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Research Article

The Space of Continuous Function Between Fuzzy Metric Spaces

Ebru Aydoğdu[1*], Abdülkadir Aygünoğlu[1], Halis Aygün[1]

[1] Kocaeli University, Faculty of Arts and Sciences, Department of Mathematics, 41380, Turkey

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Abstract

In this study, we obtain a fuzzy metric on continuous function space between fuzzy metric space. Then several properties of this function spaces such as completeness, being principal and compactness were investigated.
Keywords: Continuous function spaces, fuzzy metric space

Bulanık Metrik Uzayları Arasında Sürekli Fonksiyon Uzayları

Öz

Bu çalışmada bulanık metrik uzaylar arasındaki sürekli fonksiyon uzaylarında bir bulanık metrik elde edilmiştir. Daha sonra bu fonksiyon uzaylarında tamlık, principal olma ve kompaktlık gibi özellikler incelenmiştir.

Anahtar Kelimeler: Sürekli fonksiyon uzayları, bulanık metrik uzay

1. Introduction

Mathematical modeling of the problems in engineering, social sciences and other fields is not possible with conventional classical methods since root of these problems was based on uncertainty. Uncertainty was modelled with the notion of fuzzy logic and fuzzy set by Zadeh (Zadeh, 1965). Kramosil and Michalek (1975) gave the definition of fuzzy metric space. Then George and Veeramani (1994) redefined the notion of fuzzy metric space in slightly different way from Kramosil and Michalek to construct a Hausdorf topology from given fuzzy metric space. As an important research field in fuzzy set theory, fuzzy metric is an effective tool to solve several problems in topology, algebra, computer science, etc (see (Künzi, 2001)).

Fuzzy metric is a certain kind of mapping that associates two points with a value in [0,1] which essentially means “the degree of nearness between these points according to a parameter t". The t-parameter in the definition of fuzzy metric space is a considerable difference between the classical metric and the fuzzy metric. The parameter t allows introducing fuzzy metric concepts that cannot be defined in the classical context. For instance, when working on contractivity, D. Mihet (2007) defined p-convergence. Then, Gregori and all. (2009) investigate this

convergent concept and studied on principal fuzzy metric space. Also, the authors (Gregori et al., 2004 ; Gregori et al., 2002) present
t -uniform continuity between fuzy metric by modifying uniform continuity. Recently, the authors (Gregori et al., 2009) introduced the definition of t-continuous mapping in a natural sense for t-uniform continuous mapping. Then, in (Gregori and Minana, 2013) t-continuous mapping was characterized by p-convergent sequences. Therefore, inspired by this result, in (Gregori and Minana, 2013) they introduce the concept of p-continuity.

In this paper, we determine a fuzzy metric on continuous function space between fuzzy metric spaces. Then we investigate the completeness of continuous function space and give the relation between principal fuzzy metric space and principal continuous function space. Then we defined p-compactness and p-uniformly continuity associated with p-convergent and show that a p-continuity is equal to p-uniform continuity on a compact (p-compact) fuzzy metric space.

2. Preliminaries

Definition 2.1. Let A be a nonempty set, $*$ be a continuous t-norm and $\mu: A \times A \times (0,\infty) \to [0,1]$ be a mapping. If the listed conditions are satisfied for all $a, b, c \in A$ and $t, s > 0$, then the triplet $(U, \mu, *)$ is said to be a fuzzy metric space (shortly, FMS) :
(F1) $\mu(a,b,t) > 0$,
(F2) $\mu(a,b,t) = 1 \; iff \; a = b$,
(F3) $\mu(a,b,t) = \mu(b,a,t)$,
(F4) $\mu(a,b,t) * \mu(b,c,s) \leq \mu(a,c,t+s)$,
(F5) $\mu(a,b,\cdot): (0,\infty) \to [0,1]$ is continuous (George and Veeramani, 1995).

Remark 2.2. Let $(A,\mu,*)$ be a FMS, $a \in A$ and the open ball on $a \in A$ be $B(a,r,t) = \{b \in A: \mu(a,b,t) > 1-r\}$ for every $r \in (0,1)$ and $t > 0$. Then the family
$\{B(a,r,t): a \in A, r \in (0,1), t > 0\}$
is a base for the topology T_μ (George and Veeramani, 1995).

