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Extended Newton-type Method for Generalized Equations with Hölderian Assumptions

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Abstract
In the present paper, we consider the generalized equation $0 \in f(x)+g(x)+\mathscr{F}(x)$, where $f:\mathscr{X}\to\mathscr{Y}$ is Fréchet differentiable on a neighborhood Ω of a point $\bar{x}$ in $\mathscr{X}$, $g:\mathscr{X}\to\mathscr{Y}$ is differentiable at point $\bar{x}$ and linear as well as $\mathscr{F}$ is a set-valued mapping with closed graph acting between two Banach spaces $\mathscr{X}$ and $\mathscr{Y}$. We study the above generalized equation with the help of extended Newton-type method, introduced in [M. Z. Khaton, M. H. Rashid, M. I. Hossain, Journal of Mathematics Research, **10**(4) (2018), 1–18.], under the weaker conditions than that are used in Khaton *et al*. (2018). Indeed, semilocal and local convergence analysis are provided for this method under the conditions that the Fréchet derivative of f and the first-order divided difference of g are Hölder continuous on Ω. In particular, we show this method converges superlinearly and these results extend and improve the corresponding results in Argyros (2008) and Khaton *et al*. (2018).

Keywords: Divided difference, Extended Newton-type method, Generalized equations, Lipschitz-like mappings, Semilocal convergence.
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1. Introduction

Robinson [27, 28] introduced generalized equation problems as an universal instrument for describing, analyzing and solving various type of problems in a framed way. This form of generalized equation problems have been discussed widely. Typical examples are systems of inequalities, systems of nonlinear equations, variational inequality problems, linear and nonlinear complementary problems and etc; see for examples [7, 19, 20]. Let Ω be a subset of $\mathscr{X}$. Let f be a Fréchet differentiable function from Ω to $\mathscr{Y}$ and ∇f be its Fréchet derivative, g be a differentiable at $\bar{x}$ but it may not be differentiable in a neighborhood Ω of $\bar{x}$ and linear function from Ω to $\mathscr{Y}$, $[x,y;g]$ denote the first-order divided difference at the points x and y and $\mathscr{F}$ be a set-valued mapping from $\mathscr{X}$ to $\mathscr{Y}$ with closed graph. To find a point x in Ω, we consider the generalized equation of the following form:

$$0 \in f(x)+g(x)+\mathscr{F}(x). \tag{1.1}$$

Pietrus and Alexis [1] associated the following Newton-like method for solving (1.1):

$$\begin{aligned} 0 \;\in\;& f(x_k)+g(x_k)+(\nabla f(x_k)+[2x_{k+1}-x_k,x_k;g])(x_{k+1}-x_k) \\ &+\mathscr{F}(x_{k+1}),\ \text{for } k=0,1,\ldots \end{aligned} \tag{1.2}$$

and proved that the sequence generated by the process (1.2) converges superlinearlly. To solve the generalized equation (1.1), Rashid *et al*. [25] established the local convergence results using the weaker conditions than Alexis and Pietrus [1] for the method (1.2) and expanded the sequels by fixing a gap in the proof of [1, Theorem 1].

Furthermore, Hilout *et al*. [12] associated the following sequence for solving (1.1):

$$\begin{cases} x_0 \text{ and } x_1 \text{ are two starting points} \\ y_k = \alpha x_k + (1-\alpha)x_{k-1}; \quad \text{here } \alpha \in (0,1) \\ 0 \in f(x_k) + [y_k, x_k; f](x_{k+1} - x_k) + \mathscr{F}(x_{k+1}) \end{cases}$$

and they proved the superlinear convergence of the sequence generated by this method under the assumption that f is only differentiable and continuous at a solution x^*.

For approximating the solution of (1.1), Argyros and Hilout [4] considered the following Newton-like method :

$$0 \in f(x_k) + g(x_k) + \big(\nabla f(x_k) + [x_{k+1}, x_k; g]\big)(x_{k+1} - x_k) + \mathscr{F}(x_{k+1}), \text{for } k = 0, 1, \ldots, \tag{1.3}$$

and under Lipschitz continuity property of ∇f, they presented the quadratic convergence of the method (1.3).

Moreover, when $\mathscr{F} = \{0\}$, Argyros [2] investigated on local as well as semilocal convergence analysis for two-point Newton-like methods for solving (1.1) in a Banach space setting under very general Lipschitz type conditions. An extensive study on these issues has been investigated by Rashid [3, 19, 20, 21] and other researchers when $g = 0$. In the case when $\mathscr{F}$ is either zero mapping or nonzero mapping, a large number Newton-like iterative methods have been studied and we will not mention here all in detail.

Suppose that $x \in \mathscr{X}$ and $\mathscr{N}(x)$ is the subset of $\mathscr{X}$ which is defined as

$$\mathscr{N}(x) = \big\{ d \in \mathscr{X} : 0 \in f(x) + g(x) + (\nabla f(x) + [x+d, x; g])d + \mathscr{F}(x+d) \big\}.$$

Under some suitable conditions, Khaton *et al*. [18] introduced and studied extended Newton-type method, when ∇f is continuous and Lipschitz continuous as well as g admits first-order divided difference satisfying Lipschitzian condition. Inspired by the work of Argyros in [4], Khaton *et al*. [18] considered the following "so called" extended extended Newton-type method (see Algorithm 1):

Algorithm 1 (Extended Newton-type Method)

Step 0. Pick $\eta \in [1, \infty)$, $x_0 \in \mathscr{X}$, and put $k := 0$.
Step 1. If $0 \in \mathscr{N}(x_k)$, then stop; otherwise, go to the next Step 2.
Step 2. If $0 \notin \mathscr{N}(x_k)$, choose $d_k \in \mathscr{N}(x_k)$ such that

$$\|d_k\| \le \eta \text{ dist } (0, \mathscr{N}(x_k)).$$

Step 3. Set $x_{k+1} := x_k + d_k$.
Step 4. Replace k by $k+1$ and go to Step 1.

In contrast Algorithm 1 with the known results, we have the following conclusions: When $F = 0$ and $g = 0$, it is obvious that Algorithm 1 is turned into the known Gauss-Newton method which is a famous iterative technique for solving nonlinear least squares (model fitting) problems and has been studied widely; see for example [8, 9, 13, 15, 29, 30]. Within the case when $g = 0$, several kind of methods for solving (1.1) were established by Rashid [22, 23, 24] and also obtained their local and semilocal convergence.

