Available online: May 31, 2020

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 69, Number 1, Pages 910–928 (2020)
DOI: 10.31801/cfsuasmas.478265
ISSN 1303–5991 E-ISSN 2618-6470

http://communications.science.ankara.edu.tr

FORECASTING MORTALITY RATES WITH A GENERAL STOCHASTIC MORTALITY TREND MODEL

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Abstract. This paper presents a model, which can closely predict the future mortality rates whose efficiency is performed through the comparisons with respect to Lee-Carter and mortality trend models. This general model estimates the logit function of death rate in terms of general tendency of the mortality evolution independent of age, the mortality steepness, additional effects of childhood, youth and old age. Generalized linear model (GLM) is used to estimate the parameters. Moreover, the weighted least square (WLS) and random walk with drift (RWWD) methods are employed to project the future values of the parameters. In order to ensure the stability of the outputs and construct the confidence intervals, Monte Carlo simulation is used. The impact of the proposed model is implemented on USA, France, Italy, Japan and Israel mortality rates for both genders based on their ageing structure. A detailed comparison study is performed to illustrate modified mortality rates on the net single premiums over mortality trend model and Lee-Carter model.

1. Intoduction

Having well-constructed mortality tables and accurate future mortality rate projections are important in many areas. In the case of under or over estimation of the rates, unpredictable losses can be experienced, especially, by the life and pension companies. For this reason, modeling mortality rates accurately has gained importance in recent decades, and been used especially to measure the longevity risk. The future values of the mortality rates are projected by many methods. The main idea of these studies is to model or systematize mortality rates from the past to the future so that the actuarial calculations become proper for both present and the future. In the literature, Lee and Li (2005) proposed a multi-population mortality modeling as an extension of Lee-Carter method which is accounted as a stochastic model [10]. Also, Jarner and Kryger (2011) studied a multi-population mortality model which includes long-term trend and short- to mid-term deviations

Received by the editors: January 10, 2020; Accepted: March 06, 2020.

2010 *Mathematics Subject Classification.* 91D20, 82-05.

Key words and phrases. Mortality trend model, Lee-Carter, GLM, simulation, random walk, weighted least squares.

using time series model [8]. An application of Canada and US female mortality has been conducted by Li and Hardy (2011) with respect to basis risk in longevity index hedges [9] incorporating four extensions to the Lee-Carter model. Börger (2010) proposed one-year period longevity risk by considering the adequacy of Solvency II scenarios [1]. The adequacy of longevity shock has been analyzed by comparing the resulting capital requirement to the Value-at-Risk (VaR) based on a stochastic mortality model. On the other hand, Plat (2011) modeled the changes of long-term mortality trend from the aspects of mortality and longevity risk proposing one-year VaR measure which aims at covering the risk of the variation in the projection year as well as the risk of changes in the best estimate projection for future years [11]. Another remarkable study was conducted by Richards et al. (2014) using different methods like Lee-Carter and Cairns-Blake-Dowd in order to determine one-year period longevity risk [12]. Börger et al. (2014) proposed a new mortality trend model, which contributed to a better quantification of mortality and longevity risk over time, under modern solvency regimes [2]. In their work, mortality trend model represents young and old age effects more precisely: They use three variables separately for each group of age. The outputs of this model are employed to compare the capital requirements with respect to Solvency II standard formula.

Constructing a model that includes additional effects of specific age groups is crucial for many pension systems, since different age groups have different effects on the trend. If the parameters of specific age effects are not used, the model becomes less sensitive to inner trends of each age groups. As the proportion of a specific age group whose inner trend is not well represented increase, the gap between installments and the compensations increase, too. For this reason, in this paper, we aim (i) to create a mortality trend model which includes both stochastic and deterministic terms to project future mortality rates accurately; (ii) to propose a modified trend model which includes the impact of young ages which is crucial, especially for populations having higher proportions at younger ages; (iii) to incorporate more stochastic structure to capture the stylized facts of mortality trend by modifying the model proposed by Börger (2014). We show the effect of these models on the valuation of net single premiums. The inclusion of the childhood effect parameter as modification to the linear model having the impact of old, center and young ages on mortality rates is expected to give more sensitive estimation of mortality rates. In this aspect, to our best knowledge, this study contributes to the literature of the quantification of influence of childhood effect by determining the threshold age to describe the childhood based on population dynamics. Validation of modified mortality trend model is performed using Mean Absolute Percentage Error (MAPE), R-Squared and applied to mortality tables from selected nationalities: USA, France, Italy, Japan and Israel. The choice of these countries is made according to their ageing structure such as young, middle and elderly populations. The results of the mortality trend model, the modified mortality trend model and Lee-Carter model are compared to demonstrate the efficiency on predictions.

