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Almost Convergence and 4-Dimensional Binomial Matrix

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Abstract

In the current paper, we deal with to submit the matrix domains of the 4-dimensional binomial matrix on almost convergent and almost null double sequence spaces. Moreover, we examine some properties and tent to compute the $\alpha-$, $\beta(bp)-$ and $\gamma-$duals. Finally, some new matrix classes are characterized and some significant results are given.

Keywords: *Almost convergence, summability theory, double sequence spaces, binomial matrix, matrix domain, α-, $\beta(bp)$- and γ-duals, matrix transformations.*
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1. Introduction

The function F defind by $F:\mathbb{N}\times\mathbb{N}\to\zeta$, $(i,j)\mapsto F(i,j)=u_{ij}$ is called as *double sequence* where ζ denotes any nonempty set and $\mathbb{N}=\{0,1,2,...\}$. Ω represents the vector space of all complex valued double sequences. If Ψ is a vector subspace of Ω, then it is entitled as *double sequence space*. The sets

$$\begin{aligned}
\mathscr{M}_u &= \Big\{u=(u_{ij})\in\Omega : \|u\|_\infty=\sup_{i,j\in\mathbb{N}}|u_{ij}|<\infty\Big\},\\
\mathscr{C}_p &= \Big\{u=(u_{ij})\in\Omega : \exists L\in\mathbb{C},\forall\varepsilon>0,\exists n_\varepsilon\in\mathbb{N},\forall i,j>n_\varepsilon \ni |u_{ij}-L|<\varepsilon\Big\},\\
\mathscr{C}_{bp} &= \mathscr{M}_u\cap\mathscr{C}_p,\\
\mathscr{L}_p &= \Big\{u=(u_{ij}) : \sum_{i,j}|u_{ij}|^p<\infty\Big\}, \qquad (1\le p<\infty)
\end{aligned}$$

are well-known classical double sequence spaces, where $\mathbb{C}$ symbolizes the complex field. If $u=(u_{ij})\in\mathscr{C}_p$, it is said that u is *convergent* in the *Pringsheim's sense* (shortly *p-convergent*). In that case, L is called the *Pringsheim limit* of u and stated by $p-\lim_{i,j\to\infty}u_{ij}=L$. We say that u is *bounded* when $u\in\mathscr{M}_u$. It should be noted that $\mathscr{M}_u$ is a Banach space with $\|u\|_\infty$. Of particular interest is unlike single sequences, p-convergence does not require boundedness in double sequences. $u=(u_{ij})\in\mathscr{C}_p$ is called as *regularly convergent* (such sequences belong to $\mathscr{C}_r$) if the limits $u_i:=\lim_j u_{ij}$, $(i\in\mathbb{N})$ and $u_j:=\lim_i u_{ij}$, $(j\in\mathbb{N})$ exist, and the limits $\lim_i\lim_j u_{ij}$ and $\lim_j\lim_i u_{ij}$ exist and are equivalent to the $p-\lim$ of u. A sequence $u=(u_{ij})$ is called *double null sequence* if it converges to zero. It is known from Móricz [20] that $\mathscr{C}_{bp}$ and $\mathscr{C}_r$ are Banach space with $\|.\|_\infty$. Latterly, the space $\mathscr{L}_p$ which is a Banach space have defined and examined by Başar and Sever [4]. The space $\mathscr{L}_u$ which was defined by Zeltser [37] is the special case of $\mathscr{L}_p$ with $p=1$.
Let us take any $u\in\Omega$ and describe the sequence $S=(s_{kl})$ as $s_{kl}:=\sum_{i=0}^{k}\sum_{j=0}^{l}u_{ij}$, $\quad k,l\in\mathbb{N}$. Thus, the pair $((u_{kl}),(s_{kl}))$ is entitled as *double series*. Let Ψ be a space of double sequences, converging with respect to some linear convergence rule $\vartheta-\lim:\Psi\to\mathbb{C}$. The sum of a double series $\sum_{i,j}u_{ij}$ relating to this rule is defined by $\vartheta-\sum_{i,j}u_{ij}=\vartheta-\lim_{k,l\to\infty}s_{kl}$. Throughout this article, it is used the summation $\sum_{i,j}$ instead of $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}$ and $\vartheta\in\{p,bp,r\}$. With the notation of Zeltser [37], we describe the double sequences $e^{kl}=(e_{ij}^{kl})$ and e by $e_{ij}^{kl}=1$ if $(k,l)=(i,j)$ and $e_{ij}^{kl}=0$ for other cases, and $e=\sum_{k,l}e^{kl}$ for every $i,j,k,l\in\mathbb{N}$. If $d_{klij}=0$ for $i>k$ or $j>l$ or both for every $k,l,i,j\in\mathbb{N}$, it is said that $D=(d_{klij})$ is a *triangular matrix* and also if $d_{klkl}\neq 0$ for every $k,l\in\mathbb{N}$, then the 4-dimensional matrix D is called *triangle*.
Now, we shall deal with the matrix mapping. Let us consider double sequence spaces Ψ and Λ and the 4-dimensional complex infinite matrix $D=(d_{klij})$. If for every $u=(u_{ij})\in\Psi$, $(Du)_{kl}=\vartheta-\sum_{i,j}d_{klij}u_{ij}$ is exists and is in Λ, then it is said that D is a matrix mapping from Ψ into Λ and is written as $D:\Psi\to\Lambda$.
Let $(\Psi,\Lambda)=\{D=(d_{klij})|D:\Psi\to\Lambda\}$. Here, $D\in(\Psi,\Lambda)$ iff $D_{kl}\in\Psi^{\beta(\vartheta)}$, where $D_{kl}=(d_{klij})_{i,j\in\mathbb{N}}$ for every $k,l\in\mathbb{N}$.