The objective of this article is to continue to study the semilocal and local convergence for the extended Newton-type method under the weaker conditions than [18], that is, ∇f is (L, q)-Hölder continuous and g admits the first-order divided difference satisfying q-Hölderian condition. The Lipschitz-like property of set-valued mappings which is the main tool of this study whose concepts can be found in Aubin [5] in the context of non smooth analysis and it has been studied by a huge number of mathematicians [1, 4, 10, 12, 17]. The main result of this study is semilocal analysis for the extended Newton-type method, that is, based on the information around the initial point, the main results are the convergence criteria, which provide few suitable conditions ensuring the convergence to a solution of any sequence generated by Algorithm 1. Consequently, the results of the local convergence for the extended Newton-type method are attained.

This article is organized as follows: Some necessary notations, notions, preliminary results and a fixed-point theorem are recalled in Section 2 that are used in the subsequent sections. In Section 3, we consider the extended Newton-type method defined by Algorithm 1 to approximate the solution of (1.1). Using the concept of Lipschitz-like property for the set-valued mapping, in this section we also establish the existence and superlinear convergence of the sequence generated by Algorithm 1 in both semilocal and local cases. At the end, we give a summary of the main results and present a comparison of this study with other known results.

2. Notations and Preliminaries

In this section, we evoke some notations and take out some results that will be helpful to verify our main results. Let $\mathscr{X}$ and $\mathscr{Y}$ be two complex or real Banach spaces. Also, let $p \in \mathscr{X}$ and $\mathbb{B}(p,\alpha) = \{u \in \mathscr{X} : \|u-p\| \le \alpha\}$ denote the closed ball centered at p with radius $\alpha > 0$, and $\mathscr{F}$ be a set-valued mapping with closed graph. The domain of $\mathscr{F}$, can be stated as

$$\mathrm{dom}\mathscr{F} := \{p \in \mathscr{X} : \mathscr{F}(p) \neq \emptyset\}.$$

Let $q \in \mathscr{Y}$. Then the inverse of $\mathscr{F}$, denoted by $\mathscr{F}^{-1}$, is defined by

$$\mathscr{F}^{-1}(q) := \{p \in \mathscr{X} : q \in \mathscr{F}(p)\}.$$

The graph of $\mathscr{F}$, denoted by gph$\mathscr{F}$, is defined by

$$\mathrm{gph}\mathscr{F} := \{(p,q) \in \mathscr{X} \times \mathscr{Y} : q \in \mathscr{F}(p)\}.$$

Let M and N be two subsets of a non empty set $\mathscr{X}$ and p be a point in $\mathscr{X}$. The distance from a point p to a set M is defined by

$$\mathrm{dist}(p,M) := \inf\{\|p-m\| : m \in M\}.$$

In addition, the excess e from the set M to the set N is defined by

$$e(N,M) := \sup\{\mathrm{dist}(n,M) : n \in N\}.$$

The set $\mathscr{L}(\mathscr{X},\mathscr{Y})$ is the space of linear operators from $\mathscr{X}$ to $\mathscr{Y}$ and all the norms are denoted by $\|\cdot\|$.

Definition 2.1. *Suppose $f \in \mathscr{L}(\mathscr{X},\mathscr{Y})$. Then f is said to have the first order divided difference on the points x_1 and y_1 in $\mathscr{X}$ $(x_1 \neq y_1)$ if the following properties hold:*

(a) $[x_1,y_1;f](y_1-x_1) = g(y_1) - g(x_1)$ *for* $x_1 \neq y_1$*;*

(b) *if f is Fréchet differentiable at $x_1 \in \mathscr{X}$, then $[x_1,x_1;f] = \nabla f(x_1)$.*

Now we mention the notions of pseudo-Lipschitz and Lipchitz-like set-valued mappings, which was established by Aubin and have been studied widely. To see the more details about this topic, the reader could refer to [5, 6, 26].

Definition 2.2. *Let $\Psi : \mathscr{Y} \rightrightarrows \mathscr{X}$ be a set-valued mapping and $(\bar{q},\bar{p}) \in \mathrm{gph}\Psi$ with $\alpha_{\bar{p}}$, $\alpha_{\bar{q}}$ and ν are positive constants. Then Ψ is said to be*

(a) *Lipchitz-like on $\mathbb{B}(\bar{q},\alpha_{\bar{q}})$ relative to $\mathbb{B}(\bar{p},\alpha_{\bar{p}})$ with constant ν if the following inequality holds:*

$$e(\Psi(q_1)\cap\mathbb{B}(\bar{p},\alpha_{\bar{p}}),\Psi(q_2)) \le \nu\|q_1-q_2\| \quad \textit{for every } q_1,q_2 \in \mathbb{B}(\bar{q},\alpha_{\bar{q}}).$$

(b) *pseudo-Lipschitz around $(\bar{q},\bar{p})$ if there exist constants $\alpha'_{\bar{p}} > 0, \alpha'_{\bar{q}} > 0$ and $\nu' > 0$ such that Ψ is Lipchitz-like on $\mathbb{B}(\bar{q},\alpha'_{\bar{q}})$ relative to $\mathbb{B}(\bar{p},\alpha'_{\bar{p}})$ with constant ν'.*

The following lemma is due to Rashid *et al*. [26, Lemma 2.1], which is effective and the proof of this lemma is similar to that of [16, Theorem 1.49(i)].

Lemma 2.3. *Let $\Psi : \mathscr{Y} \rightrightarrows \mathscr{X}$ be a set-valued mapping and $(\bar{y},\bar{x}) \in \mathrm{gph}\,\Psi$. Also suppose that Ψ is Lipschitz-like on $\mathbb{B}(\bar{y},r_{\bar{y}})$ which is related to $\mathbb{B}(\bar{x},r_{\bar{x}})$ with constant μ. Then*

$$\mathrm{dist}\;(x,\Psi(y)) \le \nu\;\mathrm{dist}(y,\Psi^{-1}(x)),$$

for each $x \in \mathbb{B}(\bar{x},r_{\bar{x}})$ and $y \in \mathbb{B}(\bar{y},\frac{r_{\bar{y}}}{3})$ satisfying $\mathrm{dist}(y,\Psi^{-1}(x)) \le \dfrac{r_{\bar{y}}}{3}$, *is hold.*

Dontchev and Hager [11] proved Banach fixed point theorem, which has been employing the standard iterative concept for contracting mapping. To prove the existence of the sequence generated by Algorithm 1, the following lemma will play an important rule in this study.

Lemma 2.4. *Let $\Phi : \mathscr{X} \rightrightarrows \mathscr{X}$ be a set-valued mapping. Let $x^* \in \mathscr{X}$, $0 < \lambda < 1$ and $r > 0$ be such that*

$$\operatorname{dist}(x^*, \Phi(x^*)) < r(1-\lambda) \tag{2.1}$$

and

$$e(\Phi(x_1) \cap \mathbb{B}(x^*, r), \Phi(x_2)) \leq \lambda \|x_1 - x_2\| \quad \text{for all} \quad x_1, x_2 \in \mathbb{B}(x^*, r). \tag{2.2}$$

Then Φ has a fixed point in $\mathbb{B}(x^*, r)$, that is, there exists $x \in \mathbb{B}(x^*, r)$ such that $x \in \Phi(x)$. Furthermore, if Φ is single-valued, then there exists a fixed point $x \in \mathbb{B}(x^*, r)$ such that $x = \Phi(x)$.