This paper is organized as follows: Trend and modified trend models are presented in Section 2 along with the outline of the proposed model. Section 3 includes the application of the modified model on the mortality tables of the selected countries. Parameter estimation, projection of future mortality rates and comparison of the methods are also performed. The impact of the models on net single premium valuations is determined in Section 4. Last section concludes the paper.

2. Trend Models

The popularity of the trend models has been increased during the last decades with the Lee-Carter model [3]. There are also other trend models such as Heligman-Pollard [6], Cairns-Blake-Dowd (CBD) [14] which have significant contributions in this field. In this study, Lee-Carter is taken as benchmark model for the comparison, as it is the most commonly used method in the literature and practices.

2.1. **Lee-Carter (LC).** An extended version of the LC mortality rate model proposed by Girosi & King (2007) is [4]:

$$\log[q_{xt}] = \alpha_x + \beta_x \kappa_t + \varepsilon_{xt}, \tag{1}$$

where $q_{x,t}$ is the central rate of mortality at age x and in year t, α_x is the general tendency in the trend of mortality rates which depends on age, β_x explains the rate of decline in response, κ_t is a level of mortality index and the random error ε_{xt} follows a Normal distribution with mean 0 and variance σ_ε^2 with the following restrictions

$$\sum \beta_x = 1 \quad \text{and} \quad \sum \kappa_t = 0. \tag{2}$$

Here, x and t are age and time components, respectively. Singular Value Decomposition (SVD) is used in order to estimate β_x and κ_t parameters.

2.2. **Mortality trend model (MTM).** The mortality rate, $q_{x,t}$, is expressed as [2]:

$$\operatorname{logit} q_{x,t} = \alpha_x + \kappa_t^{(1)} + \kappa_t^{(2)}(x - x_{center}) + \kappa_t^{(3)}(x_{young} - x)^+ + \kappa_t^{(4)}(x - x_{old})^+ + \gamma_{t-x} \tag{3}$$

where $\operatorname{logit}(q_{x,t}) = \ln(\frac{q_{x,t}}{1-q_{x,t}})$ and $x^+ = \max\{x, 0\}$. Here, α_x is the location parameter, $\kappa^{(1)}$, $\kappa^{(2)}$, $\kappa^{(3)}$ and $\kappa^{(4)}$ are time dependent parameters, which represent the trend of ages, and γ is a normally distributed random error.

Börger (2014) states that including ages smaller than 20 disturbs the general trend of the mortality rate evolution and it leads to an increase in the empirical errors of the model, since mortality rates decreases over years at the childhood ages contrary to the rest of the ages [2]. Hence, small ages are generally not taken into account in the most of the similar studies. In order to increase the sensitivity to younger ages, we propose adding another additional effect parameter: childhood effect. Therefore, modified mortality trend model (M-MTM) should include the ages smaller than 20 in the analysis preventing such disturbance.

2.3. **Modified mortality trend model (M-MTM).** The model with childhood effect parameter is given as:

$$\begin{aligned}\operatorname{logit} q_{x,t} = \alpha_x + \kappa_t^{(1)} + \kappa_t^{(2)}(x - x_{center}) + \kappa_t^{(3)}(x_{child} - x)\mathbb{I}_{(x<x_{child})} \\ +\kappa_t^{(4)}(x_{young} - x)\mathbb{I}_{(x_{child}<x<x_{young})} + \kappa_t^{(5)}(x - x_{old})\mathbb{I}_{(x>x_{old})} + \gamma_{t-x}\end{aligned} \tag{4}$$

M-MTM includes another time dependent parameter $\kappa^{(5)}$ at which, indicator functions, $\mathbb{I}$, are used to define valid ranges for $\kappa^{(3)}$, $\kappa^{(4)}$, and $\kappa^{(5)}$. Having these parameters specified for the age ranges is crucial to express the special effects of the age intervals. This model is expected to incorporate the impact of young ages into the estimation and projection of the mortality rates with more precision especially for the countries having high rate of birth and young population. Moreover, since the age boundaries such as x_{child}, x_{young}, x_{center} and x_{old} can be rearrangeable based on mortality breaks, the model is also applicable for the countries having dominancy on elderly people which is shown by Hasgul (2015) [5].