Definition 2.3. Let $\{a_n\}$ be a sequence in A and $(A,\mu,*)$ be a FMS. The sequence $\{a_n\}$ converges to a if and only if for all $t > 0$ i.e.

$$lim_{n\to\infty}\mu(a_n,a,t) = 1 \text{ for all } t > 0$$

(George and Veeramani, 1994).

Definition 2.4. Let $\{a_n\}$ be a sequence in A and $(A,\mu,*)$ be a FMS. If

$$lim_{n\to\infty}\mu(a_n,a_0,t_0) = 1$$

for some $t_0 > 0$, then $\{a_n\}$ is called p-convergent to a_0 (Mihet, 2007).

Definition 2.5. Let $(A,\mu_A,*_A)$ and $(C,\mu_C,*_C)$ be two FMSs, $\{f_n\}$ be a sequence of functions from A to C and $f: A \to C$. If given $\varepsilon \in (0,1)$ and $t > 0$ there exists $n_0 \in \mathbb{N}$ such that $\mu_C(f_n(a), f(a), t) > 1-r$ for all $n \geq n_0$ and for all $a \in A$, then $\{f_n\}$ converges uniformly to f (George and Veeramani, 1997).

Definition 2.6. Let $\{a_n\}$ be sequence in A and $(A,\mu,*)$ be a FMS. If for all $\varepsilon \in (0,1)$, $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that $\mu(a_n,a_m,t) > 1-\varepsilon$ for all $n,m \geq n_0$, then $\{a_n\}$ is called a Cauchy sequence (George and Veeramani, 1994).

Definition 2.7. Let $\{a_n\}$ be Cauchy sequence in A and $(A,\mu,*)$ be a FMS. If $\{a_n\}$ is convergent sequence, then $(A,\mu,*)$ is called complete (George and Veeramani, 1994).

Definition 2.8. Let $(A,\mu,*)$ be a FMS. If the family of open sets $\{B(a,r,t): r \in (0,1)\}$ is a local base at $a \in A$, for each $a \in A$, $t > 0$, then $(A,\mu,*)$ is called principal (or, μ is principal) (Gregori et al., 2009).

Theorem 2.9. A FMS $(A,\mu,*)$ is principal if and only if every p-convergent sequence $\{a_n\}$ is convergent (Gregori et al., 2009).

Definition 2.10. Let $(A,\mu_A,*_A)$ and $(C,\mu_C,*_C)$ be two FMSs. The function $f: A \to C$ is continuous at $a_0 \in A$ $\Leftrightarrow \forall\varepsilon \in (0,1)$ and $\forall t > 0$ $\exists\delta \in (0,1)$ and $\exists s > 0$: $\mu_A(a,a_0,s) > 1-\delta$ implies $\mu_C(f(a),f(a_0),t) > 1-\varepsilon$ (Gregori et al., 2009)

Definition 2.11. Let $(A,\mu_A,*_A)$ and $(C,\mu_C,*_C)$ be two FMSs and $f: (A,\mu_A,*_A) \to (C,\mu_C,*_C)$. f is called uniformly continuous $\Leftrightarrow \forall\varepsilon \in (0,1)$ and $\forall t > 0\ \exists\delta \in (0,1)$ *and* $\exists s > 0$: $\mu_C(f(a),f(b),t) > 1-\varepsilon$ whenever $\mu_X(a,b,s) > 1-\delta$ (George and Veeramani, 1995).

Remark 2.12. Every uniformly continuous mapping $f:(A,\mu_A,*_A) \to (C,\mu_C,*_C)$ is continuous with generated topology on A and C, respectively.

Definition 2.13. A FMS $(A,\mu,*)$ is called totally bounded if for each $r \in (0,1)$ and $t > 0$, there exists a finite subset C of A, such that $A = \bigcup_{c\in C} B(c,r,t)$ (Gregori and Romaguera, 2000).

Definition 2.14. Let $(A,\mu,*)$ be a FMS. If (A,T_μ) is compact topological space, then $(A,\mu,*)$ is called compact.

Proposition 2.15. Let $(A,\mu_A,*_A)$ and $(C,\mu_C,*_C)$ be two FMSs and $\{f_n\}$ be a sequence of continuous functions from A to C. If f_n converges uniformly to a function f, then f is continuous (George and Veeramani, 1997).

3. Results

In this section, $(A,\mu_A,*_A)$ and $(C,\mu_C,*_C)$ be two FMSs and $\mathcal{C}(A,C) = \{f \mid f: A \to C \text{ continuous}\}$ be a set of continuous function.