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The domain $\Psi_D^{(\vartheta)}$ of D in a double sequence space Ψ consists of whose D-transforms are in Ψ is defined by the following way:

$$\Psi_D^{(\vartheta)} := \left\{ u=(u_{ij})\in\Omega : Du := \left(\vartheta - \sum_{ij} d_{klij}u_{ij}\right)_{k,l\in\mathbb{N}} \text{ exists and is in } \Psi \right\}.$$

In the past, many authors were interested in double sequence spaces. Now, let us give some information about these studies. In her doctoral dissertation, Zeltser [36] has fundamentally examined the topological structure of double sequences. Recently, Altay and Başar [2] have been described the spaces $\mathcal{BS}$, $\mathcal{BS}(t)$, $\mathcal{CS}_p$, $\mathcal{CS}_{bp}$, $\mathcal{CS}_r$ and $\mathcal{BV}$ of double series whose sequences of partial sums are in the spaces $\mathcal{M}_u$, $\mathcal{M}_u(t)$, $\mathcal{C}_p$, $\mathcal{C}_{bp}$, $\mathcal{C}_r$ and $\mathcal{L}_u$, respectively. After that, Talebi [24] defined and examined the space $\mathcal{E}_p^{r,s}$ for $1\leq p<\infty$ and also Yeşilkayagil and Başar [35] examined for $0<p<1$ where $\mathcal{E}_p^{r,s}=\left\{u=(u_{ij}) : E(r,s)u\in\mathcal{L}_p\right\}$. Here, $E(r,s)$ indicates the Euler mean. Tuğ and Başar [25] and Tuğ [26, 27] have defined and examined some domains of the generalized difference matrix $B(r,s,t,u)$.
For further information about double sequences the reader may refer to some of the papers [4, 11, 12, 19, 21, 22, 25, 28, 29, 33, 34, 37, 38] and references therein.
On the other hand, Bişgin [5, 6] have introduced the sequence spaces $b_0^{r,s}, b_c^{r,s}, b_p^{r,s}$ and $b_\infty^{r,s}$ of single sequences whose 2-dimensional binomial matrix $B^{r,s}$-transforms are convergent to zero, convergent, absolutely p-summable and bounded, respectively. After that in [7], Bişgin have been examined the domains of $B^{r,s}$ on f and f_0. Here, f and f_0 symbolize the spaces of every almost convergent and almost null single sequences, respectively.
Our main purpose in this article is to investigate the domains of 4-dimensional binomial matrix on the spaces of almost convergent and almost null double sequences. Let's give a brief summary about almost convergence before moving on to our main results.

2. Almost Convergence for Double Sequences

A generalization for convergence of a single sequence is almost convergence was firstly introduced by Lorentz in 1948 [18] and later, present idea was conveyed and examined for double sequences by Mòricz and Rhoades in [19]. It is said that $u\in\Omega$ is almost convergent if

$$p-\lim_{\rho,\rho'}\sup_{k,l>0}\left|\frac{1}{(\rho+1)(\rho'+1)}\sum_{i=k}^{k+\rho}\sum_{j=l}^{l+\rho'}u_{ij}-L\right|=0$$

and stated by $f_2-\lim u=L$. Every almost convergent $u\in\Omega$ are included by $\mathcal{C}_f$ which is defined by the following way:

$$\mathcal{C}_f = \left\{u=(u_{ij})\in\Omega : \exists L\in\mathbb{C}\ni \quad p-\lim_{\rho,\rho'}\sup_{k,l>0}\left|\frac{1}{(\rho+1)(\rho'+1)}\sum_{i=k}^{k+\rho}\sum_{j=l}^{l+\rho'}u_{ij}-L\right|=0, \textit{ uniformly in } k,l\right\}.$$

Moreover, the space of almost null double sequences $\mathcal{C}_{f_0}$ is obtained from $\mathcal{C}_f$ by taking $L=0$.
It is significant to say that the convergence of a double sequence does not imply its almost convergence. However, the inclusion $\mathcal{C}_{bp}\subset\mathcal{C}_f\subset\mathcal{M}_u$ is valid.

3. Almost Convergent Double Binomial Sequence Spaces

Let r, s and $r+s$ are nonzero real numbers. In this case, the 4-dimensional binomial matrix $B^{(r,s)}=(b_{klij}^{r,s})$ of orders r, s is defined as follows:

$$b_{klij}^{r,s} := \begin{cases} \frac{1}{(r+s)^{k+l}}\binom{k}{i}\binom{l}{j}s^{k+j-i}r^{l+i-j} & , \quad 0\leq i\leq k\,,\ 0\leq j\leq l, \\ 0 & , \quad \text{otherwise}, \end{cases} \tag{3.1}$$

for every $k,l,i,j\in\mathbb{N}$. As can be understood from its definition, $B^{(r,s)}$ is a triangle. In that case, we write the $B^{(r,s)}$-transform of $u\in\Omega$ as

$$v_{kl} := (B^{(r,s)}u)_{kl} = \sum_{i,j}^{k,l}\frac{1}{(r+s)^{k+l}}\binom{k}{i}\binom{l}{j}s^{k+j-i}r^{l+i-j}u_{ij}, \tag{3.2}$$

for every $k,l\in\mathbb{N}$. We will assume unless stated otherwise that the double sequences $u=(u_{ij})$ and $v=(v_{ij})$ are connected with the relation (3.2) and r, s and $r+s$ are nonzero real numbers. We would like touch on a point, when it is choosen $r+s=1$, $B^{(r,s)}$ is reduced to the 4-dimensional Euler matrix $E(r,s)$. So, our matrix $B^{(r,s)}$ generalizes the $E(r,s)$. Consider that the 4-dimensional unit matrix $I=(\delta_{klij})$ defined by

$$\delta_{klij} = \begin{cases} 1 & , \quad (k,l)=(i,j), \\ 0 & , \quad \text{otherwise}. \end{cases}$$

From the equality

$$\delta_{klij} = \sum_{m,n} b_{klmn}^{r,s}c_{mnij}^{r,s},$$

one can see that the inverse $\{B^{(r,s)}\}^{-1}=C^{(r,s)}=(c_{klij}^{r,s})$ as

$$c_{klij}^{r,s} := \begin{cases} (-1)^{k+l-(i+j)}\binom{k}{i}\binom{l}{j}s^{k-l-i}r^{l-k-j}(r+s)^{i+j} & , \quad 0\leq i\leq k, 0\leq j\leq l, \\ 0 & , \quad \text{otherwise}, \end{cases}$$

for every $k,l,i,j\in\mathbb{N}$.