The M-MTM requires certain steps to estimate the parameters based on time-varying structured data. Following algorithm illustrates that the parameter estimation is the first performed by conditional GLM with respect to the age constraints on parameters $\kappa^{(3)}$, $\kappa^{(4)}$, and $\kappa^{(5)}$. After estimation of $\kappa^{(\cdot)}$ parameters, the future values of the parameters are projected using WLS and RWWD methods. The future mortality rates are obtained by substituting the projected $\kappa^{(\cdot)}$ parameters into the mortality trend model. Finally, a Monte Carlo algorithm is employed to resemble the predictions.

Let the mortality rates be defined as $q_{x,t}$ at age x and in year t. The steps of the algorithm:

(1) Apply the logit transformation to $q_{x,t}$ as $\log(q_{x,t}/(1 - q_{x,t}))$
(2) Apply GLM to the transformed mortality rates in given intervals as in Eqn[4]
(3) Test if WLS model for $\kappa^{(1)}$ is significant
(4) Test if $\kappa^{(2)}$, $\kappa^{(3)}$, $\kappa^{(4)}$, $\kappa^{(5)}$ are stationary
(5) Apply WLS model for prediction of $\kappa^{(1)}$
(6) Apply appropriate time series model for forecasting $\kappa^{(2)}$, $\kappa^{(3)}$, $\kappa^{(4)}$, $\kappa^{(5)}$
(7) Find future prediction of mortality rates via future parameters
(8) Resemble the predictions of m-simulations by MC

The parameters in Eqn.(4), $\kappa^{(1)}$, $\kappa^{(2)}$, $\kappa^{(3)}$, $\kappa^{(4)}$ and $\kappa^{(5)}$ represent general tendency, mortality steepness, additional effects of childhood, young and old ages, respectively.

The estimation of constant parameter, α_x for a fixed x defined by

$$\alpha_x = \frac{1}{t_{max} - t_{min} + 1} \sum_{t=t_{min}}^{t_{max}} \operatorname{logit} q_{x,t}, \tag{5}$$

where t_{min} and t_{max} are starting and ending year of the time span of the data, respectively.

Estimation of $\kappa^{(\cdot)}$ is performed through a GLM, which enables more flexible fitting and compatibility with non-normal distribution of errors. By employing GLM, $\kappa_t^{(\cdot)}$ are estimated in the interval $[t_{min}, t_{max}]$. The GLM equation of the M-MTM is given as follows

$$\begin{pmatrix} logit(q_{x_{min},t}) - \alpha_{x_{min}} \\ logit(q_{x_{min}+1,t}) - \alpha_{x_{min}+1} \\ \vdots \\ logit(q_{x_{max},t}) - \alpha_{x_{max}} \end{pmatrix} \approx M \begin{pmatrix} \kappa_t^{(1)} \\ \vdots \\ \kappa_t^{(5)} \end{pmatrix},$$

where M is the coefficient matrix of the mortality trend model.

After the estimation process the next step is the projection of the future model parameters, coefficients and the future mortality rates.

The mortality tendency $\kappa^{(1)}$ is forecasted by employing WLS method which advantageously reflects the behaviour of the random errors and can be used with either linear or non-linear functions.

The first step of projection process is fitting a weighted least square model to $\kappa_t^{(1)}$. The weights (w_t) are given as:

$$w_t = \left(1 + \frac{1}{h}\right)^{t - t_{max}}; \qquad \text{for } h > 0. \tag{6}$$

Note that the weights are chosen in such a way that the last years of the data has more contribution to the model than the earlier years in the projected model with the help of weight factor, h [2]. As t gets close to t_{max}, w_t gets larger with an increasing momentum for all h greater than zero.

We add a stochastic term to the best fitted regression line $l_{t_{max}}$, where $t \in [t_{min}, t_{max}]$. Hence, the forecast for $\kappa_t^{(1)}$ is obtained as follows:

$$\kappa_t^{(1)} = l_{t-1}(t) + \varepsilon_t^{(1)}(\sigma^{(1)} + \bar{\sigma}^{(1)}), \tag{7}$$

where $\varepsilon_t^{(1)} \sim^{iid} \mathcal{N}(0,1)$ for all $t \in [t_{min}, t_{max}]$. The volatility, $\sigma^{(1)}$, is the standard deviation of the empirical errors $(\kappa_t^{(1)} - l_{t-1}(t))$ for $[t_{min}+2, t_{max}]$ obtained from the WLS estimation and the term $\bar{\sigma}^{(1)}$ is optional volatility which is assumed as zero [2]. The projection of $\kappa_t^{(1)}$ over time is done iteratively by including each new element of projection in the following forecast.

