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Rotational hypersurfaces in Euclidean 4-space with density

Yoğunluklu Öklidyen 4-uzayında dönel hiperyüzeyleri

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Rotational Hypersurfaces in Euclidean 4-space with Density

Highlights

- *Rotational hypersurfaces in Euclidean 4-space with density have considered.*
- *Weighted minimal and weighted flat rotational hypersurfaces in Euclidean 4-space with density have obtained.*
- *Some examples for these hypersurfaces have constructed.*

Graphical Abstract

In the present study, Euclidean 4-space with a positive density function $e^{x^2+y^2+z^2+t^2}$ have studied. In this context, the weighted mean and weighted Gaussian curvature functions of a rotational hypersurface in 4-dimensional Euclidean space with density have obtained.

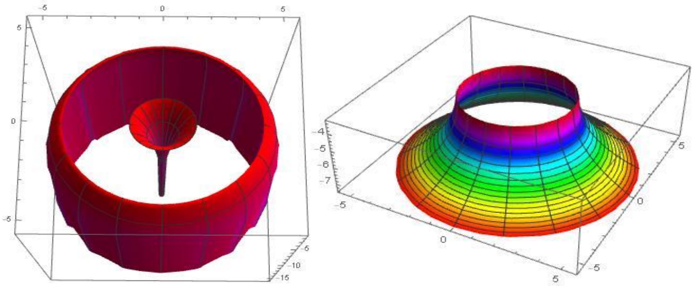

Figure. Some projections of the rotational hypersurface.

Aim

The aim of this study is to study the rotational hypersurface in 4-dimensional Euclidean space with density.

Design & Methodology

The theoretical methodology of mathematics has used to obtain the results.

Originality

All obtained results in this study are original.

Findings

The weighted mean and weighted Gaussian curvature functions of a rotational hypersurface in 4-dimensional Euclidean space with density $e^{x^2+y^2+z^2+t^2}$ have obtained and some examples for these hypersurfaces have given.

Conclusion

In this paper, we consider the rotational hypersurfaces in Euclidean 4-space with density $e^{x^2+y^2+z^2+t^2}$ and obtain the weighted minimal and weighted flat rotational hypersurfaces in this space. We think that, the results which are obtained in this study are important for differential geometers who are dealing with weighted surfaces.

Declaration of Ethical Standards

The author(s) of this article declare that the materials and methods used in this study do not require ethical committee permission and/or legal-special permission.

Yoğunluklu Öklidyen 4-Uzayında Dönel Hiperyüzeyler

Araştırma Makalesi / Research Article

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ÖZ

Bu çalışmada, pozitif yoğunluk fonksiyonu $e^{x^2+y^2+z^2+t^2}$ olan 4-boyutlu Öklid uzayı ele alınmıştır. İlk olarak, yoğunluklu 4-boyutlu Öklid uzayında bir dönel hiperyüzeyin ağırlıklı ortalama ve ağırlıklı Gauss eğrilik fonksiyonları elde edilmiştir. İkinci mertebeden lineer olmayan adi diferansiyel denklem olarak elde edilen bu fonksiyonların çözülmesiyle dönel hiperyüzeyler inşa edilmiştir. Ayrıca, yoğunluklu E^4 uzayında, ağırlıklı Gauss eğriliği ve ağırlıklı ortalama eğriliği yardımıyla dönel hiperyüzey örnekleri verilmiştir.

Anahtar Kelimeler: Yoğunluklu 4-boyutlu Öklidyen uzay, ağırlıklı ortalama eğrilik, ağırlıklı Gaussian eğriliği, dönel hiperyüzeyleri.

Rotational Hypersurfaces in Euclidean 4-Space with Density

ABSTRACT

In this paper, the Euclidean 4-space with a positive density function $e^{x^2+y^2+z^2+t^2}$ is studied. Firstly, the weighted mean and weighted Gaussian curvature functions of a rotational hypersurface in 4-dimensional Euclidean space with density are obtained. The rotational hypersurfaces are constructed by solving these obtained functions which are second-order non-linear ordinary differential equations. Besides, the examples of rotational hypersurfaces are given with the aid of the weighted Gaussian and weighted mean curvatures in E^4 with density.

Keywords: Rotational hypersurfaces, Euclidean 4-space with density, weighted mean curvatures, weighted Gaussian curvatures.

1. INTRODUCTION

Minimal and flat surfaces are the significant study areas for mathematicians, engineers, and other scientists.

The studies focus on the minimal and flat surfaces in 4-dimensional spaces that can be listed as follows: Moore has studied rotational surfaces with constant curvature in four-dimensional space and some relations have been given for them in the 1900s [1,2]. Ganchev and Milousheva have examined the Moor's studies in Minkowski 4D-space and some relations have been expressed in [7]. Complete hypersurfaces in $\mathbb{R}^4$ with constant mean curvature and scalar curvature have been classified in [3]. In [5,6], the generalized rotational surfaces and translation surfaces in 4-D Euclidean surfaces have been studied. The curvature properties of the surfaces have been investigated and some examples for them have given. Besides, it is shown that the translation surface is flat if and only if it is a hyperplane or a hypercylinder. Moruz and Mounteanu have studied Minimal translation hypersurfaces in [8]. The rotational surfaces with finite type Gauss map in Euclidean 4-space have been investigated in [4]. It is shown that the Gauss map is a finite type if and only if the rotational surface is a Clifford torus [4]. Dursun and Turgay have studied general rotational surfaces in E^4 whose meridian curves lie in 2D planes. They also have found all minimal general rotational surfaces by solving the differential equation that characterizes minimal general rotational surfaces. Besides, they have determined all pseudo-umbilical general rotational surfaces in E^4 [9]. Kahraman and Yaylı have studied Bost invariant surfaces with pointwise 1-type Gauss map in E_1^4 and they have generalized rotational surfaces of pointwise 1-type Gauss map in E_2^4 [10,11]. Güler and et al. have defined helicoidal hypersurface with the Laplace-Beltrami operator in four-space [12]. Also, Güler and et al. have studied Gauss map and the third Laplace-Beltrami operator of the rotational hypersurface in 4-space [13], second Laplace-Beltrami operator of the rotational hypersurface in 4-space [32] and Cheng-Yau operator and Gauss map of the rotational hypersurface in 4-space [33]. Yüce has studied Weingarten Map of the Hypersurface in Euclidean 4-Space [34]. Since the Gaussian curvature and the mean curvature of an n-dimensional hypersurface are important invariants to characterize the hypersurface, many authors have studied these notions for different types of hypersurfaces for a long time in different spaces, such as Euclidean, Minkowski, Galilean, and pseudo-Galilean spaces.