Definition 3.1 (See [17],[23]). *If $Du \in \mathscr{C}_p$ and $bp-\lim u = p-\lim Du$ for every $u \in \mathscr{C}_{bp}$, then D is called as RH-regular.*

We would like to point out that the 4-dimensional binomial matrix described by (3.1) is RH-regular for $r.s>0$. In the rest of the study, it will be assumed that $r.s>0$.
Now, we introduce the sequence spaces $\mathscr{B}_f^{r,s}$ and $\mathscr{B}_{f_0}^{r,s}$ by

$$\begin{aligned}
\mathscr{B}_f^{r,s} &= \left\{u=(u_{ij})\in\Omega : \exists L\in\mathbb{C}\ni p-\lim_{\rho,\rho'}\sup_{k,l>0}\left|\frac{1}{(\rho+1)(\rho'+1)}\sum_{i=k}^{k+\rho}\sum_{j=l}^{l+\rho'}\left(B^{(r,s)}u\right)_{ij}-L\right|=0,\ \textit{uniformly in } k,l\right\},\\
\mathscr{B}_{f_0}^{r,s} &= \left\{u=(u_{ij})\in\Omega : p-\lim_{\rho,\rho'}\sup_{k,l>0}\left|\frac{1}{(\rho+1)(\rho'+1)}\sum_{i=k}^{k+\rho}\sum_{j=l}^{l+\rho'}\left(B^{(r,s)}u\right)_{ij}\right|=0,\ \textit{uniformly in } k,l\right\}.
\end{aligned}$$

Theorem 3.2. *The sets $\mathscr{B}_f^{r,s}$ and $\mathscr{B}_{f_0}^{r,s}$ are linearly norm isomorphic to the spaces $\mathscr{C}_f$ and $\mathscr{C}_{f_0}$, respectively, and are Banach spaces with the norm*

$$\|u\|_{\mathscr{B}_f^{r,s}} = \sup_{\rho,\rho',k,l\in\mathbb{N}}\left|\frac{1}{(\rho+1)(\rho'+1)}\sum_{i=k}^{k+\rho}\sum_{j=l}^{l+\rho'}\left(B^{(r,s)}u\right)_{i,j}\right|. \tag{3.3}$$

Proof. Because it can be similarly shown for $\mathscr{B}_{f_0}^{r,s}$, we give the proof only for $\mathscr{B}_f^{r,s}$. For the first claim of theorem, we must see that there is a linear bijection which preserves the norm from one to the other for the spaces $\mathscr{B}_f^{r,s}$ and $\mathscr{C}_f$.
For this purpose, let us take the map $T:\mathscr{B}_f^{r,s}\to\mathscr{C}_f$, $u\mapsto v=Tu=B^{(r,s)}u$. The linearity of T is clear. Consider the equality $Tu=\theta$ which yields us that $u_{ij}=0$ for every $i,j\in\mathbb{N}$. So, $u=\theta$ and therefore, T is injective. Let us consider $v\in\mathscr{C}_f$. It is clear by defining

$$u_{kl}=\sum_{i,j=0}^{k,l}(-1)^{k+l-(i+j)}\binom{k}{i}\binom{l}{j}s^{k-l-i}r^{l-k-j}(r+s)^{i+j}v_{ij} \tag{3.4}$$

that $Tu=v$ and $u\in\mathscr{B}_f^{r,s}$ for every $k,l\in\mathbb{N}$. So, the map T is surjective. Furthermore, by bearing in mind the following equality

$$\begin{aligned}
\|u\|_{\mathscr{B}_f^{r,s}} &= \sup_{\rho,\rho',k,l\in\mathbb{N}}\left|\frac{1}{(\rho+1)(\rho'+1)}\sum_{i=k}^{k+\rho}\sum_{j=l}^{l+\rho'}\left(B^{(r,s)}u\right)_{ij}\right|\\
&= \sup_{\rho,\rho',k,l\in\mathbb{N}}\left|\frac{1}{(\rho+1)(\rho'+1)}\sum_{i=k}^{k+\rho}\sum_{j=l}^{l+\rho'}v_{ij}\right| = \|v\|_{\mathscr{C}_f}
\end{aligned}$$

that, T preserves the norm. As a result, the initial assertion of the theorem has been proved. From the Corollary 6.3.41 in [8], we reach the proof of the second part. □

Theorem 3.3. *The inclusion $\mathscr{M}_u\subset\mathscr{B}_f^{r,s}$ strictly holds.*

