau}\geq 0$, so, $\sqrt[\mu]{\left(\frac{f(t)}{t}\right)^{\mu-\tau}}\frac{df(t)}{dt}\in R^{+}\cup\{0\}$ ■

Example 5: Assume that $f(t)=-t^2$, t∈R. The result is shown in Fig.3 and

$$ {}_{\alpha}^{\partial}Kf(t)=\left(\frac{-t^2}{t}\right)^{\alpha-1}(-2t)=\left(\frac{-t^2}{t}\right)^{\frac{\mu}{\tau}-1}(-2t)=(-t)^{\frac{\mu}{\tau}-1}(-2t) $$

Fig.3 illustrates that the imaginary part is different from zero, that's why, the resultant function is a complex function.

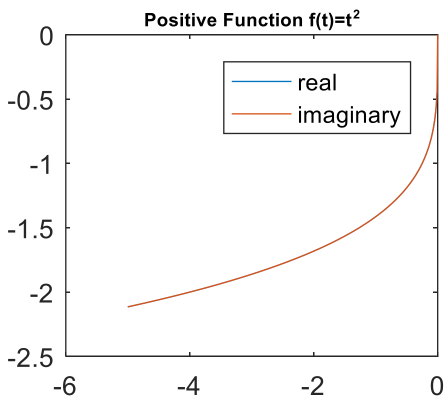


Figure 7. The results of FOD for a negative function in case of even denominator.

Example 6: Assume that $f(x,y)=-x^2y^2-x^2-y^2$ is a negative functions for all values of x and y. Fig.8(a) shows f(x,y) for non-negative values of x and y. Fig.8(b) shows f(x,y) for negative values of x and y.

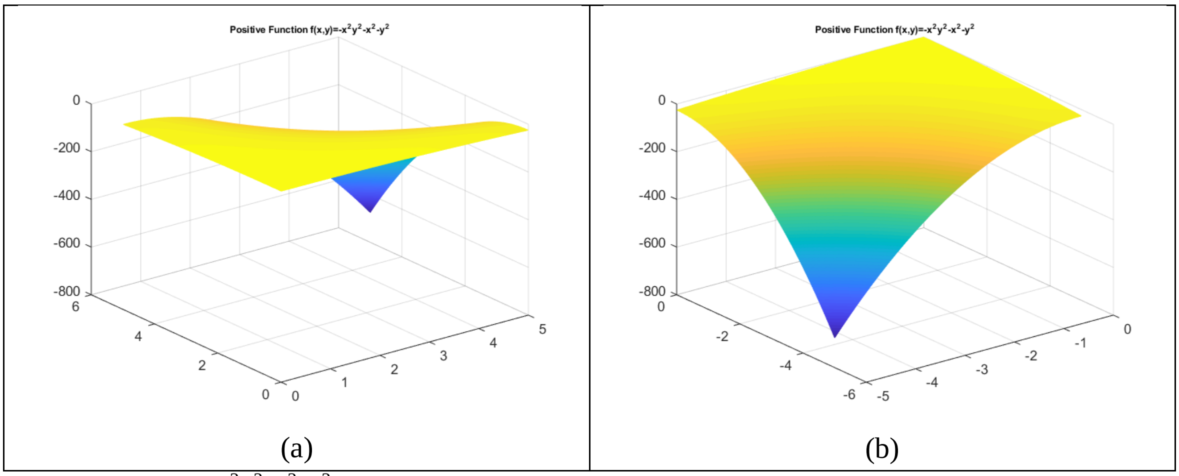


Figure 8. $f(x,y)=-x^2y^2-x^2-y^2$ where (a) $x,y \geq 0$, (b) $x,y<0$.