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Furthermore, recently, the notion of the weighted manifold which is an important topic for geometers and physicists has been studied by many scientists. Firstly, Gromov has introduced the notion of weighted mean curvature (or φ-mean curvature) of an n-dimensional hypersurface as

$$H_{\varphi} = H - \frac{1}{(n-1)}\frac{d\varphi}{dN}, \qquad (1.1)$$

where φ is density function, H and N are respectively the mean curvature and the unit normal vector field of the hypersurface [14]. A hypersurface is named weighted minimal (or φ-minimal) if its weighted mean curvature vanishes.

Also, Corvin and et al. have introduced the notion of generalized weighted Gaussian curvature on a manifold as

$$G_{\phi} = G - \triangle\, \varphi, \qquad (1.2)$$

where $\triangle$ is the Laplacian operator and G is Gaussian curvature of the hypersurface. Also, they have given a generalization of the Gauss-Bonnet formula for a 2-dimensional differentiable manifold with density [15]. A hypersurface is called weighted flat (or φ-flat) if its weighted Gaussian curvature vanishes.

After these definitions, the differential geometry of the curves and hypersurfaces on manifolds with density in Euclidean, Minkowski, and Galilean spaces has been started to be an important topic for geometers, physicists, economists, etc. For instance, in [20, 21, 29], F. Morgan and others have studied the manifolds with density, provided the generalizations of the theorem of Myers to Riemannian manifolds with density and the Perelman's proof of the Poincare conjecture, respectively.

The classification of constant weighted curvature curves in a plane with a log-linear density has been done by Hieu in [17]. Furthermore, some results on curves in the plane with log-linear density have been given by Nam in [22**].** In [28], Lopez has studied the minimal surfaces in Euclidean 3-space with a log-linear density *φ(x, y, z) = αx+βy +γz*, where *α, β,* and *γ* are real numbers not all-zero. Also, Belarbi and et al. have studied the surfaces in R^3 with density and they have given some results in a Riemannian manifold *M* with density in [16] and [26]. Next, ruled minimal surfaces in $R^3$with density e^z; helicoidal surfaces in R^3 with density $e^{-x^2-y^2}$ and weighted minimal affine translation surfaces in Euclidean space with density have been studied in [27, 23, 24], respectively. Also, some types of surfaces have been studied by geometers in other spaces such as Minkowski 3-space and Galilean 3-space with density. For instance, a helicoidal surface of type I^+ with prescribed weighted mean curvature and Gaussian curvature in Minkowski 3-space and weighted minimal translation surfaces in Minkowski 3-space with density e^z have been constructed in [25] and [30], respectively. In [31], weighted minimal translation surfaces in the Galilean 3-space with log-linear density have been classified and in [19], weighted minimal and weighted flat surfaces of revolution in Galilean 3-space with density $e^{ax^2+by^2+cz^2}$ have been investigated. Also, Altın and his friends have studied ruled surfaces and rotational surfaces in different spaces with density, in recent years (see [18, 35-38]).

In the present study, after giving some basic notions about hypersurfaces in Euclidean 4-space in the Preliminaries section; in the third section, we give the solutions of Gaussian curvature and mean curvature of rotational hypersurfaces in Euclidean 4-space. Also, we give some results and examples of the rotational hypersurfaces in this section.

In the fourth section of this paper, we obtain the weighted mean and weighted Gaussian curvatures of a rotational hypersurface in E^4 with density. Then, we solve these curvature functions which are second-order non-linear ordinary differential equations. Furthermore, we give some examples of a rotational hypersurface with different weighted Gaussian and weighted mean curvatures in E^4 with density.

2.PRELIMINARIES

In this section, some fundamental notions used in the following sections will be given.

Let $\vec{\boldsymbol{x}} = (\boldsymbol{x_1}, \boldsymbol{y_1}, \boldsymbol{z_1}, \boldsymbol{t_1})$, $\vec{\boldsymbol{y}} = (\boldsymbol{x_2}, \boldsymbol{y_2}, \boldsymbol{z_2}, \boldsymbol{t_2})$ and $\vec{\boldsymbol{z}} = (\boldsymbol{x_3}, \boldsymbol{y_3}, \boldsymbol{z_3}, \boldsymbol{t_3})$ be three vectors in $\boldsymbol{E^4}$. Then, the inner product and vector product of these vectors are given by

$$\langle \vec{x}, \vec{y} \rangle = x_1x_2 + y_1y_2 + z_1z_2 + t_1t_2 \qquad (2.1)$$