Proof. From the inequality

$$\begin{aligned}
\|u\|_{\mathscr{B}_f^{r,s}} &= \sup_{\rho,\rho',k,l\in\mathbb{N}}\left|\frac{1}{(\rho+1)(\rho'+1)}\sum_{i=k}^{k+\rho}\sum_{j=l}^{l+\rho'}\left(B^{(r,s)}u\right)_{ij}\right|\\
&\le \sup_{\rho,\rho',k,l\in\mathbb{N}}\left|\frac{1}{(\rho+1)(\rho'+1)}\sum_{i=k}^{k+\rho}\sum_{j=l}^{l+\rho'}\sum_{m=0}^{i}\sum_{n=0}^{j}b_{ijmn}^{r,s}\right||u_{mn}|\\
&\le \sup_{m,n\in\mathbb{N}}|u_{mn}|\sup_{\rho,\rho',k,l\in\mathbb{N}}\left|\frac{1}{(\rho+1)(\rho'+1)}\sum_{i=k}^{k+\rho}\sum_{j=l}^{l+\rho'}\sum_{m=0}^{i}\sum_{n=0}^{j}b_{ijmn}^{r,s}\right|\\
&= \|u\|_\infty,
\end{aligned}$$

it is seen that every sequence taken in $\mathscr{M}_u$ is in $\mathscr{B}_f^{r,s}$.
Now, let us select the sequence $u=(u_{kl})=\frac{(-s-r)^{k+l}}{r^ks^l}$ to show the strictness. In that case, we see that $u\notin\mathscr{M}_u$ but its $B^{(r,s)}$-transform $B^{(r,s)}u=\frac{(-1)^{k+l}r^ks^l}{(r+s)^{k+l}}$ in $\mathscr{M}_u\cap\mathscr{C}_p=\mathscr{C}_{bp}\subset\mathscr{C}_f$ which means that $u\in\mathscr{B}_f^{r,s}$. In the light of all this said, it is seen that $u\in\mathscr{B}_f^{r,s}-\mathscr{M}_u$ and the inclusion is strict, as claimed. □

4. Duals of the Space $\mathscr{B}_f^{r,s}$

Current section is dedicated with $\left\{\mathscr{B}_f^{r,s}\right\}^\kappa$, where $\kappa\in\{\alpha,\beta(bp),\gamma\}$. Now, we may present short information about duals at first.

The $\alpha-$, $\beta(bp)-$ and $\gamma-$duals of a $\Psi \subset \Omega$ are described as

$$\begin{aligned}
\Psi^{\alpha} &:= \left\{ t=(t_{ij}) \in \Omega : \sum_{i,j} |t_{ij}u_{ij}| < \infty \quad \text{for all} \quad (u_{ij}) \in \Psi \right\}, \\
\Psi^{\beta(\vartheta)} &:= \left\{ t=(t_{ij}) \in \Omega : \vartheta - \sum_{i,j} t_{ij}u_{ij} \quad \text{exists for all} \quad (u_{ij}) \in \Psi \right\}, \\
\Psi^{\gamma} &:= \left\{ t=(t_{ij}) \in \Omega : \sup_{k,l \in \mathbb{N}} \left| \sum_{i,j=0}^{k,l} t_{ij}u_{ij} \right| < \infty \quad \text{for all} \quad (u_{ij}) \in \Psi \right\},
\end{aligned}$$

respectively. It is well known that $\Psi^{\alpha} \subset \Psi^{\gamma}$ and if $\Psi \subset \Lambda$, then $\Lambda^{\alpha} \subset \Psi^{\alpha}$ for the double sequence spaces Ψ and Λ.

Theorem 4.1. $\left\{ \mathscr{B}_f^{r,s} \right\}^{\alpha} = \mathscr{L}_u$.

Proof. To show the inclusion $\left\{ \mathscr{B}_f^{r,s} \right\}^{\alpha} \subset \mathscr{L}_u$, assume the sequence $t=(t_{kl}) \in \left\{ \mathscr{B}_f^{r,s} \right\}^{\alpha} - \mathscr{L}_u$. So, $\sum_{k,l} |t_{kl}u_{kl}| < \infty$ for all $u=(u_{kl}) \in \mathscr{B}_f^{r,s}$. If we consider $e=\sum_{k,l} e^{kl}$, we see that $e \in \mathscr{B}_f^{r,s}$. Since $te=t \notin \mathscr{L}_u$, we obtain from the equality $\sum_{k,l} |t_{kl}e| = \sum_{k,l} |t_{kl}| = \infty$ that $t \notin \left\{ \mathscr{B}_f^{r,s} \right\}^{\alpha}$ which is a contradiction. Thus, it must be $t \in \mathscr{L}_u$ and the inclusion $\left\{ \mathscr{B}_f^{r,s} \right\}^{\alpha} \subset \mathscr{L}_u$ is valid.

For the sufficiency part, let us take the sequences $t=(t_{kl}) \in \mathscr{L}_u$ and $u=(u_{kl}) \in \mathscr{B}_f^{r,s}$. Then, there exist a double sequence $v=(v_{kl}) \in \mathscr{C}_f$ with the relation $v_{kl}=(B^{(r,s)}u)_{kl}$. Since $\mathscr{C}_f \subset \mathscr{M}_u$, then $\sup_{k,l} |v_{kl}| < M_1$, where $M_1 \in \mathbb{R}^+$. Therefore,

$$\begin{aligned}
\sum_{k,l} |t_{kl}u_{kl}| &= \sum_{k,l} |t_{kl}| \left| \sum_{i,j=0}^{k,l} (-1)^{k+l-(i+j)} \binom{k}{i} \binom{l}{j} s^{k-l-i} r^{l-k-j} (r+s)^{i+j} v_{ij} \right| \\
&\leq \sum_{k,l} |t_{kl}| \left| \frac{1}{r^k s^l} \sum_{i,j=0}^{k,l} \binom{k}{i} \binom{l}{j} (-s)^{k-i} (r+s)^i (-r)^{l-j} (r+s)^j \right| |v_{ij}| \\
&\leq M_1 \sum_{k,l} |t_{kl}| \left| \frac{1}{r^k s^l} \sum_{i=0}^{k} \binom{k}{i} (-s)^{k-i} (r+s)^i \sum_{j=0}^{l} \binom{l}{j} (-r)^{l-j} (r+s)^j \right| \\
&= M_1 \sum_{k,l} |t_{kl}|
\end{aligned}$$

and this says us that $t \in \left\{ \mathscr{B}_f^{r,s} \right\}^{\alpha}$. Thus, it is seen that $\mathscr{L}_u \subset \left\{ \mathscr{B}_f^{r,s} \right\}^{\alpha}$. □

