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Projective Synchronization of The Modified Fractional-Order Hyperchaotic Rössler System and Its Application in Secure Communication

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Abstract

In this paper, we propose a new approach to investigate the chaos projective synchronization of the modified fractional-order hyperchaotic Rössler system and its application in secure communication. The proposed communication system consists of four main elements including: modulation, master system, slave system and demodulation. The main idea of this approach is to inject the bounded or unbounded message into one of the parameters of the proposed system using the exponential function. However, the way of injecting the message in the modulation parameter must not remove the hyperchaotic character of the signal sent to the slave system. The slave system adaptively synchronizes with the master system, and the information signal can be recovered. Based on the Lyapunov stability theory, an adaptation laws and adaptive control are designed to achieve projection synchronization of the modified system. Numerical simulations are performed to show the feasibility of the proposed secure communication scheme.

1. Introduction

The concept of using chaos theory for communication systems was essentially inspired by the work of Pecora and Carroll in 1990 [1]. They discovered that two identical chaotic systems with different initial conditions can synchronize if they are properly coupled.
The chaotic transmission is a mode of secure communication that arises from the inclusion of chaos in transmission systems. The main idea of the chaotic transmission is to inject the message into a chaotic signal to hide this information and send it to the receiver system through a public channel. Thus, after the synchronization of the two chaotic systems (transmitter and receiver), the encrypted information is thus recovered at the receiver system. On the other hand, in literature, one often finds the name of the fractional derivation to the generalization of the derivation to an arbitrary order. The concepts of derivation and fractional integration are often associated with the names of Riemann-Liouville, whereas the question about the generalization of these notions is older.
With particular attention from physicists as well as engineers, a remarkable research activity has been devoted to fractional computing. Indeed, it has been found that many real physical systems are better characterized by dynamic models of fractional orders, such as diffusion systems [2], chemical systems [3], electrochemical systems [4], biological systems [5] and viscoelastic systems [6], etc. The use of classical models based on a classical derivation is therefore not appropriate. Chaos synchronization phenomena have been of particular interest in the study of chaotic and hyperchaotic dynamical systems, since they can be applied to large areas of engineering and information science, particularly in secure communication [7], control processing [8] and cryptology [9].
The basic configuration of a synchronization system consists of two chaotic or hyperchaotic systems: a transmitter system and a receiver system. Note that the two previous systems can be identical (with different initial conditions) or completely different. The transmitter system synchronizes the receiver system via one or several signals. In the literature, divers control methods have been applied to achieve synchronization, such as approximated auxiliary system [10], active control [11], adaptative control [12] and fuzzy adaptive control [13]. Using these methods, several concepts of chaotic and hyperchaotic synchronization have also been extended, such as complete synchronization [14], anti-synchronization [15], generalized synchronization [16], projective synchronization [17] and modified projective synchronization [18].

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A great deal of work has been done in recent years, exploiting chaotic and hyperchaotic signals in the context of secure communications. Indeed, their characteristics, sensitivities to the initial conditions, deterministic dynamics, ergodicity and structure complexity, are well adapted to secure transmissions [19–21].
In most of the secure communication systems proposed above, the size of the message must be small enough, otherwise an hyperchaotic system may be asymptotically stable, which may render the retrieval of the transmitted signal unsuccessful. However, in some real applications, various messages to be transmitted can be unbounded.
In [22], X Wu et al. have proposed a new secure communication scheme based on the projective generalized synchronization of a hyperchaotic system, where the signal of the message is bounded or unbounded. However, it should be mentioned that the fundamental results of the previous work apply only to integer-order hyperchaotic systems to the design of the secure communication system. So, it is very interesting to extend them to the general case of fractional order systems and the work in this area is still considered a stimulating research topic.
Motivated by the above considerations, in this paper, we propose a new simple approach to solve both the problem of projective synchronization in the modified fractional-order hyperchaotic Rössler system and that of the transmission security, where the signal of the message is bounded or unbounded.
The current manuscript is organized as follows: In Section 2, we present the system description and some preliminaries. The main result of this paper concerning a new secure communication scheme based on fractional order hyperchaotic system is mainly presented in Section 3. Therefore, in order to achieve this purpose, a modified adaptative control and a parameter update rule are designed. Numerical simulations are presented to show the viability and efficiency of the proposed method in Section 4. Finally, we conclude our paper with a short summary in Section 5.