The preceding lemma is a generalization of a fixed point theorem and it has been taken from [14], where in the second assertion the excess e is updated by Hausdorff distance.

3. Convergence Analysis

Let $f : \Omega \subseteq \mathscr{X} \to \mathscr{Y}$ be a Fréchet differentiable function on a neighborhood Ω of $\bar{x}$ with its derivative denoted by ∇f, $g\colon \Omega \to \mathscr{Y}$ which is linear and differentiable at $\bar{x}$ and let $\mathscr{F} : \mathscr{X} \rightrightarrows \mathscr{Y}$ be a set-valued mapping with closed graph. This section is dedicated to prove the existence of a sequence generated by the extended Newton-type method, represented by Algorithm 1 and show the superlinear convergence of the sequence generated by this method.

Let $x \in \mathscr{X}$. Then for each $x \in \mathscr{X}$, we get

$$\begin{aligned} g(x) + [x+d, x; g]d &= g(x) - [x+d, x; g](x - (x+d)) \\ &= g(x) - (g(x) - g(x+d)) = g(x+d). \end{aligned} \tag{3.1}$$

Define a set-valued mapping $\mathscr{G}_x$ by

$$\mathscr{G}_x(\cdot) := f(x) + g(\cdot) + \nabla f(x)(\cdot - x) + \mathscr{F}(\cdot).$$

It holds, for the formation of $\mathscr{N}(x)$ and (3.1), that

$$\mathscr{N}(x) = \left\{ d \in \mathscr{X} : 0 \in \mathscr{G}_x(x+d) \right\}.$$

In addition, for any $z \in \mathscr{X}$ and $y \in \mathscr{Y}$, we get the following identity:

$$z \in \mathscr{G}_x^{-1}(y) \text{ if and only if } y \in f(x) + g(z) + \nabla f(x)(z-x) + \mathscr{F}(z). \tag{3.2}$$

Particularly, let $(\bar{x}, \bar{y}) \in \operatorname{gph}\mathscr{G}_{\bar{x}}$. Then, the definition of closed graphness of $\mathscr{G}_{\bar{x}}$ signifies that

$$\bar{x} \in \mathscr{G}_{\bar{x}}^{-1}(\bar{y}). \tag{3.3}$$

The following outcome constitutes the equivalence between $\mathscr{G}_{\bar{x}}^{-1}$ and $(f + g + \mathscr{F})^{-1}$. This result is due to [18].

Lemma 3.1. *Let $(\bar{x}, \bar{y}) \in$ gph $(f + g + \mathscr{F})$. Suppose that ∇f is continuous around $\bar{x}$. Assume that g admits first-order divided difference. Then the followings are equivalent:*

(i) *The mapping $(f + g + \mathscr{F})^{-1}$ is pseudo-Lipschitz at $(\bar{y}, \bar{x})$;*

(ii) *The mapping $\mathscr{G}_{\bar{x}}^{-1}$ is pseudo-Lipschitz at $(\bar{y}, \bar{x})$.*

For our suitability, let $r_{\bar{x}} > 0$, $r_{\bar{y}} > 0$ and $\mathbb{B}(\bar{x}, r_{\bar{x}}) \subseteq \Omega \cap \operatorname{dom}\mathscr{F}$. Suppose that ∇f is (L, q)-Hölder continuous on $\mathbb{B}(\bar{x}, r_{\bar{x}})$, that is , there exists $L > 0$ such that

$$\|\nabla f(x) - \nabla f(x')\| \leq L \|x - x'\|^q, q \in (0, 1], \quad \text{for any } x, x' \in \mathbb{B}(\bar{x}, r_{\bar{x}}), \tag{3.4}$$

g admits a first-order divided difference satisfying q-Hölder condition, that is, there exists $\nu > 0$ such that, for all $x, y, v, w \in \mathbb{B}(\bar{x}, r_{\bar{x}})$ $(x \neq y, v \neq w)$,

$$\|[x, y; g] - [v, w; g]\| \leq \nu (\|x - v\|^q + \|y - w\|^q), \tag{3.5}$$

and the mapping $\mathscr{G}_{\bar{x}}^{-1}$ is Lipschitz-like on $\mathbb{B}(\bar{y}, r_{\bar{y}})$ relative to $\mathbb{B}(\bar{x}, r_{\bar{x}})$ with constant M, that is,

$$e(\mathscr{G}_{\bar{x}}^{-1}(y_1)\cap\mathbb{B}(\bar{x}, r_{\bar{x}}), \mathscr{G}_{\bar{x}}^{-1}(y_2)) \le M\|y_1 - y_2\| \quad \text{for any } y_1, y_2 \in \mathbb{B}(\bar{y}, r_{\bar{y}}). \tag{3.6}$$

Further, for $\bar{y}$, the closed graph property of $\mathscr{G}_{\bar{x}}$ implies that $f+g+\mathscr{F}$ is continuous at $\bar{x}$ i.e.

$$\lim_{x\to\bar{x}} \operatorname{dist}\big(\bar{y}, f(x)+g(x)+\mathscr{F}(x)\big) = 0 \tag{3.7}$$

is hold.
Let $\varepsilon_0 > 0$ and write

$$\bar{r} := \min\Big\{r_{\bar{y}} - 2\varepsilon_0 r_{\bar{x}}, \frac{r_{\bar{x}}(1-M\varepsilon_0)}{4M}\Big\}. \tag{3.8}$$

Then

$$\bar{r} > 0 \text{ if and only if } \varepsilon_0 < \min\Big\{\frac{r_{\bar{y}}}{2r_{\bar{x}}}, \frac{1}{M}\Big\}. \tag{3.9}$$

The following lemma is taken from [26, Lemma 3.1] and it plays a crucial role for convergence analysis of the extended Newton-type method.