and

$$\vec{x} \times \vec{y} \times \vec{z} = det\begin{pmatrix} e_1 & e_2 & e_3 & e_4 \\ x_1 & y_1 & z_1 & t_1 \\ x_2 & y_2 & z_2 & t_2 \\ x_3 & y_3 & z_3 & t_3 \end{pmatrix}, \qquad (2.2)$$

respectively. If

$$X: E^3 \longrightarrow E^4$$
$$(u_1, u_2, u_3) \longrightarrow X(u_1, u_2, u_3) \qquad (2.3)$$
$$= (X_1(u_1, u_2, u_3), X_2(u_1, u_2, u_3), X_3(u_1, u_2, u_3), X_4(u_1, u_2, u_3))$$

is a hypersurface in Euclidean 4-space E^4, then the normal vector field, the matrix forms of the first and second fundamental forms are

$$N = \frac{X_{u_1} \times X_{u_2} \times X_{u_3}}{\|X_{u_1} \times X_{u_2} \times X_{u_3}\|}, \qquad (2.4)$$

$$g_{ij} = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} \qquad (2.5)$$

and

$$h_{ij} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}, \qquad (2.6)$$

respectively. Here, $g_{ij} = \langle X_{u_i}, X_{u_j} \rangle$, $h_{ij} = \langle X_{u_i u_j}, N \rangle$, $X_{u_i} = \frac{\partial X}{\partial u_i}$, $X_{u_i u j} = \frac{\partial^2 X}{\partial u_i u_j}$, $\{i, j\} \in \{1,2,3\}$.

Also, the shape operator of the hypersurface (2.3) is

$$S = (a_{ij}) = (g_{ij})^{-1}.(h_{ij}), \qquad (2.7)$$

where $(g_{ij})^{-1}$ is the inverse matrix of (g_{ij}).

On the other hand, from *(2.5)-(2.7)*, the Gaussian and mean curvature of a hypersurface in E^4 are

$$K = \frac{det(h_{ij})}{det(g_{ij})} \tag{2.8}$$

and

$$3H = tr(S), \tag{2.9}$$

respectively.

Let $\gamma(x) = (x, 0, 0, f(x))$ be a profile curve in *xt*-plane defined on any open interval $I \subset R$. Then, the rotational hypersurface M in E^4 is given by

$$M: X(x,y,z) = (xcosycosz, xsinycosz, xsinz, f(x)) \tag{2.10}$$

where $f : I \subset R - \{0\} \to R$ is a C^∞ function for all $x \in I$ and $0 \leq y, z \leq 2\pi$.

The Gaussian curvature G and the mean curvature H of rotational hypersurface are obtained as follows [13, 32, 33].

$$G = -\frac{f'(x)^2 f''(x)}{x^2(1+f'(x)^2)^{5/2}}, \tag{2.11}$$

$$H = -\frac{xf''(x)+2f'(x)^3+2f'(x)}{3x(1+f'(x)^2)^{3/2}}. \tag{2.12}$$

Also, the unit normal vector N of the rotational hypersurface is

$$N = \frac{(cosycoszf'(x), sinycoszf'(x), fsinzcoszf'(x)-1)}{\sqrt{1+f'(x)^2}}. \tag{2.13}$$

3. ROTATIONAL HYPERSURFACES IN E⁴

In this section, the solutions of (*2.11)* and *(2.12)* will be given. Furthermore, some results and examples of the rotational hypersurfaces will be given.

3.1. The solution of Gaussian curvature of rotational hypersurface

To solve Eq. *(2.11)* which is a second-order nonlinear ordinary differential equation (NODE), we assume

$$A = -\frac{f'(x)^3}{x^6(1+f'(x)^2)^{3/2}}. \tag{3.1}$$

From equations *(2.11)* and *(3.1)*, we have

$$A' = -\frac{6A}{x} - \frac{3G(x)}{x^4}.$$

It is a first-order linear ordinary differential equation with respect to A and its general solution is computed as

$$A = \frac{c_1 - 3\int_1^x G(t)t^2 dt}{x^6}, \tag{3.2}$$

where $c_1 \in R$. Also, the equations *(3.1)* and *(3.2)* imply

$$(1+f'(x)^2)^{\frac{3}{2}}(c_1 - 3\int_1^x G(t)\, t^2 dt) = f'(x)^3. \tag{3.3}$$

Thus, the general solution of Eq. *(3.3)* is

$$f(x) = \pm \int \frac{(c_1 - 3\int_1^x G(t)t^2 dt)^{1/3}}{\sqrt{1-(c_1 - 3\int_1^x G(t)t^2 dt)^{2/3}}} dx + c_2,$$

where c_2 is constant and $(c_1 - 3\int_1^x G(t)\, t^2 dt)^{\frac{2}{3}} < 1$.

Conversely, let *G(x)* be a smooth function defined on an open interval $I \subset R$. Then, for any $x_0 \in I$, there exist an open subinterval $I_1 \subset R$ of x_0 $(I_1 \subset I)$ and an open interval I_2 of R containing

$$c_{1_0} = (3\int_1^x G(t)t^2 dt)(x_0)$$

such that function

$$F(x, c_1) = 1 - \left(c_1 - 3\int_1^x G(t)\, t^2 dt\right)^{2/3} > 0$$

for any $(x, c_1) \in I_1 \times I_2$. In fact, because of $F(x_0, c_{1_0}) = 1 > 0$, by the continuity of F, it is positive in a subset of $I_1 \times I_2 \subset R^2$. Therefore, for any $(x, c_1) \in I_1 \times I_2$, $c_2 \in R$ and any given smooth function *G(x)*, we can define the two-parameter family of curves

$$\gamma(x, G, c_1, c_2) = \left(x, 0, 0, \pm\int \frac{\left(c_1 - 3\int_1^x G(t)\, t^2 dt\right)^{1/3}}{\sqrt{1 - \left(c_1 - 3\int_1^x G(t)\, t^2 dt\right)^{2/3}}} dx + c_2\right).$$

By performing the one-parameter subgroup on these curves, the two-parameter family of rotational hypersurfaces with the Gaussian curvature *G(x)* can be obtained.