Lemma 4.2. *[19] The following statements are satisfied:*

(a) *$D=(d_{klij}) \in (\mathscr{C}_f, \mathscr{C}_{bp})$ iff the following conditions are satisfied:*

$$\sup_{k,l \in \mathbb{N}} \sum_{i,j} |d_{klij}| < \infty, \tag{4.1}$$

$$\exists d_{ij} \in \mathbb{C} \ni \quad bp-\lim_{k,l \to \infty} d_{klij} = d_{ij} \quad \textit{for every} \quad i,j \in \mathbb{N}, \tag{4.2}$$

$$\exists L \in \mathbb{C} \ni \quad bp-\lim_{k,l \to \infty} \sum_{i,j} d_{klij} = L, \tag{4.3}$$

$$\exists i_0 \in \mathbb{N} \ni \quad bp-\lim_{k,l \to \infty} \sum_{j} |d_{kl,i_0,j} - d_{i_0 j}| = 0, \quad \forall j \in \mathbb{N}, \tag{4.4}$$

$$\exists j_0 \in \mathbb{N} \ni \quad bp-\lim_{k,l \to \infty} \sum_{i} |d_{kli,j_0} - d_{i,j_0}| = 0, \quad \forall i \in \mathbb{N}, \tag{4.5}$$

$$bp-\lim_{k,l \to \infty} \sum_{i} \sum_{j} |\triangle_{01} d_{klij}| = 0, \tag{4.6}$$

$$bp-\lim_{k,l \to \infty} \sum_{i} \sum_{j} |\triangle_{10} d_{klij}| = 0, \tag{4.7}$$

where $\triangle_{10} d_{klij} = d_{klij} - d_{kl,i+1,j}$ and $\triangle_{01} d_{klij} = d_{klij} - d_{kli,j+1}$, $k,l,i,j \in \mathbb{N}$.

(b) *$D=(d_{klij})$ is strongly regular, that is, $D \in (\mathscr{C}_f, \mathscr{C}_{bp})_{reg}$ iff the conditions (4.1)-(4.7) are satisfied whenever $d_{ij}=0$, $\forall i,j=0,1,\ldots$ and $L=1$.*

Lemma 4.3. *[27] $D=(d_{klij}) \in (\mathscr{C}_f, \mathscr{M}_u)$ iff $D_{kl} \in \{\mathscr{C}_f\}^{\beta(\vartheta)}$ and the condition (4.1) is satisfied.*

Consider the sets $w_1 - w_7$ which are defined by the following way:

$$\begin{aligned}
w_1 &= \left\{ t=(t_{ij})\in\Omega : \sup_{k,l\in\mathbb{N}} \sum_{i,j} |\chi(k,l,i,j,m,n)| < \infty \right\},\\
w_2 &= \left\{ t=(t_{ij})\in\Omega : \exists d_{ij}\in\mathbb{C} \ni \vartheta-\lim_{k,l\to\infty} \chi(k,l,i,j,m,n) = d_{ij} \right\},\\
w_3 &= \left\{ t=(t_{ij})\in\Omega : \exists L\in\mathbb{C} \ni \vartheta-\lim_{k,l\to\infty} \sum_{i,j} \chi(k,l,i,j,m,n) = L \right\},\\
w_4 &= \left\{ t=(t_{ij})\in\Omega : \exists j_0\in\mathbb{N} \ni \vartheta-\lim_{k,l\to\infty} \sum_{i} \left|\chi(k,l,i,j_0,m,n) - d_{ij_0}\right| = 0, \quad \forall i\in\mathbb{N} \right\},\\
w_5 &= \left\{ t=(t_{ij})\in\Omega : \exists i_0\in\mathbb{N} \ni \vartheta-\lim_{k,l\to\infty} \sum_{j} \left|\chi(k,l,i_0,j,m,n) - d_{i_0j}\right| = 0, \quad \forall j\in\mathbb{N} \right\},\\
w_6 &= \left\{ t=(t_{ij})\in\Omega : \vartheta-\lim_{k,l\to\infty} \sum_{i}\sum_{j} |\triangle_{01}\chi(k,l,i,j,m,n)| = 0 \right\},\\
w_7 &= \left\{ t=(t_{ij})\in\Omega : \vartheta-\lim_{k,l\to\infty} \sum_{i}\sum_{j} |\triangle_{10}\chi(k,l,i,j,m,n)| = 0 \right\},
\end{aligned}$$

where

$$\chi(k,l,i,j,m,n) = \sum_{m=i}^{k}\sum_{n=j}^{l} (-1)^{m+n-(i+j)} \binom{m}{i}\binom{n}{j} s^{m-n-i} r^{n-m-j} (r+s)^{i+j} t_{mn}.$$

Theorem 4.4. *The following statements are satisfied:*

(i) $\left\{\mathscr{B}_f^{r,s}\right\}^{\beta(bp)} = \bigcap_{k=1}^{7} w_k$

(ii) $\left\{\mathscr{B}_f^{r,s}\right\}^{\gamma} = w_1 \cap \mathscr{CS}_\vartheta$.