2. System description and preliminaries

Consider the new hyperchaotic system [23] written by the dynamic equations:

$$\begin{cases} \dot{x}_1 = -x_2 - x_3 + x_4, \\ \dot{x}_2 = x_1 + a_1 x_2, \\ \dot{x}_3 = x_1 x_3 - a_3 x_3 + a_2, \\ \dot{x}_4 = a_4 x_1. \end{cases} \tag{2.1}$$

For the parameter values $a_2 = 0.01$, $a_3 = 5$, $a_4 = 0.1$ and $0.16 \leq a_1 \leq 0.19$, the system has large hyperchaotic region. The variation of the three largest Lyapunov exponents for different values of a_1 is given in Figure 2.1.
From the Figure 2.1, one can say that there are two positive lyapunov's exponents, when $0.16 \leq a_1 \leq 0.19$, wich means that the system is

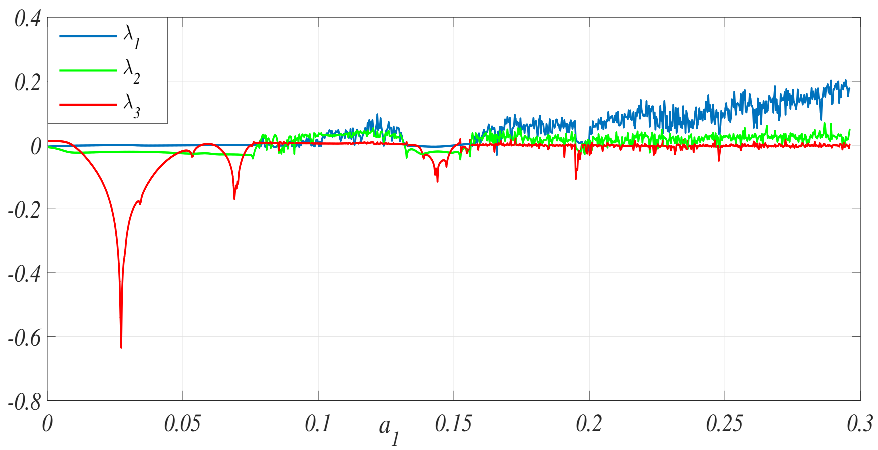

Figure 2.1: The three largest Lyapunov's exponents of system (2.1)

hyperchaotic.
The fractional version of the system (2.1) is governed by:

$$\begin{cases} D^{\alpha_1} x_1 = -x_2 - x_3 + x_4, \\ D^{\alpha_2} x_2 = x_1 + a_1 x_2, \\ D^{\alpha_3} x_3 = x_1 x_3 - a_3 x_3 + a_2, \\ D^{\alpha_4} x_4 = a_4 x_1. \end{cases} \tag{2.2}$$

where $\alpha_i \in]0,1[$, $i = 1,2,3,4$ are fractional-orders, and D^{α} is the Caputo derivative, which is defined as:

$$D^{\alpha} x(t) = J^{n-\alpha} x^{(n)}(t), \quad \alpha \in (0,1), \tag{2.3}$$

were $n = \lceil \alpha \rceil$, i.e., n is the first integer which is not less than α; $x^{(n)}$ is the general n-order derivative and J^{γ} is the γ-order Riemann–Liouville integral operator expressed as follows:

$$J^{\gamma} y = \frac{1}{\Gamma(\gamma)} \int_0^t (t-\tau)^{\gamma-1} y(\tau) d\tau, \tag{2.4}$$

where $\Gamma(.)$ is the gamma function.

Remark 2.1. *The major advantage of the Caputo definition is that the initial conditions for fractional-order differential equations take a similar form as for integer-order differential equations.*

Remark 2.2. *In system (2.1), the fractional-order system is called a commensurate fractional-order system if* $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4$, *otherwise the system is called an incommensurate fractional-order system.*

3. Main results

The Main results of this part is mainly devoted to a new secure communication scheme. This method is based on the projective synchronization (PS) of the modified fractional Rösler system, using the parametric modulation technique. Figure 3.1 describes the proposed hyperchaotic communication scheme based on parametric modulation. The signal of the message to be sent can be bounded or unbounded. The proposed communication system consists of four main elements including: modulation (using exponontial function), master system, slave system and demodulation. Finally, the original message signal transmitted can be successfully recovered by the estimated parameter and the proposed invertible function.

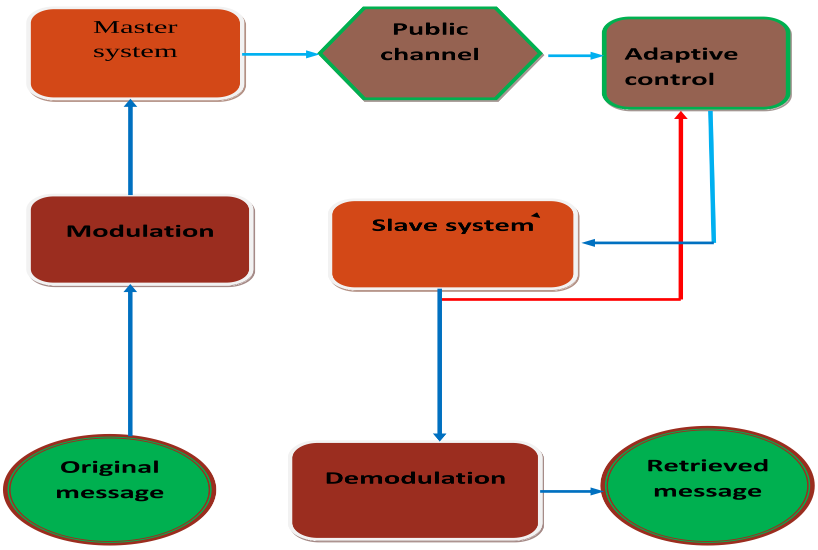

Figure 3.1: Principal diagram of the prposed secure communication.