Theorem 3.1. *Let $\gamma(x) = (x, 0, 0, f(x))$ be a profile curve of the rotational hypersurface (2.10) with the Gaussian curvature at the point $(x, 0, 0, f(x))$ given by G(x) in the Euclidean 4-space. Then, for some constants c_1 and c_2 there exists the two-parameter family of rotational hypersurface generated by plane curves*

$$\gamma(x, G(x), c_1, c_2) = (x, 0, 0, \pm \int \frac{\left(c_1 - 3\int_1^x G(t)t^2 dt\right)^{1/3}}{\sqrt{1-\left(c_1 - 3\int_1^x G(t)t^2 dt\right)^{2/3}}} dx + c_2). \tag{3.4}$$

Let G(x) be a smooth function. The two-parameter family of curves $\gamma(x, G(x), c_1, c_2)$ can be constructed and the two-parameter families of rotational hypersurfaces with the Gaussian curvature can be given by

$$X_G(x,y,z) = (xcosycosz, xsinycosz, xsinz, \pm\int \frac{\left(c_1 - 3\int_1^x G(t)t^2 dt\right)^{1/3}}{\sqrt{1-\left(c_1 - 3\int_1^x G(t)t^2 dt\right)^{2/3}}} dx + c_2). \tag{3.5}$$

Corollary 3.1. Let M be the rotational hypersurfaces in E^4 with constant Gaussian curvature $(G(x) = d_1 \in R)$. Then M can be parameterized by

$$X_G(x,y,z) = (xcosycosz, xsinycosz, xsinz, \pm\int \frac{(c_1 - d_1x^3 + d_1)^{1/3}}{\sqrt{1-(c_1 - d_1x^3 + d_1)^{2/3}}} dx + c_2),$$

where $c_1, c_2 \in R$ and $1 > (c_1 - d_1x^3 + d_1)^{\frac{2}{3}}$.

Corollary 3.2. Let M be a flat rotational hypersurfaces in E^4. Then M can be parameterized by

$$X_G(x,y,z) = \left(xcosycosz, xsinycosz, xsinz, \frac{(c_1)^{1/3}}{\sqrt{1-(c_1)^{2/3}}}x + c_2\right),$$

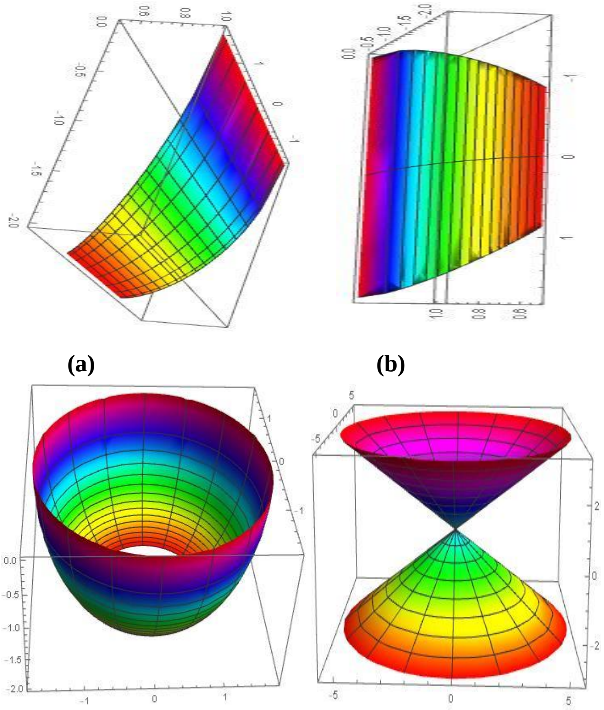
where $c_1, c_2 \in R$ and $1 > (c_1)^{2/3}$ [13,32,33].

Example 3.1. If the Gaussian curvature of rotational hypersurfaces *(3.5)* in the Euclidean 4-space is $G(x) = \frac{-1}{3x^2}$, then it can be parametrized

$$X_G(x,y,z) = (xcosycosz, xsinycosz, xsinz, \pm(-\sqrt{1-(x-1)^{\frac{2}{3}}}(2+(x-1)^{\frac{2}{3}})),$$

where $c_1 = 0, c_2 = 0$ and $1 > (x-1)^{\frac{2}{3}}$.

Figure 1 show the projections of the rotational hypersurface X_G with $G(x) = \frac{-1}{3x^2}$ and $z = \frac{\pi}{6}$ into *yzt*, *xzt*, *xyt* and *xyz*-spaces in (a), (b), (c) and (d), respectively.



(a) (b)

(c) (d)

Figure 1. Projections of the rotational hypersurface for $G(x)$ a) yzt-spaces, b) xzt-spaces, c) xyt-spaces, d) xyz-spaces

3.2. *The solution of mean curvature of rotational hypersurface*

To solve Eq. *(2.11)* which is a second-order nonlinear ordinary differential equation, we take

$$B = \frac{f'(x)}{x\sqrt{1+f'(x)^2}}. \qquad (3.6)$$

From the equations *(2.11)* and *(3.6)*, we have

$$B' = -\frac{3B}{x} - \frac{3H(x)}{x}. \qquad (3.7)$$

The solution of first-order linear ordinary differential equation *(3.7)* is

$$B = \frac{c_3 - 3\int_1^x H(t)t^2dt}{x^3}, \qquad (3.8)$$

where $c_3 \in R$. Also, from equations (3.6) and (3.8), we get

$$\sqrt{(1+f'(x)^2)}(c_3 - 3\int_1^x H(t)t^2dt) = x^2 f'(x). \qquad (3.9)$$

So, the general solution of *(3.9)* is

$$f(x) = \pm\int \frac{c_3 - 3\int_1^x H(t)t^2dt}{\sqrt{x^4 - (c_3 - 3\int_1^x H(t)t^2dt)^2}}dx + c_4,$$

where c_4 is constant and $(c_3 - 3\int_1^x H(t)t^2dt)^2 < x^4$.