Lemma 3.1. Let $(A,\mu_A,*_A)$ be compact. If $f,g: A \to C$ are continuous functions, then

$$D(f,g,t) = inf\{\mu_C(f(a),g(a),t): a \in A\} > 0.$$

Proof. Let $\varepsilon \in (0,1)$ and $t > 0$. Since A is compact, then f and g is uniformly continuous. Thus, there exists $0 < \delta < 1$ and $s > 0$ such that $\mu_A(a,b,s) > 1-\delta$ implies $\mu_C(f(a),f(b),t) > 1-\varepsilon$ and similarly implies $\mu_C(g(a),g(b),t) > 1-\varepsilon$. Since A is compact, then A is totally bounded. Thus, there exists $a_1,a_2,\dots,a_n \in A$ such that $A = \bigcup_{i=1}^{n} B(a_i,\delta,s)$.

Let $\alpha = \min \mu_C\left(f(a_i),g(a_i),\left(\frac{t}{3}\right)\right)$. For $\mu_A(a,a_i,s) > 1-\delta$ we get

$$\mu_C(f(a),g(a),t)$$
$$\geq \mu_C\left(f(a),f(a_i),\left(\frac{t}{3}\right)\right) *_Y \mu_C\left(f(a_i),g(a_i)\right.$$
$$> (1-\varepsilon) *_Y \alpha *_Y (1-\varepsilon)$$

for all $a \in A$ and $t > 0$. Hence, $\inf \mu_C(f(a),g(a),t) \geq (1-\varepsilon) *_Y \alpha *_Y (1-\varepsilon) > 0$. Thus, $D(f,g,t) > 0$.

Theorem 3.2. Let $(A,\mu_A,*_A)$ be compact. Then,

$D(f,g,t) = inf\{\mu_C(f(a),g(a),t): a \in A\}$

is a fuzzy metric on $\mathcal{C}(A,C)$.

Proof. By the Lemma 3.1 $D(f,g,t)$ is always finite, and we only must prove that D satisfies the five properties of fuzzy metric. (F1), (F2), (F3) and (F5) are obvious, and we concentrate on (F4):

Suppose that f,g,h are three functions in $\mathcal{C}(X,Y)$. According to the Lemma 3.1, there exists $a \in A$ such that $D(f,g,t) = \mu_C(f(a),g(a),t)$. Hence

$$D(f,g,t+s) = \mu_C(f(a),g(a),t+s)$$
$$\geq \mu_C(f(a),h(a),t) *_Y \mu_C(h(a),g(a),s)$$
$$\geq D(f,h,t) *_Y D(h,g,s).$$

From the Theorem 3.9 in [3], we obtain the following proposition.

Proposition 3.3. A sequence $\{f_n\}$ converges to f in $(\mathcal{C}(X,Y),D)$ if and only if $\{f_n\}$ converges uniformly to f.

Theorem 3.4. Let $(A,\mu_A,*_A)$ be compact FMS and $(C,\mu_C,*_C)$ be a complete FMS. Then $\mathcal{C}(A,C)$ is complete with the fuzzy metric D.

Proof. Suppose that $\{f_n\}$ be a Cauchy sequence in $\mathcal{C}(A,C)$. We must prove that f_n converges to a function $f \in \mathcal{C}(A,C)$.

Fix an element $a \in A$. The function values $\{f_n(a)\}$ form a Cauchy sequence in C. Since C is complete, $\{f_n(a)\}$ converges to $f(a)$ in C. This means that $\{f_n\}$ converges to a function f. We must prove that $f \in \mathcal{C}(A,C)$ and that $\{f_n\}$ converges to f in the D fuzzy metric.
Since $\{f_n\}$ is Cauchy sequence, we can for any $\varepsilon \in (0,1)$ and $t > 0$ find an $n_0 \in \mathbb{N}$ such that $D(f_n,f_m,t) > 1 - \left(\frac{\varepsilon}{2}\right)$ when $n,m \geq n_0$. This means that all $x \in X$ and all $n,m \geq n_0$, $\inf \mu_C(f_n(a),f_m(a),t) > 1 - \left(\frac{\varepsilon}{2}\right)$. If $m \to \infty$, we get for all $a \in A$ and all $n \geq n_0$

$$\mu_C(f_n(a),f(a),t)$$
$$= lim_{m\to\infty}\mu_C(f_n(a),f_m(a),t)$$
$$> 1 - \left(\frac{\varepsilon}{2}\right) > 1 - \varepsilon.$$

For all $\varepsilon \in (0,1)$ we get $lim_{n\to\infty}\mu_C(f_n(a),f(a),t) = 1$. This means that $\{f_n\}$ converges uniformly to f. According to Proposition 2.15, f is continuous and belongs to $\mathcal{C}(A,C)$, and according to Proposition 3.3, $\{f_n\}$ converges to f in $\mathcal{C}(A,C)$.

