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Research Article

Pre-Hausdorff and Hausdorff objects in the category of quantale-valued closure spaces

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Abstract

In previous papers, several T_0 and Hausdorff objects in topological categories are introduced and compared. The main objectives of this paper are to characterize $\overline{T_0}$, T_0, T_1 and pre-$\overline{T_2}$ objects in the category of quantale-valued closure space as well as to examine their mutual relationship.

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1. Introduction

It is an established fact that closure operators play a vital role not only in mathematics including algebra [34], logic [18], calculus [29] and topology [14,23], but also in physics such as quantum logic and representation theory of physical systems [2,3]. In 1940, G. Birkhoff [13] found the relations between the collection of closed sets of a closure space and complete lattice. Afterwards, their interrelations have emerged as the issues of major concerns for mathematicians [16]. Moreover, G. Aumann [4] investigated the closure structures on contact relations which have applications in social sciences.

In 1991, Baran [5] introduced T_0 and T_1 objects in a set-based topological category by using generic element method [21] which is further elucidated in topos theory. Also, he introduced pre-Hausdorff objects in an arbitrary topological category which are reduced to pre-Hausdorff topological space (X, τ), i.e., for each distinct point $x, y \in X$, if the set $\{x, y\}$ is not an indiscrete space, then the points x and y have disjoint neighborhoods [10]. The most important use of pre-Hausdorff objects is to define various forms of Hausdorff objects [11], T_3 and T_4 objects [8], regular, completely regular and normal objects [9] in arbitrary topological categories. In 1994, M. V. Mielke [27] showed that the pre-Hausdorff objects play an important role in general theory of geometric realization. Later, M.V. Mielke [28] showed that pre-Hausdorff objects are important tools for the characterization of the decidable objects in topos theory.

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With the development of fuzzy set theory, many mathematical structures have been equipped with fuzzy sets, such as fuzzy topology [24, 39], fuzzy convexity [32, 33], fuzzy convergence [30,31,40] and so on. Considering the fuzzy counterparts of closure structures, it has been generalized by introducing some suitable quantales on closure structures [24,25]. This motivates us to consider separation properties of the topological category of quantale-valued closure spaces.

The main objectives of this paper are stated as follows:

(i) to characterize $\overline{T_0}$, T_0 and T_1 in the category of quantale-valued closure space,
(ii) to provide the characterization of pre-Hausdorff and several forms of Hausdorff objects in quantale-valued closure space,
(iii) to examine how these separation axioms are related.

2. Preliminaries

In this paper, let $\mathcal{V} = (V, \otimes, k)$ be a (unital, but not necessarily commutative) quantale, i.e., a complete lattice with a monoid structure whose binary operation $\otimes$ satisfies the following properties: for all $\alpha_i, \beta \in V$, $(\bigvee_{i\in I} \alpha_i) \otimes \beta = \bigvee_{i\in I}(\alpha_i \otimes \beta)$ and $\beta \otimes (\bigvee_{i\in I} \alpha_i) = \bigvee_{i\in I}(\beta \otimes \alpha_i)$, where k is an identity (neutral) element.

A quantale $(V, \otimes, k)$ is called an *integral* quantale if $k = \top$.

In a quantale $(V, \otimes, k)$, if $p \in V$ and $p \neq \top$, then p is called the prime element if $\alpha \wedge \beta \leq p$ implies $\alpha \leq p$ or $\beta \leq p$ for all $\alpha, \beta \in V$.

Let X be a nonempty set, PX denotes the power set of X and $\mathcal{V}^X$ denotes the set of all maps from X to $\mathcal{V}$.

Definition 2.1 (cf. [25])**.** A $\mathcal{V}$-valued closure structure on set X is a map $c : PX \longrightarrow \mathcal{V}^X$ satisfying

(i) $\forall x \in A \subseteq X : k \leq (cA)(x)$ (Reflexivity),
(ii) $\forall A, B \subseteq X, x \in X$: $(\bigwedge_{y\in B}(cA)(y)) \otimes (cB)(x) \leq (cA)(x)$ (Transitivity).

The pair (X, c) is called a $\mathcal{V}$-valued closure space.

Definition 2.2 (cf. [25])**.** A $\mathcal{V}$-valued topological structure on set X is a map $c : PX \longrightarrow \mathcal{V}^X$ satisfying

(i) c is a $\mathcal{V}$-valued closure structure on X,
(ii) $\forall x \in X$ and $\emptyset$, the empty set: $(c\emptyset)(x) = \bot$,
(iii) $\forall x \in X$ and $\forall A, B \subseteq X$: $c(A \cup B)(x) = (cA)(x) \vee (cB)(x)$.

The pair (X, c) is called a $\mathcal{V}$-valued topological space.

A map $f : (X, c) \longrightarrow (Y, d)$ is called continuous (or contractive) if $(cA)(x) \leq d(fA)(fx)$ for all $A \subseteq X$ and $x \in X$.

Let **V-Cls** (resp. **V-Top**) denote the category with $\mathcal{V}$-valued closure spaces (resp. $\mathcal{V}$-valued topological spaces) as objects and contractive maps as morphisms. Note that **V-Top** is the full subcategory of **V-Cls**.

Remark 2.3 (cf. [25])**.** A $\mathcal{V}$-valued closure structure c on X satisfies the monotonicity condition, i.e., $\emptyset \neq B \subseteq A \subseteq X \implies cB \leq cA$. Furthermore, if $\mathcal{V}$ is an integral quantale or c is finitely additive, then the restriction $B \neq \emptyset$ is not needed.