As experienced from the literature, $\kappa^{(2)}$, $\kappa^{(3)}$, $\kappa^{(4)}$ and $\kappa^{(5)}$ parameters are generally non-stationary and have trends over time. Thus, RWWD is proposed to capture this stochastic pattern. In RWWD, the mean and standard deviation differences between $\kappa_t^{(\cdot)}$ and $\kappa_{t-1}^{(\cdot)}$ are assumed to be the drift, $\mu_t^{(\cdot)}$, and the volatility,

$\sigma^{(\cdot)}$, respectively. The dynamics of each parameter are given as follows:

$$\kappa_t^{(i)} = \kappa_{t-1}^{(i)} + \mu_t^{(i)} + \epsilon_t^{(i)}; \qquad \text{for } i = 2,3,4,5 \tag{8}$$

$$\text{with } \epsilon_t^{(\cdot)} \sim \mathcal{N}(0, \sigma^{(\cdot)}).$$

Finally, the future mortality rate projections are determined by Monte Carlo simulations. In addition, standard deviation of the generated samples are used in the construction of the confidence intervals (CI) for projections.

3. Implementation of the Model

MTM, M-MTM and LC Model are applied to the mortality rates of selected countries over years and the results are compared in order to test whether proposed model illustrates significancy with respect to age, time and country specific characteristics.

The reliability of the model should be examined through a validation process in order to apply the model to the selected data. Throughout the validation process, mortality rates corresponding to the last 8 years of each country are employed as in-sample justification and compared with projections. MAPE and R-Squared statistics are used in the comparison of the observed rates versus projections, and 95% confidence interval is constructed. Depending on the accuracy of in-sample forecast process, 10 years of projections are made on gender base for the selected countries.

The projections are made with the help of 'demography' package of R-code which uses the time series forecasts.

3.1. **Description of the data.** Five countries are selected with respect to their ageing structure. The main indicator for the classification of age structure is the median age. Italy (IT) and Japan (JPN) are supposed to have elderly populations with median ages 44.5 and 46, respectively. France (FR) and the United States of America (USA) are supposed as middle-aged countries having median age around 39, whereas Israel represents a young age population with median age of 30. Mortality rates are retrieved from Human Mortality Database [7]. The longest possible common (joint) time span (1960-2012) is selected. However, since Israel (ISR) mortality rates on the data sources are found to be available only after 1983, this country's mortality is studied within the years 1983-2014. Mortality rates for the ages 10 and 100 are taken into account in the study.

3.2. **Parameter estimation.** GLM has the advantage on relaxing normality assumption on dependent variable. Having two outcomes as being alive or dead in defined period allows us to assume that the distribution of mortality rates are Bernoulli distributed with the probabilities p_x and q_x, respectively. Thus, the logit function can be used as a link in order to transform response variable, $q_{x,t}$, into normal distribution which enables us to employ GLM and the parameters of the trend model in Eqn (4) are estimated. The dependence between mortality rates and

TABLE 1. The median ages and the selected time frame

Country	**Group**	**Years**	**Child***	**Young***	**Center***	**Old***	**Median**
Italy	Old	1960-2012	30; 30	55; 55	60; 60	85; 85	44.5
Japan	Old	1960-2012	30; 30	55; 55	65; 65	85; 85	46.1
France	Middle	1960-2012	35; 42	60; 55	67; 65	85; 85	40.9
USA	Middle	1960-2012	27; 25	55; 55	60; 60	85; 85	37.6
Israel	Young	1983-2014	40; 40	55; 55	60; 60	85; 85	29.9

*Age boundaries for genders (Male; Female)

its regressors are quantified (Table 2) and the correlations indicate that the general mortality trend, $\kappa^{(1)}$, has strong positive correlation to the corresponding mortality rates for all countries. On the other hand, old age effect, $\kappa^{(5)}$, expose negative correlations for all countries. Japan female (F) and France (F) cases yield the most significant correlations with the response for each parameter (old age parameter is in negative direction) compared to the others.