Proof. **(i)** Suppose that $t=(t_{kl})\in\Omega$ and $u=(u_{kl})\in\mathscr{B}_f^{r,s}$. Thus, $v=(v_{kl})\in\mathscr{C}_f$ with $B^{(r,s)}u=v$. We obtain by the relation (3.4) that

$$\begin{aligned}
z_{kl} &= \sum_{i,j=0}^{k,l} t_{ij}u_{ij}\\
&= \sum_{i,j=0}^{k,l} t_{ij} \left\{ \sum_{m,n=0}^{i,j} (-1)^{i+j-(m+n)} \binom{i}{m}\binom{j}{n} s^{i-j-m} r^{j-i-n} (r+s)^{m+n} v_{mn} \right\}\\
&= \sum_{i,j=0}^{k,l} \left\{ \sum_{m=i}^{k}\sum_{n=j}^{l} (-1)^{m+n-(i+j)} \binom{m}{i}\binom{n}{j} s^{m-n-i} r^{n-m-j} (r+s)^{i+j} t_{mn} \right\} v_{ij}\\
&= (O^{r,s}v)_{kl}
\end{aligned} \tag{4.8}$$

for every $k,l\in\mathbb{N}$, where $O^{r,s}=\left(o_{klij}^{r,s}\right)$ defined by

$$o_{klij}^{r,s} = \begin{cases} \chi(k,l,i,j,m,n) &, \quad 0\le i\le k,\ 0\le j\le l,\\ 0 &, \quad \text{otherwise}, \end{cases}$$

for every $k,l,i,j\in\mathbb{N}$. In that case, by bearing in mind (4.8), it is infered that $tu=(t_{kl}u_{kl})\in\mathscr{CS}_{bp}$ whenever $u=(u_{kl})\in\mathscr{B}_f^{r,s}$ iff $z=(z_{kl})\in\mathscr{C}_{bp}$ whenever $v=(v_{kl})\in\mathscr{C}_f$. This implies that $t=(t_{kl})\in\left\{\mathscr{B}_f^{r,s}\right\}^{\beta(bp)}$ iff $O^{r,s}\in(\mathscr{C}_f,\mathscr{C}_{bp})$ and the proof of the first part is completed in view of Lemma 4.2.

(ii) Let us select the sequences $t=(t_{kl})\in\Omega$ and $u=(u_{kl})\in\mathscr{B}_f^{r,s}$. Then, $v=(v_{kl})=B^{(r,s)}u\in\mathscr{C}_f$. Thus, it can be said that $tu\in\mathscr{BS}$ whenever $u=(u_{kl})\in\mathscr{B}_f^{r,s}$ iff $z\in\mathscr{M}_u$ whenever $v\in\mathscr{C}_f$. This means that $t\in\left\{\mathscr{B}_f^{r,s}\right\}^{\gamma}$ iff $O^{r,s}\in(\mathscr{C}_f,\mathscr{M}_u)$. In that case, it is achieved from the conditions of the Lemma 4.3 that $O_{kl}^{r,s}\in\{\mathscr{C}_f\}^{\beta(\vartheta)}$ for each fixed $k,l\in\mathbb{N}$ and

$$\sup_{k,l\in\mathbb{N}} \sum_{i,j} |\chi(k,l,i,j,m,n)| < \infty.$$

Therefore, it is obvious that $\left\{\mathscr{B}_f^{r,s}\right\}^{\gamma} = w_1\cap\mathscr{CS}_\vartheta$, as claimed. □

5. Some Matrix Classes

In the present chapter, we deal with to characterize some matrix mapping classes. Before these, it is needed to give the following lemmas which will be used in the Theorem 5.7, Corollary 5.8 and Corollary 5.10.

Lemma 5.1. *[38] The following statements are satisfied:*

(a) *$D=(d_{klij})$ is almost $\mathscr{C}_{bp}$-conservative, that is, $D\in(\mathscr{C}_{bp},\mathscr{C}_f)$ iff the condition (4.1) is satisfied and the following conditions are satisfied, too:*

$$\exists d_{ij}\in\mathbb{C}\ni bp-\lim_{\rho,\rho'\to\infty}\sigma(i,j,\rho,\rho',k,l)=d_{ij} \text{ uniformly in } k,l\in\mathbb{N} \text{ for each } i,j\in\mathbb{N}, \tag{5.1}$$

$$\exists L\in\mathbb{C}\ni bp-\lim_{\rho,\rho'\to\infty}\sum_{i,j}\sigma(i,j,\rho,\rho',k,l)=L \text{ uniformly in } k,l\in\mathbb{N}, \tag{5.2}$$

$$\exists d_{ij}\in\mathbb{C}\ni bp-\lim_{\rho,\rho'\to\infty}\sum_{i}\left|\sigma(i,j,\rho,\rho',k,l)-d_{ij}\right|=0 \text{ uniformly in } k,l\in\mathbb{N} \text{ for each } j\in\mathbb{N}, \tag{5.3}$$

$$\exists d_{ij}\in\mathbb{C}\ni bp-\lim_{\rho,\rho'\to\infty}\sum_{j}\left|\sigma(i,j,\rho,\rho',k,l)-d_{ij}\right|=0 \text{ uniformly in } k,l\in\mathbb{N} \text{ for each } i\in\mathbb{N}, \tag{5.4}$$

where $\sigma(i,j,\rho,\rho',k,l)=\sum_{m=k}^{k+\rho}\sum_{n=l}^{l+\rho'}\frac{d_{mnij}}{(\rho+1)(\rho'+1)}$.

(b) *$D=(d_{klij})$ is almost ($\mathscr{C}_{bp}$-)regular, that is, $D\in(\mathscr{C}_{bp},\mathscr{C}_f)_{reg}$ iff the conditions (4.1) and (5.1)-(5.4) are satisfied whenever $d_{ij}=0$, $\forall i,j=0,1,\ldots$ and $L=1$.*

Lemma 5.2. *[21] $D=(d_{klij})$ is almost strongly regular, that is $D\in(\mathscr{C}_f,\mathscr{C}_f)_{reg}$ iff D is almost ($\mathscr{C}_{bp}$-)regular and the following conditions are satisfied:*

$$\lim_{\rho,\rho'\to\infty}\sum_i\sum_j\left|\triangle_{10}\sigma(i,j,\rho,\rho',k,l)\right|=0 \text{ uniformly in } k,l\in\mathbb{N}, \tag{5.5}$$

$$\lim_{\rho,\rho'\to\infty}\sum_j\sum_i\left|\triangle_{01}\sigma(i,j,\rho,\rho',k,l)\right|=0 \text{ uniformly in } k,l\in\mathbb{N}, \tag{5.6}$$

where

$$\triangle_{10}\sigma(i,j,\rho,\rho',k,l)=\sigma(i,j,\rho,\rho',k,l)-\sigma(i+1,j,\rho,\rho',k,l)$$

$$\triangle_{01}\sigma(i,j,\rho,\rho',k,l)=\sigma(i,j,\rho,\rho',k,l)-\sigma(i,j+1,\rho,\rho',k,l).$$