Example 2.4. (i) For terminal quantale **1**, **1-Cls** = **1-Top** $\cong$ **Set** [25].

(ii) Consider $\mathcal{V} = (2, \wedge, \top)$, where $2 = \{\bot < \top\}$. Then **2-Cls** $\cong$ **Cls** and **2-Top** $\cong$ **Top** [25], where **Cls** is the category of closure spaces and continuous maps [15] and **Top** is the category of topological spaces and continuous maps.
(iii) If quantale $\mathcal{V} = (([0, \infty], \geq), +, 0)$ (**Lawvere's quantale**) [17], then $\mathcal{V}$-**Top** $\cong$ **App**, where **App** is the category of approach spaces and contraction maps [26]. Moreover, we have $\mathcal{V}$-**Cls** $\cong$ **Cls**$'$, where **Cls**$'$ is the category considered in [37].

(iv) Consider the quantale $\triangle_\& = (\triangle, \otimes, k)$ of all distance distribution functions $\varphi : [0,\infty] \longrightarrow [0,1]$ that satisfy $\varphi(\beta) = \sup_{\alpha<\beta} \varphi(\alpha)$ for all $\beta \in [0,\infty]$ with $(\varphi \otimes \xi)(\gamma) = \sup_{\alpha+\beta<\gamma} \varphi(\alpha)\&\xi(\beta)$, where & is Lukasiewicz operation on $[0,1]$ defined by $\alpha\&\beta = \max\{\alpha+\beta-1, 0\}$. The $\otimes$-neutral function k satisfies $k(0) = 0$ and $k(\alpha) = 1$ for all $\alpha > 0$. Then, $\triangle_\&$**-Top** $\cong$ $\mathbf{ProbApp}_\&$ [24,25] of probabilistic approach spaces defined in [19].

A functor $\mathcal{U} : \mathcal{E} \longrightarrow$ **Set** (the category of sets and functions) is called topological if (i) $\mathcal{U}$ is concrete (i.e., faithful and amnestic), (ii) $\mathcal{U}$ consists of small fibers and (iii) every $\mathcal{U}$-source has a unique initial lift or equivalently, each $\mathcal{U}$-sink has a unique final lift [1,35].

Note that a topological functor which has a left adjoint is called the discrete functor.

Lemma 2.5 (cf. [25]). *Let $\mathcal{V}$ be a quantale, (X_i, c_i) be a collection of $\mathcal{V}$-valued closure spaces and $(f_i : X \longrightarrow (X_i, c_i))_{i\in I}$ be a source. Then, for all $x \in X$ and $A \subseteq X$,*

$$(cA)(x) \quad = \quad \bigwedge_{i\in I} c_i(f_iA)(f_ix)$$

is the initial structure on X.

Lemma 2.6 (cf. [25]). *Let X be a non-empty set and (X, c) be a $\mathcal{V}$-valued closure space.*

(i) *The discrete $\mathcal{V}$-valued closure structure on X is given by*

$$\forall x \in X, \forall A \subseteq X, (c_{dis}A)(x) = \begin{cases} k, & x \in A, \\ \bot, & x \notin A. \end{cases}$$

(ii) *The indiscrete $\mathcal{V}$-valued closure structure on X is given by $(c_{ind}A)(x) = \top$.*

Note that for a quantale $\mathcal{V}$, the category **V-Cls** is a topological category over **Set** [25].

3. T_0 and T_1 Quantale-valued closure spaces

Let X be a non-empty set, $X^2 = X \times X$ and the wedge $X^2 \vee_\triangle X^2$ be two disjoint copies of X^2 identified along with the diagonal. In other words, $X^2 \vee_\triangle X^2$ is the pushout of $\triangle : X \longrightarrow X^2$ along itself. More precisely, if i_1 and $i_2 : X^2 \longrightarrow X^2 \vee_\triangle X^2$ denote the inclusion of X^2 as the first and second factor, respectively, then $i_1\triangle = i_2\triangle$ is the pushout diagram [5].

A point (x, y) in $X^2 \vee_\triangle X^2$ is denoted by $(x, y)_1$ (resp. $(x, y)_2$) if it is in the first (resp. second) component. Note that $(x, y)_1 = (x, y)_2$ iff $x = y$.

Definition 3.1 (cf. [5]). A map $A : X^2 \vee_\triangle X^2 \longrightarrow X^3$ is called a *principal axis* map provided that

$$A(x, y)_i = \begin{cases} (x, y, x), & i = 1, \\ (x, x, y), & i = 2. \end{cases}$$

Definition 3.2 (cf. [5]). A map $S : X^2 \vee_\triangle X^2 \longrightarrow X^3$ is called a *skewed axis* map provided that

$$S(x, y)_i = \begin{cases} (x, y, y), & i = 1, \\ (x, x, y), & i = 2. \end{cases}$$

Definition 3.3 (cf. [5]). A map $\nabla : X^2 \vee_\triangle X^2 \longrightarrow X^2$ is called a *folding* map provided that $\nabla(x, y)_i = (x, y)$ for $i = 1, 2$.

Definition 3.4. Let $U : \mathcal{E} \longrightarrow \textbf{Set}$ be a topological functor and $X \in Ob(\mathcal{E})$ with $U(X) = B$.

(i) X is called $\overline{T_0}$ provided that the initial lift of the U-source $\{A : B^2 \vee_\triangle B^2 \longrightarrow U(X^3) = B^3$ and $\nabla : B^2 \vee_\triangle B^2 \longrightarrow UD(B^2) = B^2\}$ is discrete, where D is the discrete functor [5].