In the proposed communication system, we plan to modulate in the unknown parameter a_1 of the system(2.2).
Let $m(t)$ be the signal of the message. Now let's define a new unknown parameter $A = A_1(t)$. In order to preserve the hyperchaotic behavior of the transmitter system studied, we propose the following parametric modulation technique:

$$A_1(t) = 0.03e^{-m(t)} + 0.16, \qquad m(t) \geq 0, \tag{3.1}$$

where $e^{(\cdot)}$ is the exponontial function.
Now, we replace the parameter a_1 of the system (2.2) by A_1, we have:

$$\begin{cases} D^{\alpha_1}x_1 = -x_2 - x_3 + x_4, \\ D^{\alpha_2}x_2 = x_1 + A_1x_2, \\ D^{\alpha_3}x_3 = x_1x_3 - a_3x_3 + a_2, \\ D^{\alpha_4}x_4 = a_4x_1, \end{cases} \tag{3.2}$$

where x_1, x_2, x_3, x_4 are chaotic signals that must be transmitted to the receiver system via a public channel. Since $A_1(t) \in [0.16, 0.19]$, the resulting system (3.2) is still hyperchaotic. Then we can take the system (3.2) as the master system.
Consider also the hyperchaotic slave system, which is supposed to be written by:

$$\begin{cases} D^{\alpha_1}y_1 = -y_2 - y_3 + y_4 + u_1, \\ D^{\alpha_2}y_2 = y_1 + \hat{A}_1y_2 + u_2, \\ D^{\alpha_3}y_3 = y_1y_3 - a_3y_3 + a_2 + u_3, \\ D^{\alpha_4}y_4 = a_4y_1 + u_4, \end{cases} \tag{3.3}$$

where $\hat{A}_1$ is the estimated parameter of A_1 and $u_i, i = 1,2,3,4$ are the controls to be determined.
Our main objective is to design a modified adaptive control u_i (for all $i = 1,2,3,4$) and a parameter $\hat{A}_1$ realizing a practical PS between the

master system (3.2) and the slave system (3.3) and finally $\hat{A}_1$ converges towards the value A_1.
To quantify this goal, the synchronization error is defined as:

$$e_i = y_i - \theta x_i, \quad i = 1,2,3,4, \tag{3.4}$$

where θ is a scaling factor defining a proportional relationship between the two synchronized systems.
Therefore, the complete synchronization and anti-synchronization are the special cases of a PS, when θ takes the values $+1$ and -1, respectively.
Let us also define the estimation error as:

$$e_{A_1} = A_1 - \hat{A}_1. \tag{3.5}$$

The error dynamics is easily obtained in the form:

$$D^{\alpha_i} e_i = D^{\alpha_i} y_i - \theta D^{\alpha_i} x_i, \quad i = 1,2,3,4. \tag{3.6}$$

Inserting(3.2) and (3.3) in (3.6) yields the following:

$$\begin{cases} D^{\alpha_1} e_1 = -e_2 - e_3 + e_4 + u_1, \\ D^{\alpha_2} e_2 = e_1 + \hat{A}_1 e_2 - \theta e_{A_1} x_2 + u_2, \\ D^{\alpha_3} e_3 = -a_3 e_3 + y_1 y_3 - \theta x_1 x_3 + a_2(1-\theta) + u_3, \\ D^{\alpha_4} e_4 = a_4 e_1 + u_4. \end{cases} \tag{3.7}$$

Differentiating (3.5) from t, we have:

$$\dot{e}_{A_1} = -0.03\dot{m}e^{-m} - \dot{\hat{A}}_1 \tag{3.8}$$

On the basis of the previous discussions, we shall state and prove the following result:

Theorem 3.1. *(Main results) If the adaptive control parameter coordinates are selected as:*

$$\begin{cases} u_1 = e_2 + e_3 - e_4 - k_1 D^{\alpha_1 - 1} e_1, \\ u_2 = -e_1 - \hat{A}_1 e_2 + \theta e_{A_1} x_2 - D^{\alpha_2 - 1}(\theta e_{A_1} x_2 + k_2 e_2), \\ u_3 = a_3 e_3 - y_1 y_3 + \theta x_1 x_3 - a_2(1-\theta) - k_3 D^{\alpha_3 - 1} e_3, \\ u_4 = -a_4 e_1 - k_4 D^{\alpha_4 - 1} e_4, \end{cases} \tag{3.9}$$

where k_i, $i = 1,2,3,4$ are positive control gains,
and the update law for the parameter estimate is taken as:

$$\dot{\hat{A}}_1 = -\theta e_2 x_2 - 0.03\dot{m}e^{-m}, \tag{3.10}$$

then the PS between the two identical systems (3.2) and (3.3) is achieved.