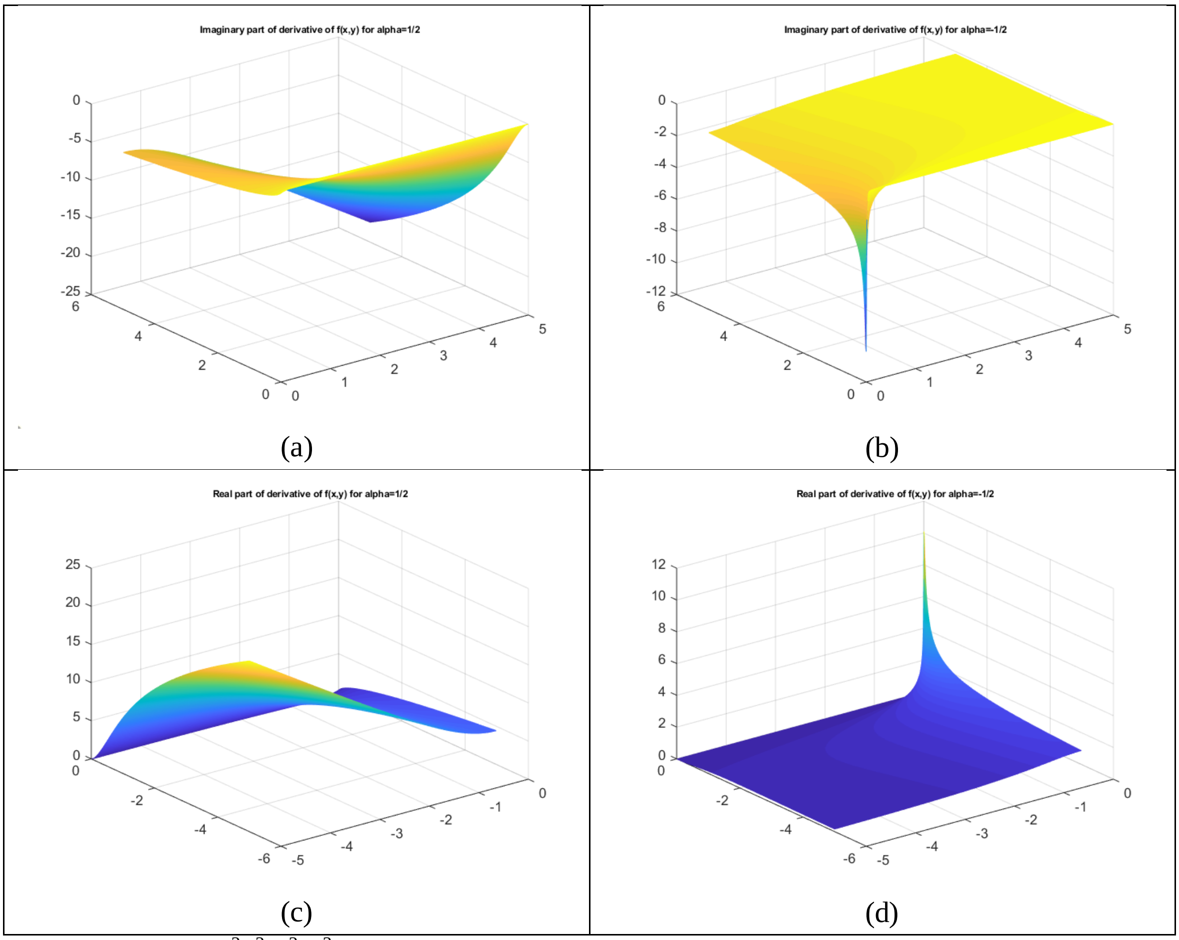


Figure 9. $f(x,y)=-x^2y^2-x^2-y^2$ where (a) imaginary part of f(x,y) is shown and real part is zero where $\alpha=\frac{1}{2}$, and $x,y \geq 0$, (b) imaginary part of f(x,y) is shown and real part is zero where $\alpha=-\frac{1}{2}$, and $x,y \geq 0$, (c) real part of f(x,y) is shown and imaginary part is zero where $\alpha=\frac{1}{2}$, and $x,y<0$, (d) real part of f(x,y) is shown and imaginary part is zero where $\alpha=-\frac{1}{2}$, and $x,y<0$.

∀x,y∈R, $f(x,y)=-x^2y^2-x^2-y^2$ is a negative function. Fig.9(a) shows that the results of fractional order derivative is a complex number for $\alpha=\frac{1}{2}$, and x,y≥0. Fig.9(b) shows that the results of fractional order derivative of f(x,y) is a complex number where $\alpha=-\frac{1}{2}$, and x,y≥0. Fig.9(c) shows that the result of fractional order derivative of f(x,y) is a real number where $\alpha=\frac{1}{2}$, and x,y<0. Fig.9(d) shows that real part of derivative is different from zero and imaginary part is zero where $\alpha=-\frac{1}{2}$, and x,y<0. This example also illustrates that complex numbers and fractional order derivatives have relationships.