(ii) X is called T_0 provided that X doesn't contain an indiscrete subspace with at least two points [38].

(iii) X is called T_1 provided that the initial lift of the U-source $\{S : B^2 \vee_\triangle B^2 \longrightarrow U(X^3) = B^3$ and $\nabla : B^2 \vee_\triangle B^2 \longrightarrow UD(B^2) = B^2\}$ is discrete [5].

Remark 3.5. In **Top** (the category of topological spaces and continuous maps), $\overline{T_0}$ and T_0 (resp. T_1) are reduced to the following statement: For each $x, y \in X$ with $x \neq y$, there exists a neighborhood of x which doesn't contain y or (resp. and) there exists a neighborhood of y which doesn't contain x [7].

Theorem 3.6. *Let (X, c) be a $\mathcal{V}$-valued closure space. (X, c) is $\overline{T_0}$ if and only if for all $x, y \in X$ with $x \neq y$, there exist $B \subseteq X$ with $x \in B$, $y \notin B$ and $C \subseteq X$ with $y \in C$, $x \notin C$ such that $c(B)(y) \wedge c(C)(x) \wedge k = \bot$, where k is the tensor-neutral element.*

Proof. Suppose (X, c) is $\overline{T_0}$. For all $x, y \in X$ with $x \neq y$, let $\{(x, y)_1\} \subseteq D \subseteq X^2 \vee_\triangle X^2$ and $(x, y)_2 \in X^2 \vee_\triangle X^2$. Note that

$$c_{dis}(\nabla D)(\nabla(x, y)_2) = c_{dis}(\nabla D)(x, y) = k,$$

$$k \leq c(\pi_1 AD)(\pi_1 A(x, y)_2) = c(\pi_1 AD)(x),$$

since $x \in \pi_1 AD$,

$$c(C)(x) = c(\pi_2 AD)(\pi_2 A(x, y)_2) = c(\pi_2 AD)(x)$$

and

$$c(B)(y) = c(\pi_3 AD)(\pi_3 A(x, y)_2) = c(\pi_3 AD)(y).$$

Since $(x, y)_2 \notin \{(x, y)_1\}$ and (X, c) is $\overline{T_0}$, by Lemma 2.5,

$$\begin{aligned}\bot &= \bigwedge \{c_{dis}(\nabla D)(\nabla(x, y)_2), c(\pi_1 AD)(\pi_1 A(x, y)_2),\\ &\qquad c(\pi_2 AD)(\pi_2 A(x, y)_2), c(\pi_3 AD)(\pi_3 A(x, y)_2)\}\\ &= \bigwedge \{k, c(B)(y), c(C)(x)\},\end{aligned}$$

and consequently, $\bigwedge \{k, c(B)(y), c(C)(x)\} = \bot$.

Conversely, let $\bar{c}$ be an initial structure on the wedge $X^2 \vee_\triangle X^2$ induced by $A : X^2 \vee_\triangle X^2 \longrightarrow U(X^3, c^3) = X^3$ and $\nabla : X^2 \vee_\triangle X^2 \longrightarrow U(X^2, c_{dis}) = X^2$, where c^3 is the product $\mathcal{V}$-valued closure structure on X^3, c_{dis} is the discrete $\mathcal{V}$-valued closure structure on $X^2 \vee_\triangle X^2$ and $\pi_j : X^3 \to X$ is the projection map for $j = 1, 2, 3$.

Suppose $u \in X^2 \vee_\triangle X^2$ and D is a non-empty subset of $X^2 \vee_\triangle X^2$.

Case I: If $\nabla u = (x, x) \in \nabla D$ for some $x \in X$, then $u = (x, x)_1$ or $u = (x, x)_2 \in D$, and it follows that $\bar{c}(D)(u) = k$, where k is the tensor neutral element.

Case II: If $\nabla u = (x, x) \notin \nabla D$, then $c_{dis}(\nabla D)(\nabla u) = \bot$ since c_{dis} is the discrete $\mathcal{V}$-valued closure structure and consequently, $\bar{c}(D)(u) = \bot$.

Case III: Suppose $\nabla u = (x, y)$ for some $x, y \in X$ with $x \neq y$ and it follows that $u = (x, y)_i$, $i = 1, 2$.

(i) If $u = (x, y)_1, (x, y)_2 \in D$, then $\nabla u \in \nabla D$ and $\pi_j A u \in \pi_j AD$ for $j = 1, 2, 3$, and consequently, $\bar{c}(D)(u) = k$.

(ii) If $u \notin D$, then $\nabla u = (x, y) \notin \nabla D$, and it follows that

$$c_{dis}(\nabla D)(\nabla u) = c_{dis}(\nabla D)(x, y) = \bot,$$

and consequently, $\bar{c}(D)(u) = \bot$.

(iii) Suppose that $u = (x, y)_1 \notin D$ but $(x, y)_2 \in D$. It follows that

$$c_{dis}(\nabla D)(\nabla(x, y)_1) = c_{dis}(\nabla D)(x, y) = k$$

and

$$k \leq c(\pi_1 AD)(\pi_1 A(x, y)_1) = c(\pi_1 AD)(x).$$

Since $x \in \pi_1 AD$, we have

$$c(B)(y) = c(\pi_2 AD)(\pi_2 A(x, y)_1) = c(\pi_2 AD)(y)$$

and

$$c(C)(x) = c(\pi_3 AD)(\pi_3 A(x, y)_1) = c(\pi_3 AD)(x).$$

By Lemma 2.5, it follows that

$$\begin{aligned} \overline{c}(D)(u) &= \bigwedge \{c_{dis}(\nabla D)(\nabla(x, y)_1), c(\pi_1 AD)(\pi_1 A(x, y)_1), \\ &\qquad c(\pi_2 AD)(\pi_2 A(x, y)_1), c(\pi_3 AD)(\pi_3 A(x, y)_1)\} \\ &= \bigwedge \{k, c(B)(y), c(C)(x)\} = \bot. \end{aligned}$$

By the assumption that $\bigwedge \{k, c(B)(y), c(C)(x)\} = \bot$.