Proof. Inserting (3.9) into (3.7), we get the error dynamic system as follows:

$$\begin{cases} D^{\alpha_1} e_1 = -k_1 D^{\alpha_1 - 1} e_1, \\ D^{\alpha_2} e_2 = -D^{\alpha_2 - 1}(\theta e_{A_1} x_2 + k_2 e_2), \\ D^{\alpha_3} e_3 = -k_3 D^{\alpha_3 - 1} e_3, \\ D^{\alpha 4} e_4 = -k_4 D^{\alpha_4 - 1} e_4. \end{cases} \tag{3.11}$$

Consider the Lyapunov function candidate as:

$$V = \frac{1}{2}\left(\sum_{i=1}^{4} e_i^2 + e_{A_1}^2\right). \tag{3.12}$$

Obviously, V is a positive semi-definite function on $\mathbb{R}^5$.
The time derivative of V along the error system (3.11) is:

$$\begin{aligned} \dot{V} &= \sum_{i=1}^{4} e_i \dot{e}_i + e_{A_1}\dot{e}_{A_1} \\ &= \sum_{i=1}^{4} e_i D^{1-\alpha_i}(D^{\alpha_i} e_i) + e_{A_1}\dot{e}_{A_1} \\ &= e_1(-k_1 e_1) - e_2(\theta e_{A_1} x_2 + k_2 e_2) + e_3(-k_3 e_3) + e_4(-k_4 e_4) + e_{A_1}(-0.03\dot{m}\exp(-m) - \dot{\hat{A}}_1) \\ &= -(k_1 e_1^2 + k_2 e_2^2 + k_3 e_3^2 + k_4 e_4^2) + e_{A_1}(-\theta e_2 x_2 - 0.03\dot{m}\exp(-m) - \dot{\hat{A}}_1). \end{aligned} \tag{3.13}$$

Substituting the adaptation law(3.10) in (3.13), we have:

$$\dot{V} = -(k_1 e_1^2 + k_2 e_2^2 + k_3 e_3^2 + k_4 e_4^2), \tag{3.14}$$

which is negative semi-definite on $\mathbb{R}^5$. Therefore, according to Lyapunov stability theory, the synchronization errors e_i, $i = 1,2,3,4$ converge asymptotically to zero, i.e. the PS between the master system (3.2) and the slave system (3.3)is achieved. This completes the proof. □

Remark 3.2. *According to the proposed transformation function (3.1), the recovered signal message should be defined by:*

$$\hat{m}(t) = \ln\left(\frac{0.03}{\hat{A}_1(t) - 0.16}\right). \tag{3.15}$$

Once the synchronization errors e_i, $i = 1,2,3,4$ approaches zero, it means:

$$\hat{A}_1(t) \to A_1(t), \text{ when } t \to \infty. \tag{3.16}$$

Hence, we have:

$$\hat{m}(t) = \ln\left(\frac{0.03}{\hat{A}_1(t) - 0.16}\right) \to m(t) = \ln\left(\frac{0.03}{A_1(t) - 0.16}\right), \text{ when } t \to \infty. \tag{3.17}$$

Therefore, it can be concluded that the message signal can be finally recovered precisely by the identified parameter and the corresponding demodulation method.

4. Numerical simulations

In this section, computer simulations will be provided to verify the feasibility of the proposed communication system. The Adams-Bashforth-Moulton method is used to solve the fractional systems.

4.1. Case of a bounded information signal

Here, the hidden message signal in the slave system is given by:

$$m(t) = 3 - \cos(2t) - 2\cos(3t). \tag{4.1}$$

Obviously, $0 \leq m(t) \leq 6$. According to the equation (3.1), we can select $A_1(t)$ as follows:

$$A_1(t) = 0.03e^{(-3+\cos(2t)+2\cos(3t))} + 0.16. \tag{4.2}$$

It follows that $A_1(0) = 0.19$.
The initial condition for the adaptation law is given by: $\hat{A}_1(0) = 0.19$.
So the initial condition of the estimation error is given by: $e_{A_1}(0) = 0$.
The initial conditions of the two systems (3.2) and (3.3) are selected respectively as:

$$x_1(0) = -0.02, \; x_2(0) = -0.01, \; x_3(0) = -0.046, \; x_4(0) = 0.02. \tag{4.3}$$

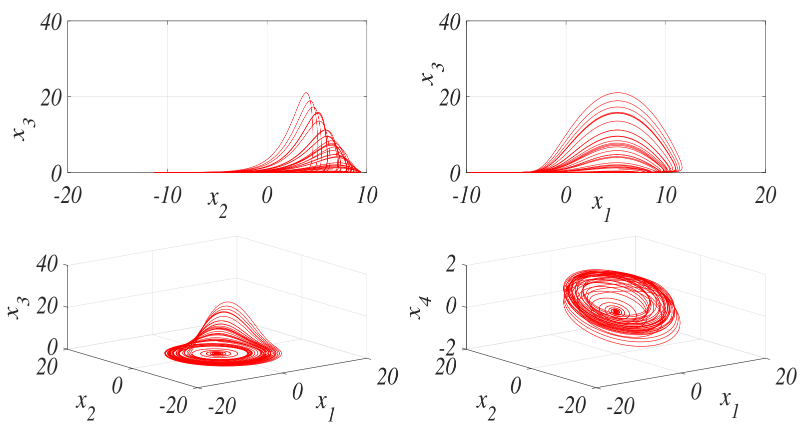

Figure 4.1: Projections of phase portraits of the resulting system (3.2). Case of the bounded information signal: $m(t) = 3 - \cos(2t) - 2\cos(3t)$

$$y_1(0) = -0.08, \; y_2(0) = -0.08, \; y_3(0) = 0.128, \; y_4(0) = 0.07 \tag{4.4}$$