Theorem 4: Assume that f(t) : R→R is a negative real function, then ${}_{\alpha}^{\partial}Kf(t)$: R→R is a complex function for odd denominator in case of $\alpha=\frac{\mu}{\tau}$.

a) Assume that f(t) is a negative function and τ is an odd number:

$$ {}_{\alpha}^{\partial}Kf(t)=\left(\frac{f(t)}{t}\right)^{\frac{\mu}{\tau}-1}\frac{df(t)}{dt}=\left(\frac{f(t)}{t}\right)^{\frac{\mu-\tau}{\tau}}\frac{df(t)}{dt}=\sqrt[\tau]{\left(\frac{f(t)}{t}\right)^{\mu-\tau}}\frac{df(t)}{dt} $$

Any odd power of a negative number concludes in complex number. It can easily be seen that $\frac{df(t)}{dt}\in R$, the Newtonian derivative of any real function is also a real function. If μ-τ is an even number, then $\left(\frac{f(t)}{t}\right)^{\mu-\tau}$ is positive so, $\sqrt[\tau]{\left(\frac{f(t)}{t}\right)^{\mu-\tau}}\in R$. If μ-τ is an odd number, and then $\left(\frac{f(t)}{t}\right)^{\mu-\tau}$ is negative so, $\sqrt[\tau]{\left(\frac{f(t)}{t}\right)^{\mu-\tau}}\in C$

The even exponent of any negative number is positive.

a) t≥0: $\left(\frac{f(t)}{t}\right)^{\mu-\tau}\in R^{-}$, so, $\sqrt[\mu]{\left(\frac{f(t)}{t}\right)^{\mu-\tau}}\frac{df(t)}{dt}\in R$.

b) t<0: $\left(\frac{f(t)}{t}\right)^{\mu-\tau}\in R^{-}$, so, $\sqrt[\mu]{\left(\frac{f(t)}{t}\right)^{\mu-\tau}}\frac{df(t)}{dt}\in R$ ■

Example 7: Assume that $f(t)=-t^2$, t∈R. The result is shown in Fig.4 and

$$ {}_{\alpha}^{\partial}Kf(t)=\left(\frac{-t^2}{t}\right)^{\alpha-1}(-2t)=\left(\frac{-t^2}{t}\right)^{\frac{\mu}{\tau}-1}(-2t)=(-t)^{\frac{\mu}{\tau}-1}(-2t) $$

Fig.10 illustrates that the imaginary part is different from zero that is why; the resultant function is a complex function.

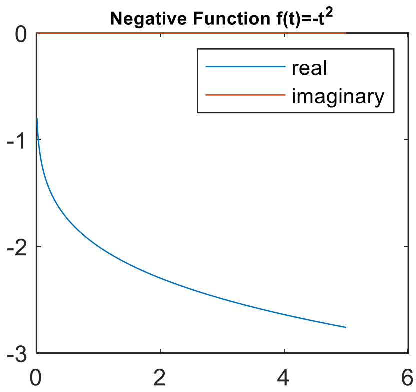


Figure 10. The results of FOD for a negative function in case of even denominator (α=1/5).

The derivative of any function has magnitude and direction, and complex numbers have magnitudes and directions. Due to this case, there should be a relationship between derivative and complex numbers. ${}^{\partial}_{\alpha}Kf(t)$ verifies these relationships.

Example 8: $\forall$x,y$\in$R, f(x,y)=-x^2y^2-x^2-y^2 is a negative function for all values of x and y. The denominator of order of derivative was selected as an odd number. Figures of f(x,y) for positive and negative values of x and y are seen in Fig.8(a) and Fig.8(b), respectively.