(iv) Similarly, if $u = (x, y)_2 \notin D$ but $(x, y)_1 \in D$, it follows that $\overline{c}(D)(u) = \bot$.

Hence, for all $u \in X^2 \vee_\triangle X^2$ and all non-empty subset D of $X^2 \vee_\triangle X^2$, we have

$$\overline{c}(D)(u) = \begin{cases} k, & u \in D, \\ \bot, & u \notin D. \end{cases}$$

By Lemma 2.6 (i), $\overline{c}$ is the discrete $\mathcal{V}$-valued closure structure on $X^2 \vee_\triangle X^2$. Thus, (X, c) is $\overline{T_0}$. □

Corollary 3.7. *Let (X, c) be a $\mathcal{V}$-valued closure space, where $\mathcal{V}$ is an integral quantale and $\mathcal{V}$ has a prime bottom element. (X, c) is $\overline{T_0}$ if and only if for all $x, y \in X$ with $x \neq y$, there exist $B \subseteq X$ with $x \in B$, $y \notin B$ and $C \subseteq X$ with $y \in C$, $x \notin C$ such that $c(B)(y) = \bot$ or $c(C)(x) = \bot$.*

Proof. It follows from Theorem 3.6 and definitions of prime bottom elements and integral quantales. □

Theorem 3.8. *Let (X, c) be a $\mathcal{V}$-valued closure space. (X, c) is T_0 if and only if for all $x, y \in X$ with $x \neq y$, $c(\{x\})(y) < \top$ or $c(\{y\})(x) < \top$.*

Proof. Suppose (X, c) is T_0. Let $D = \{x, y\}$ and c_D be the initial $\mathcal{V}$-valued closure structure induced by $i : D \longrightarrow (X, c)$. For all $x, y \in X$ with $x \neq y$, $c_D(\{x\})(y) = c(i\{x\})(i(y)) = c(\{x\})(y)$ or $c(\{y\})(x) = c(i\{y\})(i(x)) = c(\{y\})(x)$. By Lemma 2.6 (ii), it follows that $c(\{x\})(y) < \top$ or $c(\{y\})(x) < \top$. Otherwise, $c(\{x\})(y) = \top = c(\{y\})(x)$, and X contains an indiscrete subspace with at least two elements.

Conversely, let for all $x, y \in X$ with $x \neq y$, $c(\{x\})(y) < \top$ or $c(\{y\})(x) < \top$. Suppose D is an indiscrete subspace of X with at least two elements and $x, y \in D$ with $x \neq y$. Let c_D be the initial $\mathcal{V}$-valued closure structure induced by $i : D \longrightarrow (X, c)$. It follows immediately that $\top = c_D(\{x\})(y) = c(i\{x\})(i(y)) = c_D(\{x\})(y)$ and $\top = c_D(\{y\})(x) = c(i\{y\})(i(x)) = c_D(\{y\})(x)$, and consequently, $c(\{x\})(y) = \top = c(\{y\})(x)$, a contradiction to our assumption. Therefore, X doesn't contain an indiscrete subspace with at least two elements. Hence, by Definition 3.4 (ii), (X, c) is T_0. □

Theorem 3.9. *Let (X, c) be a $\mathcal{V}$-valued closure space. (X, c) is T_1 if and only if for all $x, y \in X$ with $x \neq y$, there exist $B \subseteq X$ with $x \in B$, $y \notin B$ and $C \subseteq X$ with $y \in C$, $x \notin C$ such that $c(B)(y) \wedge k = \bot = c(C)(x) \wedge k$, where k is the tensor-neutral element.*

Proof. Suppose (X,c) is T_1. For all $x,y\in X$ with $x\neq y$, let $\{(x,y)_1\}\subseteq D\subseteq X^2\vee_\triangle X^2$ and $(x,y)_2\in X^2\vee_\triangle X^2$. Note that

$$c_{dis}(\nabla D)(\nabla(x,y)_2)=c_{dis}(\nabla D)(x,y)=k,$$
$$k\leq c(\pi_1SD)(\pi_1S(x,y)_2)=c(\pi_1SD)(x),$$

since $x\in\pi_1SD$,

$$c(C)(x)=c(\pi_2SD)(\pi_2S(x,y)_2)=c(\pi_2SD)(x)$$

and

$$k\leq c(\pi_3SD)(\pi_3S(x,y)_2)=c(\pi_3SD)(y),$$

since $y\in\pi_3SD$. By the assumption that (X,c) is T_1 and by Lemma 2.5,

$$\begin{aligned}\bot &= \bigwedge\{c_{dis}(\nabla D)(\nabla(x,y)_2),c(\pi_1SD)(\pi_1S(x,y)_2),\\ &\qquad c(\pi_2SD)(\pi_2S(x,y)_2),c(\pi_3SD)(\pi_3S(x,y)_2)\}\\ &= \bigwedge\{k,c(C)(x)\},\end{aligned}$$

and consequently, $c(C)(x)\wedge k=\bot$.