The parameter θ is selected randomly as:

$$\theta = 3. \tag{4.5}$$

As a result, the initial system error conditions are given by:

$$e_1(0) = -0.02, \; e_2(0) = -0.05, \; e_3(0) = 0.01, \; e_4(0) = 0.01. \tag{4.6}$$

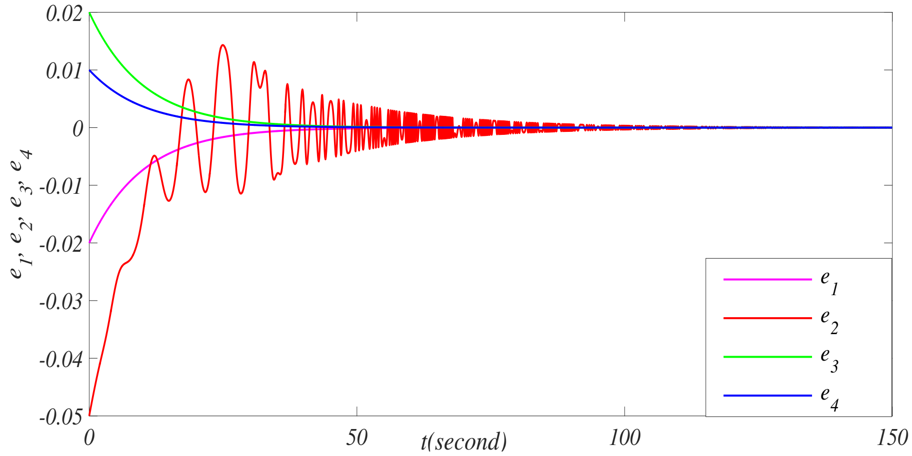

Figure 4.2: Time evolution of the synchronization errors. Case of the bounded information signal: $m(t) = 3 - \cos(2t) - 2\cos(3t)$

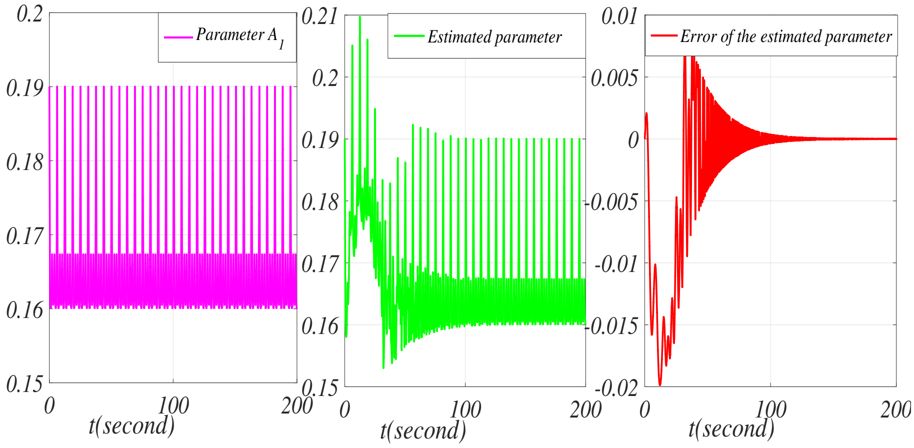

Figure 4.3: Time evolution of the parameter, the estimated parameter and the error of the estimated parameter. Case of the bounded information signal: $m(t) = 3 - \cos(2t) - 2\cos(3t)$

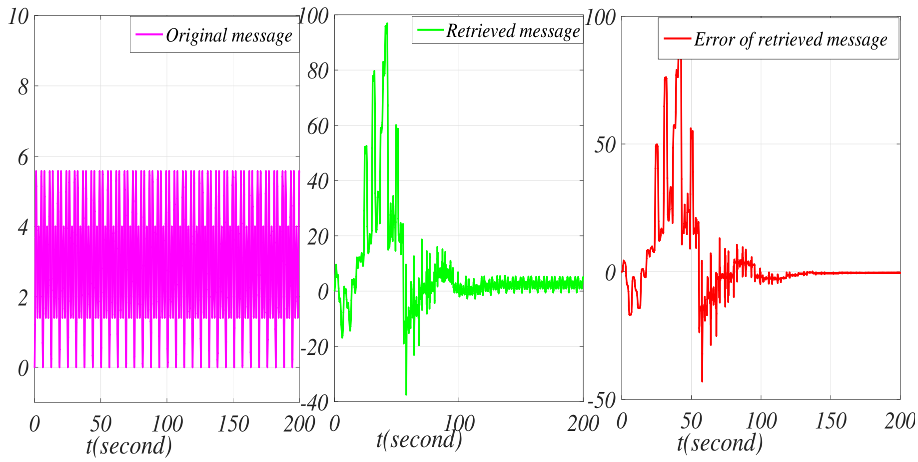

Figure 4.4: Time evolution of the original message, the retrieved message and the error of retrieved message. Case of the bounded information signal: $m(t) = 3 - \cos(2t) - 2\cos(3t)$

The gain (design) parameters are chosen as follows:

$$k_1 = k_2 = k_4 = k_4 = 0.1. \tag{4.7}$$

The orders of fractional derivatives are chosen as:

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (0.98, 0.98, 0.97, 0.97). \tag{4.8}$$

Figure 4.1 illustrates the projections of phase portraits of the resulting system (3.2). The simulation results of the proposed communication system are shown in Figures 4.2, 4.3 and 4.4.