Lemma 3.2. *Assume that $\mathscr{G}_{\bar{x}}^{-1}$ is Lipschitz-like on $\mathbb{B}(\bar{y}, r_{\bar{y}})$ relative to $\mathbb{B}(\bar{x}, r_{\bar{x}})$ with constant M and that*

$$\sup_{x', x''\in\mathbb{B}(\bar{x}, \frac{r_{\bar{x}}}{2})} \|\nabla f(x') - \nabla f(x'')\| \le \varepsilon_0 < \min\Big\{\frac{r_{\bar{y}}}{2r_{\bar{x}}}, \frac{1}{M}\Big\}. \tag{3.10}$$

Let $x \in \mathbb{B}(\bar{x}, \frac{r_{\bar{x}}}{2})$ and ε_0 be defined by (3.9). Suppose that ∇f is continuous on $\mathbb{B}(\bar{x}, \frac{r_{\bar{x}}}{2})$. Let $\bar{r}$ be defined by (3.8) such that (3.10) is true. Then $\mathscr{G}_x^{-1}$ is Lipschitz-like on $\mathbb{B}(\bar{y}, \bar{r})$ relative to $\mathbb{B}(\bar{x}, \frac{r_{\bar{x}}}{2})$ with constant $\frac{M}{1-M\varepsilon_0}$, that is,

$$e(\mathscr{G}_x^{-1}(y_1)\cap\mathbb{B}(\bar{x}, \frac{r_{\bar{x}}}{2}), \mathscr{G}_x^{-1}(y_2)) \le \frac{M}{1-M\varepsilon_0}\|y_1 - y_2\| \text{ for any } y_1, y_2 \in \mathbb{B}(\bar{y}, \bar{r}).$$

For our convenience, we would like to introduce some notations. First we define the mapping $J_x\colon \mathscr{X} \to \mathscr{Y}$, for each $x \in \mathscr{X}$, by

$$J_x(\cdot) := f(\bar{x}) + g(\cdot) + \nabla f(\bar{x})(\cdot - \bar{x}) - f(x) - g(x) - \big(\nabla f(x) + [\cdot, x; g]\big)(\cdot - x)$$

and the set-valued mapping $\Phi_x\colon \mathscr{X} \rightrightarrows \mathscr{X}$ by

$$\Phi_x(\cdot) := \mathscr{G}_{\bar{x}}^{-1}[J_x(\cdot)]. \tag{3.11}$$

Then for any $x', x'' \in \mathscr{X}$, we have

$$\begin{aligned} \|J_x(x') - J_x(x'')\| &= \|g(x') - g(x'') - [x', x; g](x' - x) + [x'', x; g](x'' - x) \\ &\quad + (\nabla f(\bar{x}) - \nabla f(x))(x' - x'')\|. \end{aligned} \tag{3.12}$$

Furthermore, let $q \in (0,1]$ and define

$$\hat{r} := \min\Big\{r_{\bar{y}} - 2Lr_{\bar{x}}^{q+1}, \frac{r_{\bar{x}}(1-MLr_{\bar{x}}^{q})}{4M}\Big\}. \tag{3.13}$$

Then

$$\hat{r} > 0 \Leftrightarrow L < \min\Big\{\frac{r_{\bar{y}}}{2r_{\bar{x}}^{q+1}}, \frac{1}{Mr_{\bar{x}}^{q}}\Big\}. \tag{3.14}$$

3.1 Superlinear Convergence

In this section we will show that the sequence generated by Algorithm 1 converges superlinearly if ∇f is (L,q)-Hölderian and g admits first-order divided difference satisfying (ν,q)-Hölder condition. In fact, the following theorem provides some sufficient conditions ensuring the convergence of the extended Newton-type method with initial point x_0.

Theorem 3.3. *Let $\eta>1$ and $q\in(0,1]$. Assume that $\mathscr{G}_{\bar{x}}^{-1}$ is Lipschitz-like on $\mathbb{B}(\bar{y},r_{\bar{y}})$ relative to $\mathbb{B}(\bar{x},r_{\bar{x}})$ with constant M and that ∇f is (L,q)- Hölder continuous on $\mathbb{B}(\bar{x},\frac{r_{\bar{x}}}{2})$ and g admits first-order divided difference that satisfies (3.5). Let $\hat{r}$ be defined by (3.13) so that (3.14) is satisfied. Let $\nu>0$, $\delta>0$ be such that*

(a) $\delta\le\min\left\{\frac{r_{\bar{x}}}{4},\,(q+5)\hat{r},\,1,\,\left(\frac{3(q+1)r_{\bar{y}}}{[L(q+2)+2\nu(q+1)](6.2^q+1)}\right)^{\frac{1}{(q+1)}}\right\}$,

(b) $(2^qM+1)[L(q+2)+2\nu(q+1)]\left(\eta(q+1)\delta^q+4^{1-q}r_{\bar{x}}^q\right)\le(q+1)$,

(c) $\|\bar{y}\|<\frac{[L(q+2)+2\nu(q+1)]}{3(q+1)}\delta^{q+1}$.

Suppose that

$$\lim_{x\to\bar{x}}\operatorname{dist}(\bar{y},f(x)+g(x)+\mathscr{F}(x))=0. \tag{3.15}$$

Then there exist some $\hat{\delta}>0$ such that any sequence $\{x_n\}$ generated by Algorithm 1 with initial point x_0 in $\mathbb{B}(\bar{x},\hat{\delta})$ converges superlinearly to a solution x^* of (1.1).

Proof. According to the assumption (a) $4\delta\le r_{\bar{x}}$ and $\eta>1$, by assumption (b) we can write the inequality as follows

$$\begin{aligned}(2^qM+1)(q+5)[L(q+2)+2\nu(q+1)]\delta^q &= (2^qM+1)[L(q+2)+2\nu(q+1)]\left((q+1)\delta^q+4\delta^q\right)\\ &\le(2^qM+1)[L(q+2)+2\nu(q+1)]\left(\eta(q+1)\delta^q+4\delta^q\right)\\ &\le(2^qM+1)[L(q+2)+2\nu(q+1)]\left(\eta(q+1)\delta^q+4^{1-q}r_{\bar{x}}^q\right)\\ &\le(q+1).\end{aligned} \tag{3.16}$$

Furthermore, using assumption (a) $4\delta\le r_{\bar{x}}$ and assumption(b) we can reduce the inequality as follows

$$\begin{aligned}\eta M[L(q+2)+2\nu(q+1)]\delta^q &<\eta 2^qM[L(q+2)+2\nu(q+1)](q+5)\delta^q\\ &\le(2^qM+1)[L(q+2)+2\nu(q+1)](\eta(q+1)\delta^q+4\delta^q)-2^qML4\delta^q\\ &\le(2^qM+1)[L(q+2)+2\nu(q+1)](\eta(q+1)\delta^q+4^{1-q}r_{\bar{x}}^q)-2^qML4^{1-q}r_{\bar{x}}^q\\ &\le(q+1)-2^qML4^{1-q}r_{\bar{x}}^q.\end{aligned}$$