Theorem 3.5. Let $(C,\mu_C,*_C)$ be principal FMS. Then $\mathcal{C}(A,C)$is principal.

Proof. Let $(C,\mu_C,*_C)$ be principal and $\{f_n\}$ be a sequence of functions which p-convergent. There exists $t_0 > 0$ such that $lim_{n\to\infty}D(f_n,f,t_0) = 1$. This means that for all $a \in A$, $lim_{n\to\infty}\mu(f_n(a),f(a),t_0) = 1$. Since C is principal, then for all $t > 0$, $lim_{n\to\infty}\mu(f_n(a),f(a),t) = 1$. This means that $lim_{n\to\infty}inf_{a\in A}\mu(f_n(a),f(a),t) = 1$, hence $\{f_n\}$ converges to f.

Definition 3.6. Let $\{a_n\}$ be a sequence in A where $(A,\mu,*)$ is a FMS. If $\{a_n\}$ has a p-convergent subsequence, then $(A,\mu,*)$ is called p-compact.

Definition 3.7. Let $(A,\mu,*)$ be a FMS and $t > 0$. If there exists a finite subset C of A, such that $X = \bigcup_{\{c\in C\}} B(c,r,t)$ for each $r \in (0,1)$, then $(A,\mu,*)$ is called p-totally bounded.

Theorem 3.8. If $(A,\mu_A,*_A)$ is p-compact FMS, then $(A,\mu_A,*_A)$ is p-totally bounded.

Proof. Suppose that $(A, \mu_A, *_A)$ is not p-totally bounded. Then there exists $r \in (0,1)$ such that A cannot be covered by finitely balls of radius r for all $t > 0$. Take $a_1 \in A$. Since $B(a_1, r, t)$ does not cover A, there exists at least one point in $A - B(a_1, r, t)$. Choose one such and call it a_2. Since $B(a_1, r, t) \cup B(a_2, r, t)$ does not cover A, there exists at least one point in $A - [B(a_1, r, t) \cup B(a_2, r, t)]$. Continuing this process, we find a sequence $\{a_n\}$ satisfying $a_{n+1} \in A \setminus \bigcup_{i=1}^{n} B(a_i, r, t)$ for each $n \in \mathbb{N}$. However, such sequence cannot have a p-convergent subsequence, since $\mu(a_n, a_m, t) < 1 - r$ for all n, m. This is a contradiction. Therefore A is a p-totally bounded.

Definition 3.9. Let $(A, \mu_A, *_A)$ and $(C, \mu_C, *_C)$ be two FMSs and $f: A \to C$. If for each $\varepsilon \in (0,1)$ there exists $t > 0$ and $\delta \in (0,1)$ such that $\mu_A(a, b, t) > 1 - \delta$ implies $\mu_C(f(a), f(b), t) > 1 - \varepsilon$, then f is called p-continuous at $a \in A$.

Definition 3.10. Let $(A, \mu_A, *_A)$ and $(C, \mu_C, *_C)$ be two FMSs. $f: (A, \mu_A, *_A) \to (C, \mu_C, *_C)$ is called p-uniformly continuous if for each $\varepsilon \in (0,1)$ there exists $t > 0$ and $\delta \in (0,1)$ such that $\mu_A(a, b, t) > 1 - \delta$ implies $\mu_C(f(a), f(b), t) > 1 - \varepsilon$.

Theorem 3.11. Let $f: (A, \mu_A, *_A) \to (C, \mu_C, *_C)$ be p-continuous for a $t_1 > 0$ and $(A, \mu_A, *_A)$ be p-compact FMS for a $t_2 > 0$. Then f is p-uniformly continuous if $t_1 \geq t_2$.

Theorem 3.12. Let $f: (A, \mu_A, *_A) \to (C, \mu_C, *_C)$ be p-continuous and $(A, \mu_A, *_A)$ be a compact FMS. Then f is p-uniformly continuous.

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