Remark 4.1. *From the Figure 4.2, we can easily see that the errors synchronisation e_i, $i = 1,2,3,4$ converge asymptotically towards zero quickly, i.e., the PS between the master system and the slave system is obtained.*
On the other hand, Figure 4.4 describes the original message signal $m(t)$, the recovered message signal $\hat{m}(t)$ and the signal error via the demodulator (3.15).
From these figures, we can easily see that the error of the parameter converges quickly to zero, when $t \geq 100s$, which shows that the reconstructed signal $\hat{m}(t)$ coincides with the original message signal $m(t)$ with good precision, and the goal of secure communication is achieved.

4.2. Case of an unbounded information signal

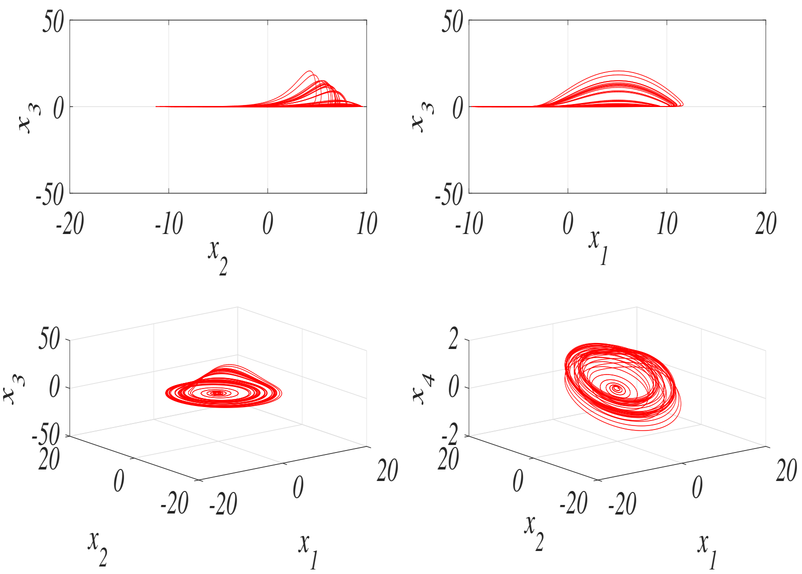

Figure 4.5: Different hyperchaotic attractors of the resulting system (3.2). Case of unbounded information signal: $m(t) = 0.05(t + \sin(t))$

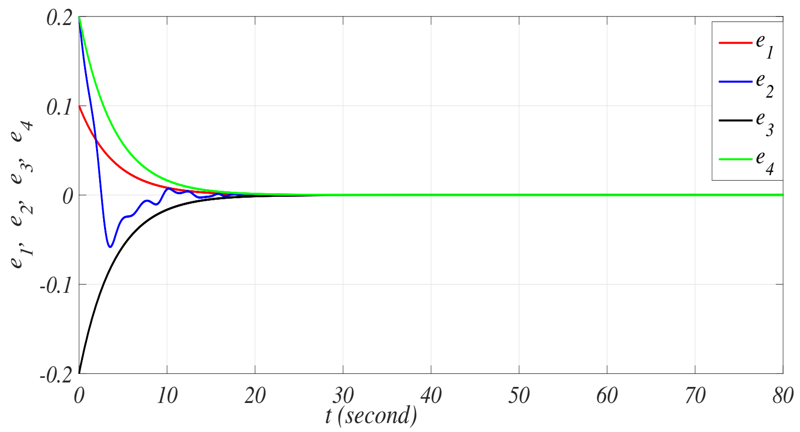

Figure 4.6: Time evolution of the synchronization errors. Case of unbounded information signal: $m(t) = 0.05(t + \sin(t))$

In this case, the message signal is taken as follows:

$$m(t) = 0.05(t + sin(t)), \tag{4.9}$$

According to the equation(3.1), $A_1(t)$ can be obtained as follows:

$$A_1(t) = 0.03e^{(-0.05(t+sin(t)))} + 0.16. \tag{4.10}$$

It follows that $A_1(0) = 0.19$.
The initial condition for the adaptation law is given by: $\hat{A}_1(0) = 0.19$.