Conversely, let *H(x)* be a smooth function defined on an open interval $I \subset R$ and

$$F(x,c_1) = x^4 - \left(c_3 - 3\int_1^x H(t)t^2dt\right)^2 > 0$$

be a function defined on $I_1 \times R \subset R^2$. For any $x_0 \in I$, there exists

$$c_{3_0} = (3\int_1^x H(t)t^2dt)(x_0).$$

So, we can find an open subinterval $x_0 \in I_1 \subset I$ and an open interval $c_{3_0} \in I_2 \subset R$. That is the function $F(x,c_3)$ for any $(x,c_3) \in I_1 \times I_2$. In fact,

$F(x_0, c_{3_0}) = x_0{}^4 > 0$, by the continuity of F, it is positive in a subset of $I_1 \times I_2 \subset R^2$. Therefore, for any $(x,c_3) \in I_1 \times I_2$, $c_2 \in R$ and any given smooth function *H(x)*, we can define the two-parameter family of curves

$$\gamma(x,H,c_3,c_4) = \left(x,0,0, \pm\int \frac{c_3 - 3\int_1^x H(t)t^2dt}{\sqrt{x^4 - (c_3 - 3\int_1^x H(t)t^2dt)^2}}dx + c_4\right).$$

Consequently, we can obtain a two-parameter family of rotational hypersurfaces with the mean curvature *H(x).*

Theorem 3.2. *Let $\gamma(x) = (x,0,0,f(x))$ be a profile curve of the rotational hypersurface (2.10) with the mean curvature at the point $(x,0,0,f(x))$ given by H(x) in the Euclidean 4-space. Then, for some constants c_3 and c_4 there exists the two-parameter family of rotational hypersurface generated by plane curves*

$$\gamma(x,H(x),c_3,c_4) = \left(x,0,0, \pm\int \frac{c_3 - 3\int_1^x H(t)t^2dt}{\sqrt{x^4 - (c_3 - 3\int_1^x H(t)t^2dt)^2}}dx + c_4\right). \qquad (3.10)$$

Let H(x) be a smooth function. The two-parameter family of curves $\gamma(x,H(x),c_3,c_4)$ can be constructed and the two-parameter families of rotational hypersurfaces with the mean curvature can be given by

$$X_H(x,y,z) = (xcosycosz, xsinycosz, xsinz,$$

$$= \pm \int \frac{c_3 - 3\int_1^x H(t)t^2 dt}{\sqrt{x^4 - (c_3 - 3\int_1^x H(t)t^2 dt)^2}} dx + c_4). \qquad (3.11)$$

Corollary 3.3. Let M be the rotational hypersurfaces in E^4 with constant mean curvature ($H(x) = d_2 \in R$). Then M can be parameterized by

$$X_H(x,y,z) = (xcosycosz, xsinycosz, xsinz,$$
$$\pm \int \frac{c_3 - d_2 x^3 + d_2}{\sqrt{x^4 - (c_3 - d_2 x^3 + d_2)^2}} dx + c_4),$$

where $c_3, c_4 \in R$ and $x^4 > (c_3 - d_2 x^3 + d_2)^2$.

Corollary 3.4. Let M be a minimal rotational hypersurfaces in E^4. Then M can be parameterized by

$$X_H(x,y,z) = (xcosycosz, xsinycosz, xsinz, \pm \int \frac{c_3}{\sqrt{x^4 - (c_3)^2}} dx + c_4),$$

where c_3 and $c_4 \in R$ [13,32,33].

Example 3. If the mean curvature of rotational hypersurfaces *(3.11)* in the Euclidean 4-space is $H(x) = \frac{2\sin(x) + x\cos(x)}{-3x}$, then it can be parametrized

$$X_H(x,y,z) = (xcosycosz, xsinycosz, xsinz,$$
$$\pm(-ln(\cos(x)))), \qquad (3.12)$$

where $c_3 = \sin(1)$, $c_4 = 0$ and $\frac{\pi}{2} > x > \frac{-\pi}{2}$.

Figure 2 show the projections of the rotational hypersurface X_H with $H(x) = \frac{2\sin(x) + x\cos(x)}{-3x}$ and $z = \frac{\pi}{6}$ into *yzt*, *xzt*, and *xyt*-spaces in (a), (b) and (c), respectively.

(Here, we take " $\pm$ " in the equation (3.12) as "+".)