Similarly, if $\{(x,y)_2\}\subseteq D\subseteq X^2\vee_\triangle X^2$ and $(x,y)_1\in X^2\vee_\triangle X^2$, then

$$\begin{aligned}\bot &= \bigwedge\{c_{dis}(\nabla D)(\nabla(x,y)_1),c(\pi_jSD\})(\pi_jS(x,y)_1),j=1,2,3\}\\ &= \bigwedge\{k,c(B)(y)\},\end{aligned}$$

and consequently, $c(B)(y)\wedge k=\bot$.

Conversely, let $\overline{c}$ be an initial structure on the wedge $X^2\vee_\triangle X^2$ induced by $S:X^2\vee_\triangle X^2\longrightarrow U(X^3,c^3)=X^3$ and $\nabla:X^2\vee_\triangle X^2\longrightarrow U(X^2,c_{dis})=X^2$, where c^3 is the product $\mathcal{V}$-valued closure structure on X^3, c_{dis} is the discrete $\mathcal{V}$-valued closure structure on $X^2\vee_\triangle X^2$ and $\pi_j:X^3\to X$ is the projection map for $j=1,2,3$.

Let $u\in X^2\vee_\triangle X^2$ and D be a non-empty subset of $X^2\vee_\triangle X^2$, and for all $x,y\in X$ with $x\neq y$, there exist $B\subset X$ with $x\in B$, $y\notin B$ and $C\subset X$ with $y\in C$, $x\notin C$ such that $c(B)(y)\wedge k=\bot=c(C)(x)\wedge k$.

Case I: If $\nabla u=(x,x)\in\nabla D$ for some $x\in X$, then $u=(x,x)_1$ or $u=(x,x)_2\in D$ and consequently, $\overline{c}(D)(u)=k$, where k is the tensor neutral element.

Case II: If $\nabla u=(x,x)\notin\nabla D$, then $c_{dis}(\nabla D)(\nabla u)=\bot$ since c_{dis} is the discrete $\mathcal{V}$-valued closure structure and consequently, $\overline{c}(D)(u)=\bot$.

Case III: Suppose $\nabla u=(x,y)$ for some $x,y\in X$ with $x\neq y$ and it follows that $u=(x,y)_i$, $i=1,2$.

(i) If $u=(x,y)_i\in D$ for $i=1,2$, then $\nabla u\in\nabla D$ and $\pi_jSu\in\pi_jSD$ for $j=1,2,3$, and consequently,

$$\overline{c}(D)(u) = \bigwedge\{c_{dis}(\nabla D)(\nabla u),c(\pi_jSD)(\pi_jSu):j=1,2,3\}=k.$$

(ii) If $u\notin D$, then $\nabla u=(x,y)\notin\nabla D$, and it follows that

$$c_{dis}(\nabla D)(\nabla u)=c_{dis}(\nabla D)(x,y)=\bot,$$

and consequently, $\overline{c}(D)(u)=\bot$.

(iii) Suppose that $u=(x,y)_1\notin D$ but $\{(x,y)_2\}\in D$. It follows that

$$c_{dis}(\nabla D)(\nabla u)=c_{dis}(\nabla D)(\nabla(x,y)_1)=k,$$
$$k\leq c(\pi_1SD)(\pi_1Su)=c(\pi_1SD)((\pi_1S(x,y)_1)=c(\pi_1SD)((x)$$

since $x\in\pi_1SD$,

$$c(B)(y)=c(\pi_2SD)(\pi_2Su)=c(\pi_2SD)(\pi_2S(x,y)_1)=c(\{x\})(y)$$

and

$$k \leq c(\pi_3 SD)(\pi_3 Su) = c(\pi_3 SD)(\pi_3 S(x,y)_1) = c(\pi_3 SD)(y)$$

since $y \in \pi_3 SD$. By Lemma 2.5,

$$\begin{aligned}\overline{c}(D)(u) &= \bigwedge \{c_{dis}(\nabla D)(\nabla(x,y)_1), c(\pi_1 D)(\pi_1 S(x,y)_1), \\ &\qquad c(\pi_2 SD)(\pi_2 S(x,y)_1), c(\pi_3 SD)(\pi_3 S(x,y)_1)\} \\ &= \bigwedge \{k, c(B)(y)\} = \bot\end{aligned}$$

since $c(B)(y) \wedge k = \bot$.

(iv) Similarly, if $u = (x,y)_2 \notin D$ but $(x,y)_1 \in D$, then

$$\begin{aligned}\overline{c}(D)(u) &= \bigwedge \{c_{dis}(\nabla D)(\nabla(x,y)_2), c(\pi_1 D)(\pi_1 S(x,y)_2), \\ &\qquad c(\pi_2 SD)(\pi_2 S(x,y)_2), c(\pi_3 SD)(\pi_3 S(x,y)_2)\} \\ &= \bigwedge \{k, c(C)(x)\} = \bot\end{aligned}$$

since $k \wedge c(C)(x) = \bot$, and consequently, $\overline{c}(D)(u) = \bot$.