Since $q\in(0,1]$ then, we get $2^qML4^{1-q}r_{\bar{x}}^q\ge(q+1)MLr_{\bar{x}}^q$. Now using (3.16) in the above equation and it becomes

$$\eta M[L(q+2)+2\nu(q+1)]\delta^q\le(q+1)-(q+1)MLr_{\bar{x}}^q. \tag{3.17}$$

Putting

$$s:=\frac{\eta M[L(q+2)+2\nu(q+1)]\delta^q}{(q+1)(1-MLr_{\bar{x}}^q)}.$$

Then, from (3.17) we have that

$$s\le1. \tag{3.18}$$

Pick $0<\hat{\delta}\le\delta$ such that, for each $x_0\in\mathbb{B}(\bar{x},\hat{\delta})$,

$$\operatorname{dist}(0,f(x_0)+g(x_0)+F(x_0))\le\frac{[L(q+2)+2\nu(q+1)]}{3(q+1)}\delta^{q+1}. \tag{3.19}$$

Note that since (3.15) holds and assumption (c) is true, we assume that such $\hat{\delta}$ exists, which satisfies (3.19). Let $x_0 \in \mathbb{B}(\bar{x}, \hat{\delta})$. By induction we will show that Algorithm 1 generates at least one sequence and such sequence $\{x_n\}$ generated by Algorithm 1 satisfies the following statements:

$$\|x_n - \bar{x}\| \leq 2\delta \tag{3.20}$$

$$\text{and} \qquad \|d_n\| \leq s\left(\frac{1}{3}\right)^{(q+1)^n} \delta, \tag{3.21}$$

hold for every $n = 0, 1, 2, \ldots.$

Define

$$r_x := \frac{(q+5)M}{4(q+1)}\Big([L(q+2)+2\nu(q+1)]\|x-\bar{x}\|^{(q+1)} + (q+1)\|\bar{y}\|\Big) \quad \text{for each } x \in X. \tag{3.22}$$

From (3.16) we get

$$2^q M[L(q+2)+2\nu(q+1)]\delta^q \leq \frac{q+1}{q+5}. \tag{3.23}$$

$$\text{and} \quad [L(q+2)+2\nu(q+1)]\delta^q \leq \frac{q+1}{q+5}. \tag{3.24}$$

Hence by the combination of $\delta \leq (q+5)\hat{r}$ in assumption (a) and inequality (3.24), we get

$$\begin{aligned} \|\bar{y}\| &< \frac{[L(q+2)+2\nu(q+1)]\delta^{q+1}}{3(q+1)} \\ &\leq \frac{(q+1)}{(q+1)\cdot(q+5)} \cdot \frac{(q+5)\hat{r}}{3} = \frac{\hat{r}}{3}. \end{aligned} \tag{3.25}$$

Utilizing (3.23) and assumption (c) together with (3.24), we get from (3.22) that

$$\begin{aligned} r_x &\leq \frac{(q+5)M}{4(q+1)}\Big([L(q+2)+2\nu(q+1)]\|\bar{x}-x_0\|^{q+1} + \frac{[L(q+2)+2\nu(q+1)]}{3}\delta^{q+1}\Big) \\ &< \frac{(q+5)M}{12(q+1)}\Big(3[L(q+2)+2\nu(q+1)](2\delta)^{q+1} + 2^q[L(q+2)+2\nu(q+1)]\delta^{q+1}\Big) \\ &= \frac{(q+5)M}{12(q+1)}[L(q+2)+2\nu(q+1)]\delta^{q+1}(3.2.2^q+2^q) \\ &= \frac{(q+5)(6\cdot 2^q+2^q)M}{12(q+1)}[L(q+2)+2\nu(q+1)]\delta^{q+1} \\ &= \frac{(q+5)7\cdot 2^q M}{12(q+1)}[L(q+2)+2\nu(q+1)]\delta^{p+1} \\ &= \frac{7(q+5)}{12(q+1)}\cdot\frac{(q+1)}{(q+5)}\delta < \frac{7}{12}\delta < 2\delta \quad \text{for each } x \in \mathbb{B}(\bar{x}, 2\delta). \end{aligned} \tag{3.26}$$

Observe that (3.20) is trivial for $n = 0$.

At first, we need to prove $\mathcal{N}(x_0) \neq \emptyset$ to show that (3.21) holds for $n = 0$. The nonemptyness of $\mathcal{N}(x_0)$ will ensure us to deduce the existence of the point x_1. We will apply Lemma 2.4 to the map Φ_{x_0} with $\eta_0 = \bar{x}$ for completing this. We have to show that Lemma 2.4 holds with $r := r_{x_0}$ and $\lambda := \dfrac{q+1}{q+5}$ satisfying both assertions (2.1) and (2.2). We get from (3.3) that $\bar{x} \in \mathcal{G}_{\bar{x}}^{-1}(\bar{y}) \cap \mathbb{B}(\bar{x}, 2\delta)$. According to the definition of the excess e and (3.11), defined as the mapping of Φ_{x_0}, we have that

$$\begin{aligned} \operatorname{dist}(\bar{x}, \Phi_{x_0}(\bar{x})) &\leq e(\mathcal{G}_{\bar{x}}^{-1}(\bar{y}) \cap \mathbb{B}(\bar{x}, r_{x_0}), \Phi_{x_0}(\bar{x})) \\ &\leq e(\mathcal{G}_{\bar{x}}^{-1}(\bar{y}) \cap \mathbb{B}(\bar{x}, 2\delta), \Phi_{x_0}(\bar{x})) \\ &\leq e(\mathcal{G}_{\bar{x}}^{-1}(\bar{y}) \cap \mathbb{B}(\bar{x}, r_{\bar{x}}), \mathcal{G}_{\bar{x}}^{-1}[J_{x_0}(\bar{x})]). \end{aligned} \tag{3.27}$$

Since ∇f is (L,q)-Hölder continuous and g admits first-order divided difference satisfies Hölderian condition, for every $x \in \mathbb{B}(\bar{x}, 2\delta) \subseteq \mathbb{B}(\bar{x}, \frac{r_{\bar{x}}}{2})$, we have that

$$\begin{aligned} \|J_{x_0}(x)-\bar{y}\| &= \|f(\bar{x})+g(x)+\nabla f(\bar{x})(x-\bar{x})-f(x_0)-g(x_0) \\ &\quad -(\nabla f(x_0)+[x,x_0;g])(x-x_0)-\bar{y}\| \\ &\leq \|f(\bar{x})-f(x_0)-\nabla f(x_0)(\bar{x}-x_0)\|+\|\big(\nabla f(x_0)-\nabla f(\bar{x})\big)(\bar{x}-x)\| \\ &\quad +\|g(x)-g(x_0)-[x,x_0;g](x-x_0)\|+\|\bar{y}\| \\ &\leq \frac{L}{q+1}\|\bar{x}-x_0\|^{q+1}+\|[x_0,x;g]-[x,x_0;g]\|\|x-x_0\|+ \\ &\quad L\|x_0-\bar{x}\|^q\|\bar{x}-x\|+\|\bar{y}\| \qquad (3.28) \\ &\leq \frac{L}{q+1}\|\bar{x}-x_0\|^{q+1}+\nu\big(\|x_0-x\|^q+\|x-x_0\|^q\big)\|x-x_0\|+ \\ &\quad L\|x_0-\bar{x}\|^q\|\bar{x}-x\|+\|\bar{y}\| \\ &\leq \frac{L}{q+1}(2\delta)^{q+1}+L(2\delta)^q\cdot 2\delta+\nu\big((2\delta)^q+(2\delta)^q\big)\cdot 2\delta+\|\bar{y}\| \\ &\leq \frac{L(q+2)+2\nu(q+1)}{q+1}\delta^{q+1}\cdot 2^{q+1}+\|\bar{y}\|. \qquad (3.29) \end{aligned}$$