Fig.11(a) illustrates that the real part of fractional order derivative (Karcı derivative) for $\alpha = \frac{1}{5}$ and x,y$\geq$0 is different from zero and imaginary part is zero. Fig.11(b) depicts the real part of fractionalorder derivative for $\alpha = -\frac{1}{5}$ and x,y$\geq$0 is different from zero and imaginary part is zero. Fig.11(c) illustrates that the real part of fractional order derivative for $\alpha = -\frac{1}{5}$ and x,y<0 is different from zero and imaginary part of derivative is zero. Fig.11(d) depicts that the real part of fractional order derivative for $\alpha = \frac{7}{5}$ and x,y<0 is different from zero and imaginary part of derivative is zero.

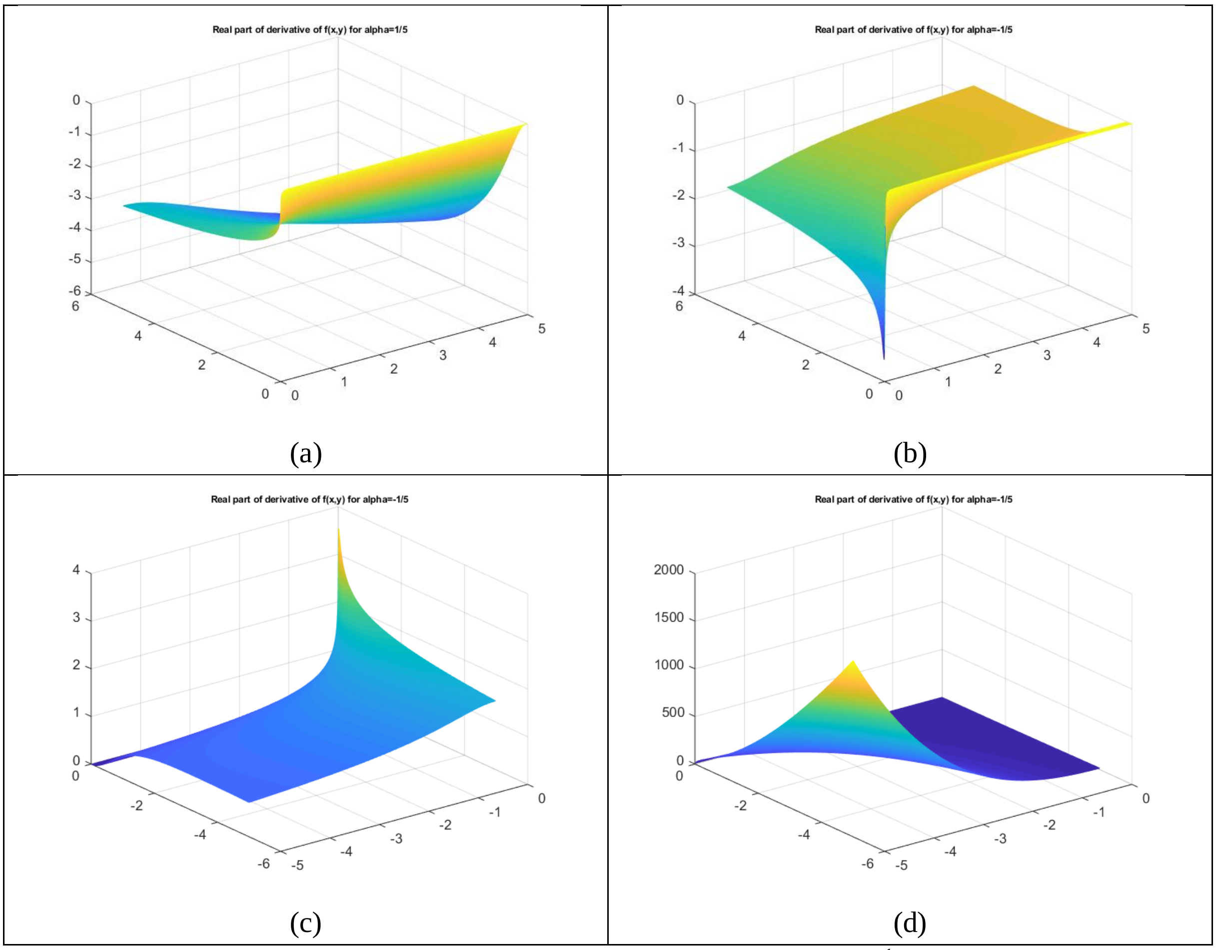


Figure 11. f(x,y)=-x^2y^2-x^2-y^2 where (a) real part of derivative for $\alpha = \frac{1}{5}$ and x,y$\geq$0, (b) real part of derivative for $\alpha = -\frac{1}{5}$ and x,y$\geq$0, (c) real part of derivative for $\alpha = -\frac{1}{5}$ and x,y<0, (d) real part of derivative for $\alpha = \frac{7}{5}$ and x,y<0.