Hence, for all $u \in X^2 \vee_\triangle X^2$ and all non-empty subset D of $X^2 \vee_\triangle X^2$, we have

$$\overline{c}(D)(u) = \begin{cases} k, & u \in D, \\ \bot, & u \notin D. \end{cases}$$

By Lemma 2.6 (i), $\overline{c}$ is the discrete $\mathcal{V}$-valued closure structure on $X^2 \vee_\triangle X^2$. Thus, (X,c) is T_1. □

Corollary 3.10. *Let (X,c) be a $\mathcal{V}$-valued closure space, where $\mathcal{V}$ is an integral quantale. (X,c) is T_1 if and only if for all $x,y \in X$ with $x \neq y$, there exist $B \subseteq X$ with $x \in B$, $y \notin B$ and $C \subseteq X$ with $y \in C$, $x \notin C$ such that $c(B)(y) = \bot = c(C)(x)$.*

Proof. It follows from Theorem 3.9 and the definition of integral quantales. □

Example 3.11. Consider $V = [0,1]$ (the real unit interval) with $\leq$ as the partial order, the product $\cdot$ as the quantale operation and 1 as the identity element. Then $\mathcal{V} = (([0,1], \leq), \cdot, 1)$ is a quantale. Let $X = \{a,b,c\}$ and $c : P(X) \longrightarrow \mathcal{V}^X$ be a map defined by for all $x \in X$ and all non-empty subset A of X,

$$(cA)(x) = \begin{cases} 1, & x \in A, \\ 1/3, & x \notin A. \end{cases}$$

Clearly, (X,c) is a $\mathcal{V}$-valued closure space. By Theorem 3.8, (X,c) is T_0 but by Theorems 3.6 and 3.9, (X,c) is neither $\overline{T_0}$ nor T_1.

4. Pre-Hausdorff and Hausdorff quantale-valued closure spaces

Definition 4.1. Let $U : \mathcal{E} \longrightarrow$ **Set** be a topological functor and $X \in Ob(\mathcal{E})$ with $U(X) = B$.

(i) X is called Pre-$\overline{T_2}$ provided that the initial lifts of U-sources $\{A : B^2 \vee_\triangle B^2 \longrightarrow U(X^3) = B^3$ and $S : B^2 \vee_\triangle B^2 \longrightarrow U(X^3) = B^3\}$ coincide [5, 10].

(ii) X is called $\overline{T_2}$ provided that X is $\overline{T_0}$ and Pre-$\overline{T_2}$ [5, 11].

(iii) X is called NT_2 provided that X is T_0 and Pre-$\overline{T_2}$ [11].

Remark 4.2. In **Top** (the category of topological spaces and continuous maps), $\overline{T_2}$ and NT_2 are reduced to Hausdorff topological space (X,τ), i.e., for each $x,y \in X$ with $x \neq y$, there exists a neighborhood U_x of x which doesn't contain y and there exists a neighborhood U_y of y which doesn't contain x such that $U_x \cap U_y = \emptyset$ [11].

Theorem 4.3. *Let (X,c) be a $\mathcal{V}$-valued closure space, where $\mathcal{V}$ is an integral quantale. (X,c) is Pre-$\overline{T_2}$ if and only if for all $x,y \in X$ with $x \neq y$, there exist $B \subseteq X$ with $x \in B$, $y \notin B$ and $C \subseteq X$ with $y \in C$, $x \notin C$ such that*

$$c(B)(y) \wedge c(C)(x) = c(B)(y) = c(C)(x).$$

Proof. Suppose (X,c) is Pre-$\overline{T_2}$. Let $\pi_j : X^3 \longrightarrow X$, $j = 1,2,3$ be the projection map. For all $x,y \in X$ with $x \neq y$, let $u = (x,y)_1 \in X^2 \vee_\triangle X^2$ and $\{(x,y)_2\} \subseteq D \subseteq X^2 \vee_\triangle X^2$. Note that

$$c(\pi_1 AD)(\pi_1 A(x,y)_1) = c(\pi_1 AD)(x) = k = \top = c(\pi_1 SD)(\pi_1 S(x,y)_1)$$

since $x \in \pi_1 AD$ and $x \in \pi_1 SD$,

$$c(\pi_2 AD)(\pi_2 A(x,y)_1) = c(\pi_2 AD)(y)$$

since $y \notin \pi_2 AD$ and $x \in \pi_2 AD$,

$$c(\pi_2 SD)(\pi_2 S(x,y)_1) = c(\pi_2 SD)(y)$$

since $y \notin \pi_2 SD$ and $x \in \pi_2 SD$. It follows that

$$c(\pi_2 AD)(\pi_2 A(x,y)_1) = c(\pi_2 AD)(y) = c(\pi_2 SD)(\pi_2 S(x,y)_1) = c(B)(y),$$
$$c(C)(x) = c(\pi_3 AD)(\pi_3 A(x,y)_1) = c(\pi_3 AD)(x)$$

and

$$c(\pi_3 SD)(\pi_3 S(x,y)_1) = c(\pi_3 SD)(y) = k = \top$$

since $y \in \pi_3 SD$. This implies

$$\begin{aligned}\bigwedge\{c(\pi_j AD)(\pi_j A(x,y)_1); j = 1,2,3\} &= \bigwedge\{c(\pi_1 AD)(x), c(\pi_2 AD)(y), c(\pi_3 AD)(x)\} \\ &= \bigwedge\{c(B)(y), c(C)(x)\}.\end{aligned}$$

Similarly,

$$\begin{aligned}\bigwedge\{c(\pi_j SD)(\pi_j S(x,y)_1); j = 1,2,3\} &= \bigwedge\{c(\pi_1 SD)(x), c(\pi_2 SD)(y), c(\pi_3 SD)(y)\} \\ &= c(B)(y).\end{aligned}$$