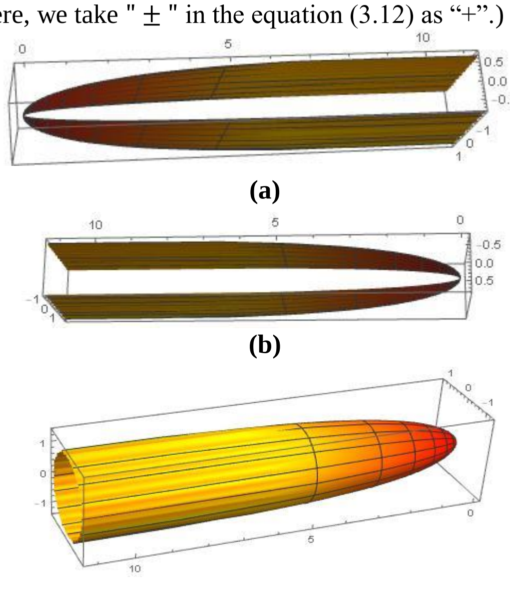


Figure 2. Projections of the rotational hypersurface for $H(x)$ a) yzt-spaces, b) xzt-spaces, c) xyt-spaces

4. ROTATIONAL HYPERSURFACES IN E^4 WITH DENSITY

In this section, the weighted mean and weighted Gaussian curvatures of a rotational hypersurface in 4-D Euclidean space with density will be given. Also, these curvatures which are the second-order non-linear ordinary differential equation will be solved. Besides, the examples of a rotational hypersurface with different weighted Gaussian and weighted mean curvature in E^4 with density will be given.

4.1. Weighted Gaussian Curvatures of Rotational Hypersurfaces in E^4 with Density $e^{ax^2+by^2+cz^2+dt^2}$

From *(1.2)*, *(2.10)* and *(2.11)*, the weighted Gaussian curvature of the rotational hypersurface in Euclidean 4-space with density $e^{ax^2+by^2+cz^2+dt^2}$ is obtained as

$$G_\varphi(x) = -\left(\frac{f'(x)^2 f''(x) + 2x^2(a+b+c+d)\left(1+f'(x)^2\right)^{5/2}}{x^2(1+f'(x)^2)^{5/2}}\right), \quad (4.1)$$

where a, b, c and d are not all zero constants. When calculations similar to ones in the subsection *(3.1)* are carried out, the following theorem is obtained.

Theorem 4.1. *Let $\gamma(x) = (x, 0, 0, f(x))$ be a profile curve of the rotational hypersurface (2.10) with the weighted Gaussian curvature at the point $(x, 0, 0, f(x))$ given by $G_\varphi(x)$ in the Euclidean 4-space with density $e^{ax^2+by^2+cz^2+dt^2}$. Then, for some constants c_5 and c_6 there exists the two-parameter family of rotational hypersurface generated by plane curves*

$$\gamma(x, G_\varphi(x), c_5, c_6) =$$
$$(x, 0, 0, \pm \int \frac{\left(c_5 - 3\int_1^x (G_\varphi(t) + 2(a+b+c+d))t^2 dt\right)^{1/3}}{\sqrt{1 - \left(c_5 - 3\int_1^x (G_\varphi(t) + 2(a+b+c+d))t^2 dt\right)^{2/3}}} dx + c_6) . \qquad (4.2)$$

Conversely, let $G_\varphi(x)$ be a smooth function. Two-parameter family of curves $\gamma(x, G_\varphi(x), c_5, c_6)$ can be constructed and so the two-parameter families of rotational hypersurfaces with the weighted Gaussian curvature can be given by

$$X_{G_\varphi}(x,y,z) = (xcosycosz, xsinycosz, xsinz,$$
$$\pm \int \frac{\left(c_5 - 3\int_1^x (G_\varphi(t) + +2(a+b+c+d))t^2 dt\right)^{1/3}}{\sqrt{1 - \left(c_5 - 3\int_1^x (G_\varphi(t) + +2(a+b+c+d))t^2 dt\right)^{2/3}}} dx + c_6) \qquad (4.3)$$

where c_5 and c_6 is constant and

$$(c_5 - 3\int_1^x (G_\varphi(t) + 2(a+b+c+d))t^2 dt)^{2/3} < 1.$$

Corollary 4.1. Let M be the rotational hypersurfaces in E^4 with density $e^{ax^2+by^2+cz^2+dt^2}$ with constant weighted Gaussian curvature ($G_\varphi(x) = d_3 \in R$). Then M can be parameterized by

$$X_{G_\varphi}(x,y,z) = (xcosycosz, xsinycosz, xsinz,$$
$$\pm \int \frac{\left(c_5 - [2(a+b+c+d) + d_3](x^3 - 1)\right)^{1/3}}{\sqrt{1 - (c_5 - [2(a+b+c+d) + d_3](x^3 - 1))^{2/3}}} dx + c_6),$$

where $c_5, c_6 \in R$ and

$$1 > (c_5 - [2(a+b+c+d) + d_3](x^3 - 1))^{2/3}$$

Example 4.1. Consider rotational hypersurfaces with the weighted Gaussian curvature

$G_\varphi(\mathrm{x}) = \frac{\sin(x)}{3x^2} - 2(a+b+c+d)$, in the Euclidean 4-space with density $e^{ax^2+by^2+cz^2+dt^2}$. So, we get the rotational hypersurfaces in equation *(3.5)* as

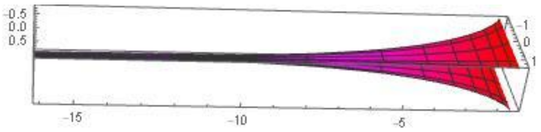
$$X_{G_\varphi}(x,y,z) = (xcosycosz, xsinycosz, xsinz,$$

$$\pm\frac{1}{2}(3\,\mathrm{arcsinh}\left(\frac{1+2(\cos(x))^{\frac{2}{3}}}{\sqrt{3}}\right) + \sqrt{3}(\ln\left(1-(\cos(x))^{\frac{2}{3}}\right) - \ln(3+$$

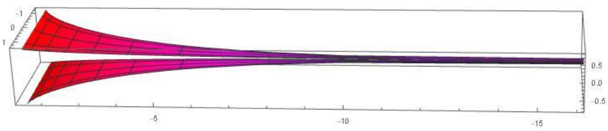
$$3(\cos(x))^{\frac{2}{3}} + 2\sqrt{3(1+(\cos(x))^{\frac{2}{3}}+(\cos(x))^{\frac{4}{3}}})))),$$

where $c_5 = \cos(1)$, $c_6 = 0$.