Theorem 5: Assume that ${}^{\partial}_{\alpha}Kf(t)$ is fractional order derivative of real function f(t), ${}^{\partial}_{\alpha}Kf(t)$ is a complex function whatsoever, f(t) has single term.

Proof: It is known that any polynomial relation/function can be emphasized as an exponential relation/function. The Taylor series for exponential, sinus and cosinus are as follow.

$$e^{\theta} = 1 + \frac{\theta}{1!} + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \cdots$$

$$sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$$

$$cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots$$

So,

$$^{\partial}_{\alpha}Kf(t) = e^{i\theta} = cos\theta + isin\theta = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)$$

=x+iy ■

Miscellaneous Examples: This section includes some mixed examples for revealing the relationships between complex numbers and fractional order derivatives defined by Karcı ($^{\partial}_{\alpha}Kf(x,y)$). The Newtonian derivative has order as 1 and the imaginary part vanishes. When the derivative gets order different from 1, the imaginary part reveals. In order to illustrates these cases. Two examples for positive and negative functions were used and their details were given below.

Example 9: $\forall$x,y,$\in$R, f(x,y)=$x^2y^2+x^2+y^2$ where -5≤x,y≤5. f(x,y) is a positive function for all values of x and y (Fig.12). Fig.13(a) and Fig.13(b) show that real and imaginary parts of derivative of f(x,y) for $\alpha = \frac{\mu}{\tau} = \frac{1}{2}$ where $^{\partial}_{\alpha}Kf(x,y) = \frac{^{\partial}_{\alpha}Kf(x,y)}{^{\partial}_{\alpha}K(x)} = 2xy^2 + 2x$. Fig.13(c) and Fig.13(d) show that real and imaginary parts of derivative of f(x,y) for $\alpha = \frac{\mu}{\tau} = -\frac{1}{2}$. Fig.13(e) and Fig.13(f) show that real and imaginary parts of derivative of f(x,y) for $\alpha = \frac{\mu}{\tau} = \frac{7}{2}$. Fig.13(g) and Fig.13(h) show that real and imaginary parts of derivative of f(x,y) for $\alpha = \frac{\mu}{\tau} = -\frac{7}{2}$.

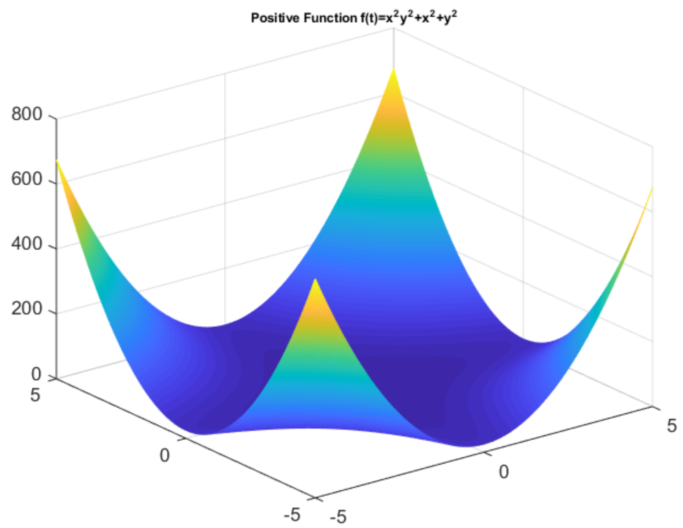


Figure 12. f(x,y)=$x^2y^2+x^2+y^2$ where -5≤x,y≤5.