Since (X,c) is Pre-$\overline{T_2}$, we have

$$\bigwedge\{c(\pi_j AD)(\pi_j A(x,y)_1); j = 1,2,3\} = \bigwedge\{c(\pi_j SD)(\pi_j S(x,y)_1); j = 1,2,3\},$$

and consequently,

$$\bigwedge\{c(B)(y), c(C)(x)\} = c(B)(y).$$

Let $u = (x,y)_2 \in X^2 \vee_\triangle X^2$ and $\{(x,y)_1\} \subseteq D \subseteq X^2 \vee_\triangle X^2$. By a similar verification, we have $\bigwedge\{c(B)(y), c(C)(x)\} = c(C)(x)$ and consequently,

$$\bigwedge\{c(B)(y), c(C)(x)\} = c(C)(x) = c(B)(y).$$

Conversely, let $\overline{c}_A$ and $\overline{c}_S$ be the two initial $\mathcal{V}$-valued closure structures on $X^2 \vee_\triangle X^2$ induced by $A : X^2 \vee_\triangle X^2 \longrightarrow U(X^3, c^3) = X^3$ and $S : X^2 \vee_\triangle X^2 \longrightarrow U(X^3, c^3) = X^3$ respectively, where c^3 is the product $\mathcal{V}$-valued closure structure on X^3 induced by the projection map $\pi_j : X^3 \longrightarrow X$ for $j = 1,2,3$. We need to show that for all $u \in X^2 \vee_\triangle X^2$ and all non-empty subset D of $X^2 \vee_\triangle X^2$, $\overline{c}_A(D)(u) = \overline{c}_S(D)(u)$.

Case (I): If $u \in D$, then $\overline{c}_A(D)(u) = \overline{c}_S(D)(u)$ since $\overline{c}_A(D)(u) = \overline{c}_S(D)(u) = k = \top$.

Case (II): Suppose $u \notin D$ and they are in the same component of $X^2 \vee_\triangle X^2$. This implies that $u = (x,y)_i$, and $\{(z,w)_i\} \subseteq D$ for $i = 1,2$, where $x,y,z,w \in X$. For $i = 1$, we have

$$c(\pi_1 AD)(\pi_1 Au) = c(\pi_1 AD)(x),$$
$$c(\pi_2 AD)(\pi_2 Au) = c(\pi_2 AD)(\pi_2 A(x,y)_1) = c(\pi_2 AD)(y),$$

and

$$c(\pi_3 AD)(\pi_3 Au) = c(\pi_3 AD)(\pi_3 A(x,y)_1) = c(\pi_3 AD)(x).$$

Note that

$$\begin{aligned}\overline{c}_A(AD)(Au) &= \bigwedge\{c(\pi_j AD)(\pi_j Au) : j = 1,2,3\} \\ &= \bigwedge\{c(\pi_1 AD)(\pi_1 A(x,y)_1), c(\pi_2 AD)(\pi_2 A(x,y)_1), \\ &\qquad c(\pi_3 AD)(\pi_3 A(x,y)_1)\} \\ &= \bigwedge\{c(\pi_1 AD)(x), c(\pi_2 AD)(y)\}.\end{aligned}$$

and

$$\begin{aligned}\overline{c}_S(SD)(Su) &= \bigwedge\{c(\pi_j SD)(\pi_j Su) : j = 1,2,3\} \\ &= \bigwedge\{c(\pi_1 SD)(x), c(\pi_2 SD)(y)\}.\end{aligned}$$

This implies $\overline{c}_A(AD)(Au) = \overline{c}_S(SD)(Su)$.

For $i = 2$, it follows that

$$\overline{c}_A(AD)(Au) = \overline{c}_A(AD)(A(x,y)_2) = \overline{c}_S(SD)(S(x,y)_2) = \overline{c}_S(SD)(Su).$$

Case (III): Suppose $u \notin D$ and they are in the different components of $X^2 \vee_\triangle X^2$. We have the following cases.

(a) If $u = (x,y)_1$ or $(y,x)_1$ and $\{(x,y)_2\} \subseteq D$ or $\{(y,x)_2\} \subseteq D$ for all $x \neq y$.

Suppose $u = (x,y)_1$ and $\{(x,y)_2\} \subseteq D$ (resp. $\{(y,x)_2\} \subseteq D$). Then by Remark 2.5, it follows that

$$\begin{aligned}\overline{c}_A(AD)(Au) &= \bigwedge\{C(\pi_j AD)(\pi_j Au) : j = 1,2,3\} \\ &= \bigwedge\{c(\pi_1 AD)(\pi_1 A(x,y)_1), c(\pi_2 AD)(\pi_2 A(x,y)_1), c(\pi_3 AD)(\pi_3 A(x,y)_1)\} \\ &= \bigwedge\{c(\pi_2 AD)(y), c(\pi_3 AD)(x), \top\} \\ &= c(B)(y) \wedge c(C)(x) \quad (resp.\ c(B)(y))\end{aligned}$$

and

$$\begin{aligned}\overline{c}_S(SD)(Su) &= \bigwedge\{C(\pi_j SD)(\pi_j Su) : j = 1,2,3\} \\ &= \bigwedge\{c(\pi_1 SD)(\pi_1 S(x,y)_1), c(\pi_2 SD)(\pi_2 S(x,y)_1), (\pi_3 SD)(\pi_3 S(x,y)_1)\} \\ &= \bigwedge\{\top, c(B)(y)\} \\ &= c(B)(y) \quad (resp.\ c(B)(y) \wedge c(C)(x)).\end{aligned}$$

By the assumption, we have $\overline{c}_A(AD)(Au) = \overline{c}_S(SD)(Su)$.