Lemma 5.3. *[38] The following statements are satisfied:*

(a) *$D=(d_{klij})$ is almost $\mathscr{C}_r$-conservative, that is, $D\in(\mathscr{C}_r,\mathscr{C}_f)$ iff the conditions (4.1), (5.1), (5.2) are satisfied and the following conditions are satisfied, too:*

$$\exists j_0\in\mathbb{N}\ni bp-\lim_{\rho,\rho'\to\infty}\sum_i\sigma(i,j_0,\rho,\rho',k,l)=x_{j_0} \text{ uniformly in } k,l\in\mathbb{N}, \tag{5.7}$$

$$\exists i_0\in\mathbb{N}\ni bp-\lim_{\rho,\rho'\to\infty}\sum_j\sigma(i_0,j,\rho,\rho',k,l)=y_{i_0} \text{ uniformly in } k,l\in\mathbb{N}. \tag{5.8}$$

(b) *$D=(d_{klij})$ is almost $\mathscr{C}_r$-regular, that is, $D\in(\mathscr{C}_r,\mathscr{C}_f)_{reg}$ iff the conditions (4.1), (5.1), (5.2), (5.7) and (5.8) are satisfied whenever $d_{ij}=x_{j_0}=y_{i_0}=0$, $\forall i,j=0,1,\ldots$ and $L=1$.*

Lemma 5.4. *[38] The following statements are satisfied:*

(a) *$D=(d_{klij})$ is almost $\mathscr{C}_p$-conservative, that is, $D\in(\mathscr{C}_p,\mathscr{C}_f)$ iff the conditions (4.1), (5.1), (5.2) are satisfied and the following conditions are satisfied, too:*

$$\forall i\in\mathbb{N},\exists j_0\in\mathbb{N}\ni \quad d_{klij}=0,\quad \forall j>j_0 \text{ and } k,l\in\mathbb{N}, \tag{5.9}$$

$$\forall j\in\mathbb{N},\exists i_0\in\mathbb{N}\ni \quad d_{klij}=0,\quad \forall i>i_0 \text{ and } k,l\in\mathbb{N}. \tag{5.10}$$

(b) *$D=(d_{klij})$ is almost $\mathscr{C}_p$-regular, that is, $D\in(\mathscr{C}_p,\mathscr{C}_f)_{reg}$ iff the conditions (4.1), (5.1), (5.2), (5.9) and (5.10) are satisfied whenever $d_{ij}=0$, $\forall i,j=0,1,\ldots$ and $L=1$.*

Lemma 5.5. *[33] $D=(d_{klij})\in(\mathscr{M}_u,\mathscr{C}_f)$ iff the condition (4.1) is satisfied and the following conditions are satisfied, too:*

$$\exists d_{ij}\in\mathbb{C}\ni \quad f_2-\lim_{k,l\to\infty} d_{klij}=d_{ij} \quad \text{for every} \quad i,j\in\mathbb{N}, \tag{5.11}$$

$$\forall k,l,j\in\mathbb{N},\exists M_2\in\mathbb{N}\ni \quad \sigma(i,j,\rho,\rho',k,l)=0, \quad \forall\rho,\rho',i>M_2, \tag{5.12}$$

$$\forall k,l,i\in\mathbb{N},\exists M_3\in\mathbb{N}\ni \quad \sigma(i,j,\rho,\rho',k,l)=0, \quad \forall\rho,\rho',j>M_3. \tag{5.13}$$

Lemma 5.6. *[28] The following statements are satisfied:*

(a) *Let $0<q\leq 1$. Then, $D=(d_{klij})\in(\mathscr{L}_q,\mathscr{C}_f)$ iff the condition (5.11) is satisfied and the following condition is satisfied, too:*

$$\sup_{k,l,i,j\in\mathbb{N}}\left|d_{klij}\right|<\infty. \tag{5.14}$$

(b) *Let $1<q<\infty$. Then, $D=(d_{klij})\in(\mathscr{L}_q,\mathscr{C}_f)$ iff the condition (5.11) is satisfied and the following condition is satisfied, too:*

$$\sup_{k,l\in\mathbb{N}}\sum_{i,j}\left|d_{klij}\right|^q<\infty. \tag{5.15}$$

Theorem 5.7. *Suppose that $D=(d_{klij})$ be a 4-dimensional matrix. In that case, $D\in(\mathscr{B}_f^{r,s},\mathscr{M}_u)$ iff*

$$D_{kl}\in\{\mathscr{B}_f^{r,s}\}^{\beta(\vartheta)} \tag{5.16}$$

and

$$\sup_{k,l\in\mathbb{N}}\sum_{i,j}\left|\sum_{a=i}^{\infty}\sum_{b=j}^{\infty}\binom{a}{i}\binom{b}{j}s^{a-b-i}r^{b-a-j}(r+s)^{i+j}d_{klab}\right|<\infty. \tag{5.17}$$