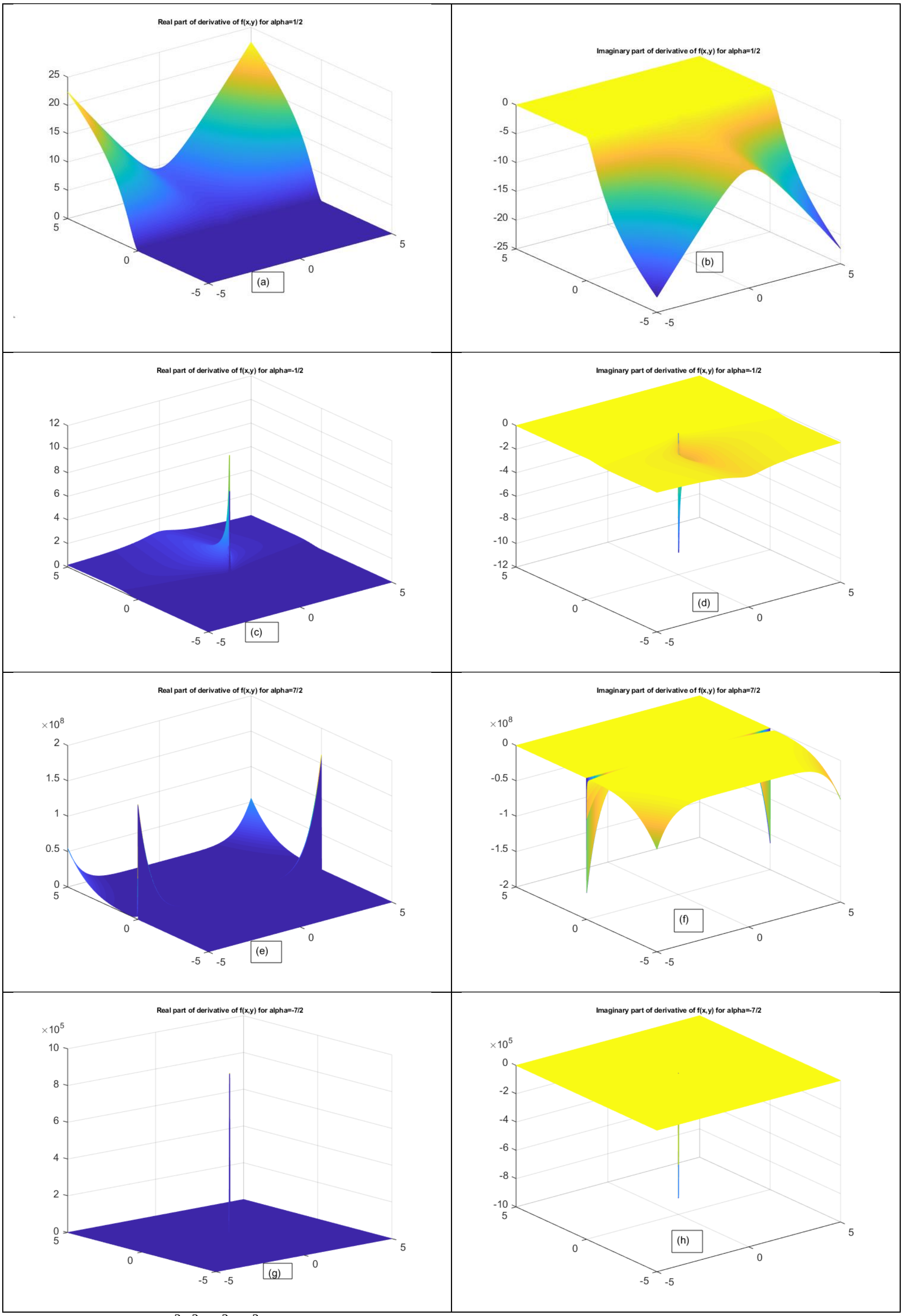


Figure 13. $f(x,y)=x^2y^2+x^2+y^2$, $-5\leq x,y\leq 5$, and its fractional order derivatives.

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3. Conclusions

$\frac{\partial}{\alpha}Kf(t)$ is a non-linear operator, since most of the events in nature are not linear in their nature, so this derivative can model events more realistic than Newtonian derivative. In this paper, the aim is to finds out the relationships between FOD and complex numbers. It is a natural logic, since complex numbers and derivative are vectorial magnitudes.

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