Similarly, if $u = (y,x)_1$ and $\{(x,y)_2\} \subseteq D$ or $\{(y,x)_2\} \subseteq D$ for all $x \neq y$. It follows that $\overline{c}_A(AD)(Au) = \overline{c}_S(SD)(Su)$.

(b) If $u = (x,y)_2$ or $(y,x)_2$ and $\{(x,y)_1\} \subseteq D$ or $\{(y,x)_1\} \subseteq D$ for all $x \neq y$.

Let $u = (x,y)_2$ and $\{(x,y)_1\} \subseteq D$, (resp. $\{(y,x)_1\} \subseteq D$). Then it follows from Remark 2.5 that

$$\begin{aligned}\overline{c}_A(AD)(Au) &= \bigwedge\{C(\pi_j AD)(\pi_j Au) : j = 1,2,3\} \\ &= \bigwedge\{c(\pi_1 AD)(\pi_1 A(x,y)_2), c(\pi_2 AD)(\pi_2 A(x,y)_2), c(\pi_3 AD)(\pi_3 A(x,y)_2)\} \\ &= \bigwedge\{\top, c(\pi_2 AD)(x), c(\pi_3 AD)(y)\} \\ &= c(B)(y) \wedge c(C)(x) \quad (resp.\ c(C)(x))\end{aligned}$$

and

$$\begin{aligned}\bar{c}_S(SD)(Su) &= \bigwedge\{C(\pi_j SD)(\pi_j Su) : j = 1, 2, 3\} \\ &= \bigwedge\{c(\pi_1 SD)(\pi_1 S(x,y)_2), c(\pi_2 SD)(\pi_2 S(x,y)_2), c(\pi_3 SD)(\pi_3 S(x,y)_2)\} \\ &= \bigwedge\{\top, c(C)(x)\} \\ &= c(C)(x) \quad (resp.\ c(B)(y) \wedge c(C)(x)).\end{aligned}$$

Similarly, if $u = (y, x)_2$ and $\{(x, y)_1\} \subseteq D$ or $\{(y, x)_1\} \subseteq D$ for all $x \neq y$, then it follows that $\bar{c}_A(AD)(Au) = \bar{c}_S(SD)(Su)$.

(c) For any three (resp. four) distinct points x, y, z (resp. w) $\in X$, similar to above cases, $\bar{c}_A(AD)(Au) = \bar{c}_S(SD)(Su)$.

Therefore, for all $u \in X^2 \vee_\triangle X^2$ and all non-empty subset D of $X^2 \vee_\triangle X^2$, $\bar{c}_A(D)(u) = \bar{c}_S(D)(u)$. Thus, (X, c) is Pre-$\overline{T_2}$. □

Theorem 4.4. *Let $\mathcal{V}$ be an integral quantale, and let (X, c) be a $\mathcal{V}$-valued closure space. (X, c) is $\overline{T_2}$ if and only if (X, c) is a discrete $\mathcal{V}$-valued closure space.*

Proof. It follows from the definition of integral quantales, Definition 4.1 (ii), Lemma 2.6 (i) and Theorems 3.6 and 4.3. □

Theorem 4.5. *Let $\mathcal{V}$ be an integral quantale, and let (X, c) be a $\mathcal{V}$-valued closure space. The followings are equivalent.*

(i) *(X, c) is T_1.*
(ii) *(X, c) is $\overline{T_2}$.*
(iii) *(X, c) is a discrete $\mathcal{V}$-valued closure space.*

Proof. The proof follows from Lemma 2.6 (i), and Theorems 3.9 and 4.4. □

Theorem 4.6. *Let $\mathcal{V}$ be an integral quantale, and let (X, c) be a $\mathcal{V}$-valued closure space. (X, c) is NT_2 if and only if there exist $x, y \in X$ with $x \neq y$,*

$$c(\{y\})(x) = c(\{x\})(y) < \top.$$

Proof. It follows from Definition 4.1 (iii) and Theorems 3.8 and 4.3. □

Remark 4.7. (I) For any arbitrary topological category, there is no relation between T_0 and $\overline{T_0}$, and between $\overline{T_2}$ and NT_2. For example,

(a) In category **Cls** of closure spaces and continuous maps, Pre-$\overline{T_2} = NT_2 = \overline{T_2} \Rightarrow T_1 = \overline{T_0} \Rightarrow T_0$ [12].

(b) In category **CHY** of of Cauchy spaces and Cauchy continuous maps, $T_0 = \overline{T_0} = T_1 = \overline{T_2} \implies$ Pre-$\overline{T_2}$ [22].

(c) In **ConFCO** (the category of constant filter convergence spaces and continuous maps), $\overline{T_2} = NT_2 \Rightarrow T_0 = \overline{T_0} = T_1$ but in **ConLFCO** (the category of constant local filter convergence spaces and continuous maps), $T_0 \implies \overline{T_0} = T_1$ and $T_0 = NT_2 \implies \overline{T_2}$ [6].

(d) In **L-App** (category of L-gauge space (resp. L-distance approach space) and contraction maps) [20], local T_1, i.e., T_1 at p implies local $\overline{T_0}$, i.e., $\overline{T_0}$ at p [36].

(II) In **V-Cls** with $\mathcal{V}$ as an integral quantale, by Theorems 3.6-3.9 and 4.5, $\overline{T_2} = T_1 \implies \overline{T_0} \implies T_0$ but converse is not true in general by Example 3.11. Moreover, by Theorems 4.3-4.6, if (X, c) is $\overline{T_2}$, then it is Pre-$\overline{T_2}$ and NT_2.

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