Now through the assumptions (a) $\frac{[L(q+2)+2\nu(q+1)](6\cdot 2^q+1)}{3(q+1)}\delta^{q+1} \leq r_{\bar{y}}$ and (c), (3.28) gives that

$$\begin{aligned} \|J_{x_0}(x)-\bar{y}\| &\leq \frac{[L(q+2)+2\nu(q+1)]}{q+1}2^{q+1}\delta^{q+1}+\frac{[L(q+2)+2\nu(q+1)]}{3(q+1)}\delta^{q+1} \\ &= \frac{[L(q+2)+2\nu(q+1)](3.2.2^q+1)}{3(q+1)}\delta^{q+1} \\ &< \frac{[L(q+2)+2\nu(q+1)](6\cdot 2^q+1)}{3(q+1)}\delta^{q+1} \\ &\leq r_{\bar{y}}. \qquad (3.30) \end{aligned}$$

This means that $J_{x_0}(x) \in \mathbb{B}(\bar{y}, r_{\bar{y}})$. Moreover, let $x=\bar{x}$ in (3.28). Then it is easily proved that

$$J_{x_0}(\bar{x}) \in \mathbb{B}(\bar{y}, r_{\bar{y}})$$

and

$$\|J_{x_0}(\bar{x})-\bar{y}\| \leq \frac{[L+2\nu(q+1)]}{q+1}\|\bar{x}-x_0\|^{q+1}+\|\bar{y}\|. \qquad (3.31)$$

By using the Lipschitz-like property of $\mathscr{G}_{\bar{x}}^{-1}$ and (3.31) in (3.27), we obtain

$$\begin{aligned} \operatorname{dist}(\bar{x}, \Phi_{x_0}(\bar{x})) &\leq M\|\bar{y}-J_{x_0}(\bar{x})\| \\ &\leq \frac{M[L(q+2)+2\nu(q+1)]}{q+1}\|\bar{x}-x_0\|^{q+1}+M\|\bar{y}\| \\ &\leq \frac{4}{q+5}r_{x_0}=\Big(1-\frac{q+1}{q+5}\Big)r_{x_0} \\ &= (1-\lambda)r; \end{aligned}$$

i,e,. the statement (2.1) of Lemma 2.4 is hold.

Now, it is evident to show that statement (2.2) of Lemma 2.4 holds. Let $x', x'' \in \mathbb{B}(\bar{x}, r_{x_0})$. Then we have that $x', x'' \in \mathbb{B}(\bar{x}, r_{x_0}) \subseteq \mathbb{B}(\bar{x}, 2\delta) \subseteq \mathbb{B}(\bar{x}, r_{\bar{x}})$ by (3.26) and $J_{x_0}(x'), J_{x_0}(x'') \in \mathbb{B}(\bar{y}, r_{\bar{y}})$ by (3.30). This together with Lipschitz-like property of $\mathscr{G}_{\bar{x}}^{-1}$ follows as

$$\begin{aligned} e(\Phi_{x_0}(x') \cap \mathbb{B}(\bar{x}, r_{x_0}), \Phi_{x_0}(x'')) &\leq e(\Phi_{x_0}(x') \cap \mathbb{B}(\bar{x}, 2\delta), \Phi_{x_0}(x'')) \\ &\leq e(\mathscr{G}_{\bar{x}}^{-1}[J_{x_0}(x')] \cap \mathbb{B}(\bar{x}, r_{\bar{x}}), \mathscr{G}_{\bar{x}}^{-1}[J_{x_0}(x'')]) \\ &\leq M\|J_{x_0}(x')-J_{x_0}(x'')\|. \qquad (3.32) \end{aligned}$$

Now, using the definition of first order divided difference of g in (3.12) we obtain

$$\begin{aligned}
\|J_{x_0}(x') - J_{x_0}(x'')\| &= \|g(x') - g(x'') - [x', x_0; g](x' - x_0) + [x'', x_0; g](x'' - x_0) \\
&\quad + (\nabla f(\bar{x}) - \nabla f(x_0))(x' - x'')\| \\
&\leq \|g(x') - g(x'') + [x', x_0; g](x_0 - x') - [x'', x_0; g](x_0 - x'')\| \\
&\quad + \|\nabla f(\bar{x}) - \nabla f(x_0)\| \|x' - x''\| \\
&\leq \|g(x') - g(x'') + g(x_0) - g(x') - g(x_0) + g(x'') \\
&\quad + \|\nabla f(\bar{x}) - \nabla f(x_0)\| \|x' - x''\| \\
&\leq \|\nabla f(\bar{x}) - \nabla f(x_0)\| \|x' - x''\| \leq L\|\bar{x} - x_0\|^q \|x' - x''\| \\
&\leq L.2^q \delta^q \|x' - x''\|. \qquad (3.33)
\end{aligned}$$

It follows from (3.32), that

$$e(\Phi_{x_0}(x') \cap \mathbb{B}(\bar{x}, r_{x_0}), \Phi_{x_0}(x'')) \leq ML.2^q \delta^q \|x' - x''\|.$$

Since $\nu, M, L > 0$ and $q \in (0,1]$, then we can write $2^q ML\delta^q < 2^q M[L(q+2) + 2\nu(q+1)]\delta^p$ and hence the above inequality becomes

$$\begin{aligned}
e(\Phi_{x_0}(x') \cap \mathbb{B}(\bar{x}, r_{x_0}), \Phi_{x_0}(x'')) &\leq 2^q M[L(q+2) + 2\nu(q+1)]\delta^p \|x' - x''\| \\
&\leq \frac{q+1}{q+5} \|x' - x''\| \\
&= \lambda \|x' - x''\|.
\end{aligned}$$

Thus the statement (2.2) of Lemma 2.4 is also hold. Hence, both statements (2.1) and (2.2) of Lemma 2.4 are accomplished. Finally, it shows that Lemma 2.4 is adequate to presume the position of a point $\hat{x}_1 \in \mathbb{B}(\bar{x}, r_{x_0})$ such that $\hat{x}_1 \in \Phi_{x_0}(\hat{x}_1)$ which implies that $0 \in f(x_0) + g(x_0) + (\nabla f(x_0) + [\hat{x}_1, x_0; g])(\hat{x}_1 - x_0) + \mathscr{F}(\hat{x}_1)$ and hence $\mathscr{N}(x_0) \neq \emptyset$.