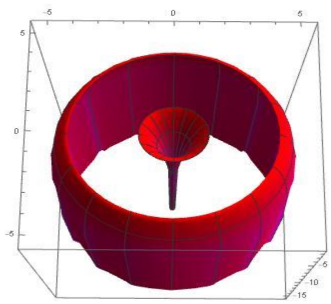
Figure 3 show the projections of the rotational hypersurface X_{G_φ} with the $G_\varphi(\mathrm{x}) = \frac{\sin(x)}{3x^2} - 2(a+b+c+d)$ and $z = \frac{\pi}{6}$ into *yzt*, *xzt*, and *xyt*-spaces in (a), (b) and (c), respectively.



(a)



(b)



(c)

Figure 3. Projections of the rotational hypersurface for $G_\varphi(\mathrm{x})$ a) yzt-spaces, b) xzt-spaces, c) xyt-spaces

4.2. Weighted Mean Curvatures of Rotational Hypersurfaces in E^4 with Density $e^{ax^2+by^2+cz^2}$

From *(1.1)*, *(2.12)* and *(2.13)*, the weighted mean curvature of the rotational hypersurface in Euclidean 4-space with density $e^{ax^2+by^2+cz^2}$ is obtained as

$$H_\varphi = -\left(\frac{A.f'(x)+4xf''(x)+Af'(x)^3}{12x(1+f'(x)^2)^{3/2}}\right), \qquad (4.4)$$

where a, b, c are not all zero constants and

$$A = 8 + 2ax^2 + 2bx^2 + 4cx^2 + 2(a-b)x^2cos[2y]$$
$$+(a-b)x^2Cos[2(y-z)] + 2ax^2cos[2z]$$
$$+2bx^2cos[2z] - 4cx^2cos[2z]$$
$$+ax^2cos[2(y+z)] - bx^2cos[2(y+z)].$$

Especially, if we take *a*=*b*=*c* in the weighted mean curvature *(4.4)* of the rotational hypersurface in Euclidean 4-space with density $e^{ax^2+ay^2+az^2}$, then we obtain

$$H_\varphi = -\left(\frac{2(1+ax^2)f'(x)+xf''(x)+2(1+ax^2)f'(x)^3}{3x(1+f'(x)^2)^{3/2}}\right), \qquad (4.5)$$

where $0 \neq a \in R$.

To solve the second-order nonlinear ordinary differential eq. *(4.5)*, let take

$$C = \frac{f'(x)}{x^2\sqrt{1+f'(x)^2}}. \qquad (4.6)$$

From equations *(4.5)* and *(4.6)*, we have

$$C' = -\left(\frac{4}{x} + 2ax\right)C - \frac{3H_\varphi(x)}{x^2}. \qquad (4.7)$$

The solution of first-order linear ordinary differential equation *(4.7)* is

$$C = \frac{c_7 - 3\int_1^x e^{at^2}H_\varphi(t)t^2dt}{x^4e^{ax^2}}, \qquad (4.8)$$

where $c_7 \in R$. Also, from equations *(4.6)* and *(4.8)*, we get

$$\sqrt{(1+f'(x)^2)}(c_7 - 3\int_1^x e^{at^2}H_\varphi(t)t^2dt) = x^2e^{ax^2}f'(x) \quad . \qquad (4.9)$$

So, the general solution of equation *(4.9)* is

$$f(x) = \pm\int \frac{c_7 - 3\int_1^x e^{at^2}H_\varphi(t)t^2dt}{\sqrt{x^4e^{2ax^2} - (c_7 - 3\int_1^x e^{at^2}H_\varphi(t)t^2dt)^2}}dx + c_8,$$

where c_8 is constant and

$$(c_7 - 3\int_1^x e^{at^2}H_\varphi(t)t^2dt)^2 < x^4e^{ax^2}.$$

Conversely, let H_φ *(x)* be a smooth function defined on an open interval $I \subset R$ and

$$F(x,c_7) = x^4e^{2ax^2} - (c_7 - 3\int_1^x e^{at^2}H_\varphi(t)t^2dt)^2 > 0$$

be a function defined on $I_1 \times R \subset R^2$. For any $x_0 \in I$, there exists

$$c_{7_0} = (3\int_1^x e^{at^2}H_\varphi(t)t^2dt))(x_0).$$

So, we can find an open subinterval $x_0 \in I_1 \subset I$ and an open interval $c_{7_0} \in I_2 \subset R$. That is the function $F(x,c_7)$ for any $(x,c_7) \in I_1 \times I_2$. In fact, because $F(x_0, c_{7_0}) = x^4e^{ax^2} > 0$, by the continuity of *F*, it is positive in a subset of $I_1 \times I_2 \subset R^2$. Therefore, for any $(x,c_7) \in I_1 \times I_2$, $c_2 \in R$ and any given smooth function H_φ (x), we can define the two-parameter family of curves

$$\gamma(x, H_\varphi, c_7, c_8) =$$

$$\left(x, 0, 0, \pm\int \frac{c_7 - 3\int_1^x e^{at^2}H_\varphi(t)t^2dt}{\sqrt{x^4e^{2ax^2} - \left(c_7 - 3\int_1^x e^{at^2}H_\varphi(t)t^2dt\right)^2}}dx + c_8\right).$$

Consequently, we can obtain a two-parameter family of rotational hypersurfaces in E^4 with density $e^{ax^2+ay^2+az^2}$ with the weighted mean curvature $H_\varphi(x)$.