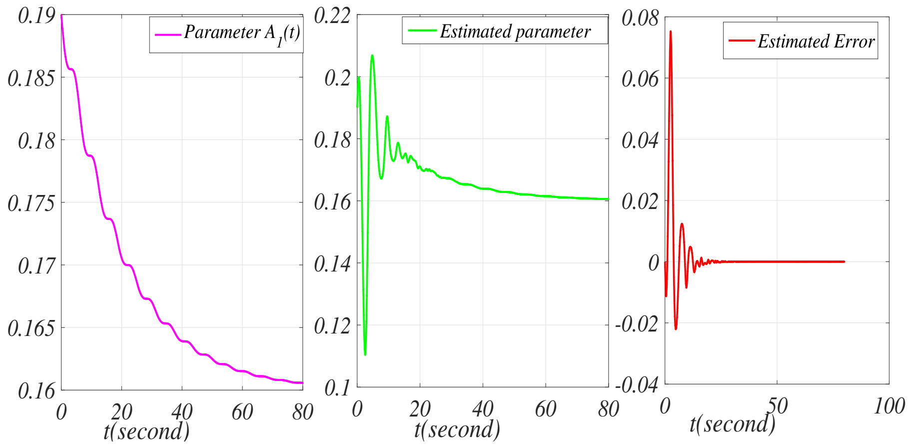

Figure 4.7: Time evolution of the parameter, the estimated parameter and the error of the estimated parameter. Case of unbounded information signal: $m(t) = 0.05(t+\sin(t))$

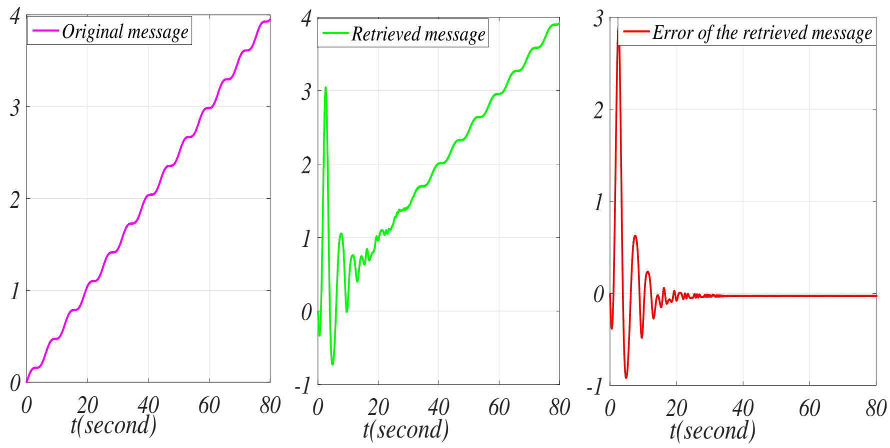

Figure 4.8: Time evolution of the original message, the retrieved message and the error of the retrieved message. Case of unbounded information signal: $m(t) = 0.05(t+\sin(t))$

So the initial condition of the estimation error is given by $e_{A_1}(0) = 0$.
The initial conditions of the two systems (3.2) and (3.3) are selected respectively as:

$$x_1(0) = 0.1,\ x_2(0) = -0.1,\ x_3(0) = -0.2,\ x_4(0) = 0.2. \tag{4.11}$$

$$y_1(0) = 0.3,\ y_2(0) = 0,\ y_3(0) = -0.6,\ y_4(0) = 0.6. \tag{4.12}$$

The scale parameter θ is randomly selected as:

$$\theta = 2. \tag{4.13}$$

Therefore, the initial system error conditions are given by:

$$e_1(0) = 0.1,\ e_2(0) = 0.2,\ e_3(0) = -0.2,\ e_4(0) = 0.2. \tag{4.14}$$

The gain parameters are chosen as follows:

$$k_1 = k_3 = k_4 = 0.25,\ k_2 = 0.5. \tag{4.15}$$

The orders of fractional derivatives are chosen as:

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (0.98, 0.98, 0.98, 0.97). \tag{4.16}$$

The different hyperchaotic attractors of the resulting system (3.2) is shown in Figure 4.5 . The simulation results of the proposed communication system are shown in Figures 4.6, 4.7 and 4.8.

Remark 4.2. *From the Figure 4.6 , its easy to show that all of the synchronization errors e_i $i = 1, 2, 3, 4$, approach to zero quickly. Therefore, the proposed systems are globally synchronized.*
The original message signal $m(t)$, the recovered message signal $\hat{m}(t)$ and the signal error $\hat{m}(t) - m(t)$ are shown in Figure 4.8, which shows that the reconstructed signal $\hat{m}(t)$ coincides with the original message signal $m(t)$ with good precision, and the goal of secure communication is achieved.

5. Conclusion

In the present paper, a new approach for hyperchaotic secure communication method is included by using the parametric modulation technique. Two kinds of secure communication schemes in the case that the hidden message is bounded or unbounded are presented for the possible application in real engineering. We think that we have achieved two important goals. First one, using Lyapunov method, a modified adaptative controller and update law for a parameter estimate are introduced to achieve PS of fractional-order hyperchaotic systems. In particular, the errors system converge to zero quickly, which helps to find the time required. The most important part of this analysis is the proper design of modulation technique so that the message signals in both cases (bounded or unbounded) can be successfully and secretly transmitted via four main elements, namely: modulation, master system, slave system and demodulation. Finally, numerical simulations were provided to verify the effectiveness and feasibility of the proposed secure communication scheme.