Proof. Let $D\in(\mathscr{B}_f^{r,s},\mathscr{M}_u)$. Then, Du exists and in $\mathscr{M}_u$ for every $u\in\mathscr{B}_f^{r,s}$. In that case, we understand that $D_{kl}\in\{\mathscr{B}_f^{r,s}\}^{\beta(\vartheta)}$. By taking into account the equality (3.4), the (m,n)th rectangular partial sum of the series $\sum_{i,j}d_{klij}u_{ij}$ obtained as

$$\begin{aligned}(Du)_{kl}^{[m,n]} &= \sum_{i,j=0}^{m,n} d_{klij}u_{ij}\\ &= \sum_{i,j=0}^{m,n}\left[\sum_{a=i}^{m}\sum_{b=j}^{n}(-1)^{a+b-(i+j)}\binom{a}{i}\binom{b}{j}s^{a-b-i}r^{b-a-j}(r+s)^{i+j}d_{klab}\right]v_{ij}\end{aligned} \tag{5.18}$$

for every $k,l,m,n\in\mathbb{N}$. Let us define the 4-dimensional matrix $H=\left(h_{klij}\right)$ as $h_{klij}:=$

$$\begin{cases}\displaystyle\sum_{a=i}^{\infty}\sum_{b=j}^{\infty}(-1)^{a+b-(i+j)}\binom{a}{i}\binom{b}{j}s^{a-b-i}r^{b-a-j}(r+s)^{i+j}d_{klab} & , \quad 0\leq k\leq i, 0\leq l\leq j,\\ 0 & , \quad \text{otherwise}\end{cases}$$

for every $k,l,i,j\in\mathbb{N}$. In that case, by taking ϑ-limit on (5.18) as $m,n\to\infty$, it is seen that $Du=Hv$. Therefore, if we take into account the fact that $D=(d_{klij})\in(\mathscr{B}_f^{r,s},\mathscr{M}_u)$ if and only if $H\in(\mathscr{C}_f,\mathscr{M}_u)$ with the Lemma 4.3, then it is obvious that the condition (5.17) holds.
Conversely, suppose that the conditions (5.16) and (5.17) hold. Let us choose the sequence $u\in\mathscr{B}_f^{r,s}$ with $v\in\mathscr{C}_f$ with the relation (3.2). Since the condition (5.16) holds, Du exists. By using the relation (3.4), one can derive from the (m,n)th partial sum of the series $\sum_{i,j}d_{klij}u_{ij}$ for every $k,l,i,j\in\mathbb{N}$ that

$$\sum_{i,j=0}^{m,n}d_{klij}u_{ij}=\sum_{i,j=0}^{m,n}\left[\sum_{a=i}^{m}\sum_{b=j}^{n}(-1)^{a+b-(i+j)}\binom{a}{i}\binom{b}{j}s^{a-b-i}r^{b-a-j}(r+s)^{i+j}d_{klab}\right]v_{ij}.$$

By taking ϑ-limit in the equality above as $m,n\to\infty$, it can be easily obtain from the following equality for every $k,l\in\mathbb{N}$

$$\sum_{i,j}d_{klij}u_{ij}=\sum_{i,j}h_{klij}v_{ij}$$

that $Du=Hv$. From the condition (5.17), it is known that $H\in(\mathscr{C}_f,\mathscr{M}_u)$ and thus we get $D\in(\mathscr{B}_f^{r,s},\mathscr{M}_u)$ which completes the proof. □

Corollary 5.8. *Suppose that $D=(d_{klij})$ be a 4-dimensional matrix. In that case the following statements are satisfied:*

(i) $D\in\left(\mathscr{B}_f^{r,s},\mathscr{C}_{bp}\right)$ *iff the conditions (4.1)-(4.7) are satisfied with h_{klij} in place of d_{klij},*

(ii) $D\in\left(\mathscr{B}_f^{r,s},\mathscr{C}_f\right)_{reg}$ *iff the conditions (4.1), (5.1)-(5.6) are satisfied with h_{klij} in place of d_{klij}.*

Lemma 5.9. *[34] Let $B=(b_{klij})$ be a triangle matrix. In that case, $D=(d_{klij})\in(\Psi,\Lambda_B)$ iff $BD\in(\Psi,\Lambda)$.*

Now, let us define the 4-dimensional matrix $G=(g_{klij})$ by

$$g_{klij}=\sum_{m,n=0}^{k,l} b_{klmn}^{r,s} d_{mnij}$$

for every $k,l,i,j\in\mathbb{N}$ and give the following corollary.

Corollary 5.10. *Suppose that $D=(d_{klij})$ be a 4-dimensional matrix. In that case the following statements are satisfied:*

(i) $D\in\left(\mathscr{C}_{bp},\mathscr{B}_f^{r,s}\right)$ *iff the conditions (4.1), (5.1)-(5.4) are satisfied with g_{klij} in place of d_{klij},*

(ii) $D\in\left(\mathscr{C}_{r},\mathscr{B}_f^{r,s}\right)$ *iff the conditions (4.1), (5.1), (5.2), (5.7) and (5.8) are satisfied with g_{klij} in place of d_{klij},*

(iii) $D\in\left(\mathscr{C}_{p},\mathscr{B}_f^{r,s}\right)$ *iff the conditions (4.1), (5.1), (5.2), (5.9) and (5.10) are satisfied with g_{klij} in place of d_{klij},*

(iv) $D\in\left(\mathscr{M}_{u},\mathscr{B}_f^{r,s}\right)$ *iff the condition (4.1) and (5.11)-(5.13) are satisfied with g_{klij} in place of d_{klij},*

(v) $D\in\left(\mathscr{L}_{q},\mathscr{B}_f^{r,s}\right)$ *iff the conditions (5.11) and (5.14) are satisfied for $0<q\leq 1$ with g_{klij} in place of d_{klij},*

(vi) $D\in\left(\mathscr{L}_{q},\mathscr{B}_f^{r,s}\right)$ *iff the conditions (5.11) and (5.15) are satisfied for $1<p<\infty$ with g_{klij} in place of d_{klij},*

(vii) $D\in\left(\mathscr{C}_{f},\mathscr{B}_f^{r,s}\right)_{reg}$ *iff the conditions (4.1) and (5.1)-(5.6) are satisfied with g_{klij} in place of d_{klij}.*

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