Next, it is sufficient to prove that (3.21) holds for $n = 0$. As ∇f is (L,q)- Hölder continuous on $\mathbb{B}(\bar{x}, \frac{r_{\bar{x}}}{2})$, we have for all $x', x'' \in \mathbb{B}(\bar{x}, \frac{r_{\bar{x}}}{2})$, that

$$Lr_{\bar{x}}^q \geq \sup_{x', x'' \in \mathbb{B}(\bar{x}, \frac{r_{\bar{x}}}{2})} \|\nabla f(x') - \nabla f(x'')\|. \qquad (3.34)$$

Observe the assumption (a) that $\hat{r} > 0$. Therefore, from (3.13) and (3.34)imply that Lemma 3.2 is satisfied with $\varepsilon_0 := Lr_{\bar{x}}^p$. According to our assumption $\mathscr{G}_{\bar{x}}^{-1}$ is Lipschitz-like on $\mathbb{B}(\bar{y}, r_{\bar{y}})$ relative to $\mathbb{B}(\bar{x}, r_{\bar{x}})$. Then, it implies from Lemma 3.2 that, $\mathscr{G}_{x_0}^{-1}$ is Lipschitz-like on $\mathbb{B}(\bar{y}, \hat{r})$ relative to $\mathbb{B}(\bar{x}, \frac{r_{\bar{x}}}{2})$ with constant $\frac{M}{1 - MLr_{\bar{x}}^q}$ as $x_0 \in \mathbb{B}(\bar{x}, \hat{\delta}) \subseteq \mathbb{B}(\bar{x}, \delta) \subseteq \mathbb{B}(\bar{x}, \frac{r_{\bar{x}}}{2})$ by assumption (a) and the choice of $\hat{\delta}$. On the other hand, (3.19) follows as

$$\begin{aligned}
\text{dist}(0, \mathscr{G}_{x_0}(x_0)) &= \text{dist}(0, f(x_0) + g(x_0) + \mathscr{F}(x_0)) \\
&\leq \frac{\hat{r}}{3}.
\end{aligned}$$

Inequality (3.25) shows that $0 \in \mathbb{B}(\bar{y}, \frac{\hat{r}}{3})$ and observe before that $x_0 \in \mathbb{B}(\bar{x}, \frac{r_{\bar{x}}}{2})$. Hence using Lemma 2.3, we get

$$\begin{aligned}
\text{dist}(x_0, \mathscr{G}_{x_0}^{-1}(0)) &\leq \frac{M}{1 - MLr_{\bar{x}}^q} \text{dist}(0, \mathscr{G}_{x_0}(x_0)) \\
&= \frac{M}{1 - MLr_{\bar{x}}^q} \text{dist}(0, f(x_0) + g(x_0) + \mathscr{F}(x_0)).
\end{aligned}$$

This together with (3.1), gives

$$\begin{aligned}
\text{dist}(0, \mathscr{N}(x_0)) &= \text{dist}(x_0, \mathscr{G}_{x_0}^{-1}(0)) \\
&\leq \frac{M}{1 - MLr_{\bar{x}}^q} \text{dist}(0, f(x_0) + g(x_0) + \mathscr{F}(x_0)). \qquad (3.35)
\end{aligned}$$

According to Algorithm 1 and using (3.35), (3.19) and then assumption (a), we have

$$\begin{aligned}
\|d_0\| &\leq \eta \ \mathrm{dist}(0,\mathscr{N}(x_0)) \\
&\leq \frac{\eta M}{(1-MLr_{\bar{x}}^q)} \ \mathrm{dist}(0,f(x_0)+g(x_0)+\mathscr{F}(x_0)) \\
&\leq \frac{\eta M[L(q+2)+2\nu(q+1)]\delta^{q+1}}{3(q+1)(1-MLr_{\bar{x}}^q)} = s\Big(\frac{1}{3}\Big)\delta.
\end{aligned}$$

This means that

$$\|x_1-x_0\| = \|d_0\| \leq s\Big(\frac{1}{3}\Big)\delta,$$

and therefore, (3.21) is true for $n=0$.

Suppose $x_1,x_2,\ldots,x_k$ are formed and (3.20), and (3.21) hold for $n=0,1,2,\ldots,k-1$. We show that there exists x_{k+1} such that (3.20) and (3.21) also hold for $n=k$. Since (3.20) and (3.21) are true for each $n \leq k-1$, we have the following inequality:

$$\|x_k-\bar{x}\| \leq \sum_{i=0}^{k-1}\|d_i\| + \|x_0-\bar{x}\| \leq s\delta \sum_{i=0}^{k-1}\Big(\frac{1}{3}\Big)^{(q+1)^i} + \delta \leq 2\delta.$$

This implies (3.20) holds for $n=k$. Now with all the same argument as we did for the case when $n=0$, we can prove that $\mathscr{N}(x_k) \neq \emptyset$, that is, the point x_{k+1} exists and $\mathscr{G}_{x_k}^{-1}$ is Lipschitz-like on $\mathbb{B}(\bar{y},\hat{r})$ relative to $\mathbb{B}(\bar{x},\frac{r_{\bar{x}}}{2})$ with constant $\frac{M}{1-MLr_{\bar{x}}^q}$. Therefore, we have that