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Availability of data and materials

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Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

References

[1] T. L. Carroll, L. M. Pecora, *S*ynchronizing chaotic circuits, IEEE Trans. Circuits Syst., **38** (4) (1991), 453-456.
[2] I. S. Jesus, J. T. Machado, *F*ractional control of heat diffusion systems, Nonlinear Dynamics, **54** (3) (2008), 263-282.
[3] F. Tlacuahuac, L. T. Biegler, *O*ptimization of fractional order dynamic chemical processing systems, Industrial and Engineering Chemistry Research, **53** (13) (2014), 5110-5127.
[4] R. Darling, J. Newman, *O*n the short behaviour of porous interaction electrodes, J. of the Electrochemical Society, **144** (1997), 3057-3063.
[5] R. T. Hernandez, V. R. Ramirez, G. A. Iglesias-Silva, M. U. Diwekar, *A* fractional calculus approach to the dynamic optimization of biological reactive systems, Part I: Fractional models for biological reactions, Chemical Engineering Science, **117** (2014), 217-228.
[6] R. L. Bagley, R. A. Calico, *F*ractional order state equations for the control of viscoelastically damped structures, Journal of Guid Control Dyn., **14** (2) (1991), 304-311.
[7] I. M. Olga, A. K. Alexey, R. H. Alexander, *G*eneralized synchronization of chaos for secure communication: remarkable stability to noise, Physics Letters A, **374** (29) (2010), 2925-2931.
[8] M. S. Abdelouahab, N. Hamri, *F*ractional-order Hybrid Optical System and its Chaos Control Synchronization, Electronic Journal of Theoretical Physics, **11** (30) (2014), 49-62.
[9] E. I. Gonzalez, C. Hernandez, *D*ouble hyperchaotic encryption for security in biometric systems, Nonlinear Dynamics and Systems Theory, **13** (1) (2013), 55-68.
[10] T. Menacer, N. Hamri, *S*ynchronization of different chaotic fractional-order systems via approached auxiliary system the modified Chua oscillator and the modified Van der Pol-Dufing oscillator, Electronic Journal of Theoretical Physics, **28** (25) (2011), 253-266.
[11] H. E. Guitian, L. U. O. Maokang, *D*ynamic behavior of fractional order Dufing chaotic system and its synchronization via singly active control, Appl. Math. Mech.-Engl. Ed., **33** (5) (2012), 567-582.
[12] Q. Gan, Y. Yang, S. Fan, Y. Wang, *S*ynchronization of stochastic Fuzzy cellular neural networks with leakage delay based on adaptive control, Differ. Equ. Dyn. Syst., **22** (2014), 319-332.
[13] A. Bouzeriba, A. Boulkroune, T. Bouden, *P*rojective synchronization of two different fractional-order chaotic systems via adaptive fuzzy control, Neural Comput. Applic., (2016), 1349-1360.
[14] T. L. Carroll, L. M. Pecora, *S*ynchronizing chaotic circuits, IEEE Trans. Circuits Syst., **38** (4) (1991), 453-456.
[15] M. Rehan, *S*ynchronization and anti-synchronization of chaotic oscillators under input saturation, Appl. Math. Model., **37** (2013), 6829-6837.
[16] S. Kaouache, M. S. Abdelouahab, *G*eneralized synchronization between two chaotic fractional non-commensurate order systems with different dimensions, Nonlinear Dynamics and Systems Theory, **18** (3) (2018), 273-284.
[17] R. Manieri, J. Rehacek, *P*rojective synchronization in three-dimensional chaotic systems, Phys. Rev. Lett., **82** (15) (1999), 3042-3045.
[18] G. H. Li, *M*odified projective synchronization of chaotic system, Chaos Solitons Fractals, **32** (5) (2007), 1786-1790.
[19] S. Liu, F. Zhang, *C*omplex function projective synchronization of complex chaotic system and its applications in secure communication, Nonlinear Dyn., **76** (2014), 1087-1097.
[20] X. Wu, H. Wang, H. Lu, *M*odified generalized projective synchronization of a new fractional-order hyperchaotic system and its application in secure communication, Nonlinear Anal. RWA, **13** (2012), 1441-1450.
[21] C. J. Cheng, *R*obust synchronization of uncertain unified chaotic systems subject to noise and its application to secure communication, Appl. Math. Comput., **219** (2012), 2698-712.
[22] W. Xiangjun, F. Zhengye, K. Jürgen, *A* secure communication scheme based generalized function projective synchronization of a new 5*D* hyperchaotic system, Phys. Scr., **90** (4) (2015), Article ID 045210, 12 pages, doi:10.1088/0031-8949/90/4/045210 .
[23] S. Kaouache, M. S. Abdelouahab, *M*odified Projective Synchronization between Integer Or der and Fractional Order Hyperchaotic Systems, Jour. of Adv. Research in Dynamical and Control Systems, **10** (5) (2018), 96-104.