Theorem 4.2. *Let* $\gamma(x) = \big(x, 0, 0, f(x)\big)$ *be a profile curve of the rotational hypersurface (2.10) with the weighted mean curvature at the point* $\big(x, 0, 0, f(x)\big)$ *given by* $H_\varphi(x)$ *in the Euclidean 4-space with density* $e^{ax^2+ay^2+az^2}$. *Then, for some constants* c_7, c_8 *, there exists the two-parameter family of rotational hypersurface generated by plane curves*

$$\gamma\big(x, H_\varphi(x), c_7, c_8\big) =$$

$$\left(x, 0, 0, \pm\int \frac{c_7 - 3\int_1^x e^{at^2} H_\varphi(t)t^2 dt}{\sqrt{x^4 e^{2ax^2} - (c_7 - 3\int_1^x e^{at^2} H_\varphi(t)t^2 dt)^2}} dx + c_8\right). \quad (4.10)$$

Conversely, let $H_\varphi(x)$ *be a smooth function. Then, we can construct the two-parameter family of curves* $\gamma(x, H_\varphi(x), c_3, c_4)$ *and so the two-parameter families of rotational hypersurfaces in* E^4 *with density* $e^{ax^2+ay^2+az^2}$ *with the weighted mean curvature can be given by*

$$X_{H_\varphi}(x, y, z) = (xcosycosz, xsinycosz, xsinz,$$

$$\pm\int \frac{c_7 - 3\int_1^x e^{at^2} H_\varphi(t)t^2 dt}{\sqrt{x^4 e^{2ax^2} - (c_7 - 3\int_1^x e^{at^2} H_\varphi(t)t^2 dt)^2}} dx + c_8). \quad (4.11)$$

Corollary 4.2. Let M be the rotational hypersurfaces in E^4 with density $e^{ax^2+ay^2+az^2}$ with constant weighted mean curvature ($H_\varphi(x) = d_4 \in R$). Then M can be obtain by using Mathematica, as follows

$$\mathrm{X}_{\mathrm{H}_\varphi}(\mathrm{x}, \mathrm{y}, \mathrm{z}) = (\mathrm{xcosycosz}, \mathrm{xsinycosz}, \mathrm{xsinz},$$

$$\pm\int \frac{M}{\sqrt{x^4 e^{2ax^2} - (M)^2}} dx + c_8),$$

where $c_7, c_8 \in R$, $x^4 > (M)^2$ and

$$M = c_7 + \frac{3d_4(2\sqrt{a}(e^a - e^{ax^2}x) + \sqrt{\pi}(-Erfi[\sqrt{a}] + Erfi[\sqrt{a}x]))}{4a^{3/2}}$$

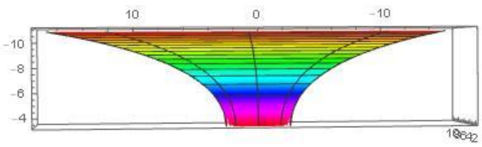
Example 4.2. Consider a rotational hypersurfaces with the weighted mean curvature $H_\varphi(x) = \frac{1+2ax^2}{-3x^2}$ in the Euclidean 4-space with density $e^{ax^2+ay^2+az^2}$. So we get the rotational hypersurfaces in equation *(4.11)* as

$$X_{H_\varphi}(x, y, z) = (xcosycosz, xsinycosz, xsinz,$$

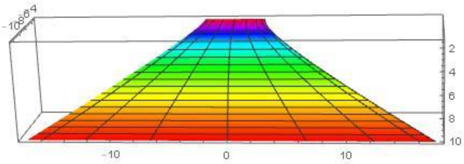
$$\pm\big(ln\big(x + \sqrt{x^2 - 1}\big)\big),$$

where $c_7 = e^a, c_8 = 0$ and $x \geq 1$.

Figure 4 show the projections of the rotational hypersurface X_{H_φ} with the $H_\varphi(\mathrm{x}) = \frac{1+2ax^2}{-3x^2}$ and $z = \frac{\pi}{6}$ into *yzt*, *xzt*, and *xyt*-spaces in (a), (b) and (c), respectively.



(a)



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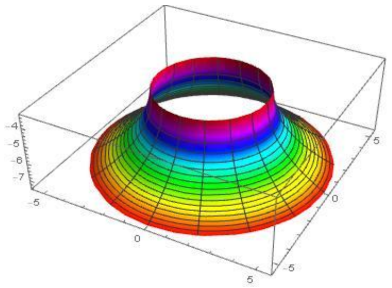

(c)

Figure 4. Projections of the rotational hypersurface for H_φ(x) a) yzt-spaces, b) xzt-spaces, c) xyt-spaces

5. CONCLUSION

The surface theory has an important place in 4-dimensional spaces as in 3-dimensional spaces. So, in the study, we consider the rotational hypersurfaces in Euclidean 4-space with density and obtain the weighted minimal and weighted flat rotational hypersurfaces in this space. We think that the results which are obtained in this study are important for differential geometers who are dealing with weighted surfaces. In fact, the results which are stated in this study better be handled in different four or higher dimensional spaces.

DECLARATION OF ETHICAL STANDARDS

I hereby declare that the materials and methods I use in the work of this article do not require ethics committee approval and / or legal-specific permission.

AUTHORS' CONTRIBUTIONS

Mustafa ALTIN: Prepared the entire article.

CONFLICT OF INTEREST

There is no conflict of interest in this study.

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