$$\begin{aligned}
\|x_{k+1}-x_k\| &= \|d_k\| \leq \eta \ \mathrm{dist}(0,\mathscr{N}(x_k)) \\
&\leq \eta \ \mathrm{dist}(x_k,\mathscr{G}_{x_k}^{-1}(0)) \\
&= \frac{\eta M}{1-MLr_{\bar{x}}^q} \ \mathrm{dist}(0,f(x_k)+g(x_k)+\mathscr{F}(x_k)) \\
&\leq \frac{\eta M}{1-MLr_{\bar{x}}^q}\|f(x_k)+g(x_k)-f(x_{k-1})-g(x_{k-1}) \\
&\quad -\big(\nabla f(x_{k-1})+[x_k,x_{k-1};g]\big)(x_k-x_{k-1})\| \\
&\leq \frac{\eta M}{1-MLr_{\bar{x}}^q}\big(\|f(x_k)-f(x_{k-1})-\nabla f(x_{k-1})(x_k-x_{k-1})\| \\
&\quad +\|g(x_k)-g(x_{k-1})-[x_k,x_{k-1};g](x_k-x_{k-1})\|\big) \\
&\leq \frac{\eta M}{(q+1)(1-MLr_{\bar{x}}^q)}\big(L\|x_k-x_{k-1}\|^{q+1}+ \\
&\quad (q+1)\|[x_{k-1},x_k;g]-[x_k,x_{k-1};g]\|\|x_k-x_{k-1}\|\big) \\
&\leq \frac{\eta M}{(q+1)(1-MLr_{\bar{x}}^q)}\big(L\|x_k-x_{k-1}\|^{q+1}+ \\
&\quad (q+1)\nu(\|x_{k-1}-x_k\|^q+\|x_k-x_{k-1}\|^q)\|x_k-x_{k-1}\|\big) \\
&\leq \frac{\eta M[L+2\nu(q+1)]}{(q+1)(1-MLr_{\bar{x}}^q)}\|d_{k-1}\|^{q+1} \\
&\leq \frac{\eta M[L(q+2)+2\nu(q+1)]}{(q+1)(1-MLr_{\bar{x}}^q)}\|d_{k-1}\|^{q+1} \\
&\leq \frac{\eta M[L(q+2)+2\nu(q+1)]}{(q+1)(1-MLr_{\bar{x}}^q)}\Big(s\Big(\frac{1}{3}\Big)^{(q+1)^{k-1}}\delta\Big)^{q+1} \\
&\leq s\Big(\frac{1}{3}\Big)^{(q+1)^k}\delta.
\end{aligned}$$

This implies that (3.21) holds for $n=k$ and therefore the proof of the theorem is complete. □

Consider the special case when $\bar{x}$ is a solution of (1.1) (that is, $\bar{y}=0$) in Theorem 3.3. We have the following corollary, which describes the local superlinear convergence result for the extended Newton-type method.

Corollary 3.4. *Suppose that $\bar{x}$ is a solution of (1.1). Let $q \in (0,1]$ and $\eta > 1$ and let $\mathscr{G}_{\bar{x}}^{-1}$ be pseudo-Lipschitz around $(0,\bar{x})$. Let $\tilde{r} > o$ and suppose that ∇f is (L,q)-Hölder continuous on $\mathbb{B}(\bar{x},\tilde{r})$ and g admits first-order divided difference satisfying Hölderian condition on $\mathbb{B}(\bar{x},\tilde{r})$. Assume that*

$$\lim_{x\to\bar{x}} \operatorname{dist}(0,\mathscr{G}_x(x)) = 0. \tag{3.36}$$

Then, with an initial point x_0, there exists some $\hat{\delta} > 0$ such that any sequence $\{x_n\}$ generated by Algorithm 1 converges superlinearly to a solution x^* of (1.1).

Proof. Suppose that $\mathscr{G}_{\bar{x}}^{-1}$ is pseudo-Lipschitz around $(0,\bar{x})$. Then by definition of pseudo-Lipschitz continuty, there exist constants $M,\tilde{r}$ and r_0 such that $\mathscr{G}_{\bar{x}}^{-1}$ is Lipschitz-like on $\mathbb{B}(\bar{y},r_0)$ relative to $\mathbb{B}(\bar{x},\tilde{r})$ with constant M. Then, for each $0 < r_{\bar{x}} \le \tilde{r}$, we have that

$$e(\mathscr{G}_{\bar{x}}^{-1}(y_1)\cap\mathbb{B}(\bar{x},r_{\bar{x}}),\mathscr{G}_{\bar{x}}^{-1}(y_2) \le M\|y_1-y_2\| \text{ for any } y_1,\ y_2 \in \mathbb{B}(0,r_0),$$

that is, $\mathscr{G}_{\bar{x}}^{-1}$ is Lipschitz-like on $\mathbb{B}(\bar{y},r_0)$ relative to $\mathbb{B}(\bar{x},r_{\bar{x}})$ with constant M. Let $L \in (0,1]$, $q \in (0,1]$ and $\nu > 0$. By the (L,q)-Hölder continuty of ∇f we can select $r_{\bar{x}} \in (0,\tilde{r})$ such that $\frac{r_{\bar{x}}}{2} \le \tilde{r}$, $r_0 - 2Lr_{\bar{x}}^{q+1} > 0$, $MLr_{\bar{x}}^q < 1$ and

$$Lr_{\bar{x}}^q \ge \sup_{x',x''\in\mathbb{B}(\bar{x},\frac{r_{\bar{x}}}{2})} \|\nabla f(x') - \nabla f(x'')\|.$$

Then, define

$$\hat{r} := \min\left\{r_0 - 2Lr_{\bar{x}}^{q+1}, \frac{r_{\bar{x}}(1-MLr_{\bar{x}}^q)}{4M}\right\} > 0.$$

and

$$\min\left\{\frac{r_{\bar{x}}}{4},\,(q+5)\hat{r},\,\frac{3(q+1)r_0}{[L(q+2)+2\nu(q+1)](6.2^q+1)}\right\} > 0$$

Thus, we can choose $0 < \delta \le 1$ such that

$$\delta \le \min\left\{\frac{r_{\bar{x}}}{4},\,(q+5)\hat{r},\,\frac{3(q+1)r_0}{[L(q+2)+2\nu(q+1)](6.2^q+1)}\right\}$$

and

$$(2^qM+1)[L(q+2)+2\nu(q+1)]\Big(\eta(q+1)\delta^q + 4^{1-q}r_{\bar{x}}^q\Big) \le (q+1).$$

Now it is routine to check that conditions (a)-(c) of Theorem 3.3 are satisfied. Thus we can apply Theorem 3.3 to complete the proof. □

4. Conclusion

The semilocal and local convergence results are presented for the extended Newton-type method when $\eta > 1$, $\mathscr{G}_{\bar{x}}^{-1}$ is Lipschitz-like, ∇f satisfies Hölderian condition and g admits first-order divided difference satisfying the Hölder condition defined by (3.5). In particular, we have presented semilocally superlinear convergence analysis for extended Newton-type method in Theorem 3.3 while the locally superlinear convergence analysis for extended Newton-type method is presented in Corollary 3.4. This result extends and improves the corresponding ones [4, 18].

Moreover, according to our main results, we have the following conclusions:

(i) If we set $q = 0$ in Theorem 3.3, it gives the semilocal linear convergence result for the extended Newton-type method and this result coincides with the result presented in [18, Theorem 3.1]. On the other hand, if we put $q = 0$ in Corollary 3.4, this result provides locally linear convergence result which is similar with the result presented in [18, Corollary 3.1].

(ii) If we put $q = 1$ in Theorem 3.3, it yields the semilocal quadratic convergence result for the extened Newton-type method and this result is analogous to the outcome presented in [18, Theorem 3.2]. Furthermore, if we give $q = 1$ in Corollary 3.4, it gives the local quadratic convergence result for this method which is resembling the work presented in [18, Corollary 3.2].

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