TABLE 2. Correlation coefficients between response variable and regressors in M-MTM

MALE	**IT**	**JPN**	**USA**	**FR**	**ISR**
general ($\kappa^{(1)}$)	0.9835	0.9967	0.9944	0.9903	0.9845
center ($\kappa^{(2)}$)	-0.4786	0.8148	0.6722	0.686	0.4841
child ($\kappa^{(3)}$)	0.2978	0.8523	0.8041	0.9749	0.6438
young ($\kappa^{(4)}$)	-0.6998	0.1016	-0.5707	0.0994	-0.6241
old ($\kappa^{(5)}$)	-0.7027	-0.8527	-0.9246	-0.8354	-0.6139

FEMALE	**IT**	**JPN**	**USA**	**FR**	**ISR**
general ($\kappa^{(1)}$)	0.9916	0.9979	0.9961	0.9868	0.9906
center ($\kappa^{(2)}$)	0.7347	0.9616	0.1592	0.9437	0.5068
child ($\kappa^{(3)}$)	0.8278	0.897	0.8884	0.979	0.6371
young ($\kappa^{(4)}$)	0.1404	0.9689	0.3562	0.6889	-0.245
old ($\kappa^{(5)}$)	-0.8931	-0.9455	-0.8584	-0.9526	-0.7567

The parameter estimates are obtained according to the proposed algorithm. Findings for some selected years are illustrated for USA male (M) data for a slice of the time span in Table 3. We see that the change in parameters over years is recognizable, especially in $\kappa^{(3)}$, $\kappa^{(4)}$ and κ^5 which correspond to childhood, young and old ages, respectively. Even if the changes in $\kappa^{(3)}$, $\kappa^{(4)}$ and κ^5 are greater than in $\kappa^{(1)}$ proportionally, it is important to note that $\kappa^{(1)}$ is the leading variable which has major effect on the mortality rate and makes it dramatically decrease over time

TABLE 3. Estimated $\kappa^{(\cdot)}$ parameters for USA (M) mortality rates

Parameter	**1960**	**1961**	**1962**	**.**	**2010**	**2011**	**2012**
$\kappa^{(1)}$	-3.7819	-3.8041	-3.7866	.	-4.4662	-4.4751	-4.4866
$\kappa^{(2)}$	0.0851	0.0852	0.0851	.	0.0805	0.0804	0.0799
$\kappa^{(3)}$	0.0053	0.0044	0.0028	.	-0.0221	-0.0212	-0.0244
$\kappa^{(4)}$	-0.0064	-0.0064	-0.0068	.	0.0024	0.0030	0.0032
$\kappa^{(5)}$	-0.0280	-0.0304	-0.0234	.	0.0422	0.0419	0.0452

(Figure 1). In other words, mortality rates generally decrease with respect to the major effect of $\kappa^{(1)}$.

In order to compare the parameters of different populations with each other, some transformations on the parameters should be conducted. Hence, some adjustments on $\kappa^{(1)}$ and $\kappa^{(2)}$ are done as follows [2] :

$$\kappa_t^{(2)} \Longleftarrow \kappa_t^{(2)} + \varphi_1,$$

$$\alpha_x \Longleftarrow \alpha_x - \varphi_1(x - x_{center}),$$

$$\varphi_2 = \alpha_{center},$$

$$\alpha_x \Longleftarrow \alpha_x - \varphi_2,$$

$$\kappa_t^{(1)} \Longleftarrow \kappa_t^{(1)} + \varphi_2,$$

where φ is the slope of the fitted regression line of α_x for $x \in \{x_{child}, ..., x_{old}\}$. These adjustments do not detort the results. However, they bring the parameters onto a comparable scale so that the explanation on the parameters become legitimate. $\kappa^{(\cdot)}$ estimates for the selected countries (M, F: 1960-2012) are illustrated in Figures 1-5. These graphs depict that the mortality rates of all countries in the study decrease over time and females generally tend to live longer compared to males which can be inferred from general tendency parameter ($\kappa^{(1)}$). While childhood parameter ($\kappa^{(3)}$) decreases over time, the old age parameter ($\kappa^{(5)}$) increases. In other words, these graphs shows us that the mortality rates of people aged below the boundary of childhood parameter ($\kappa^{(3)}$) decrease more than the amount the general tendecy ($\kappa^{(1)}$) proposes. However, the mortality rate of people aged above the boundary of old age parameter ($\kappa^{(5)}$) decreases less than the amount general tendency proposes.

This illustrates even though the mortality trend model has a decreasing pattern, the contribution of each age class may differ in the amount of the decay keeping up the pace of the decay based on mortality structure.

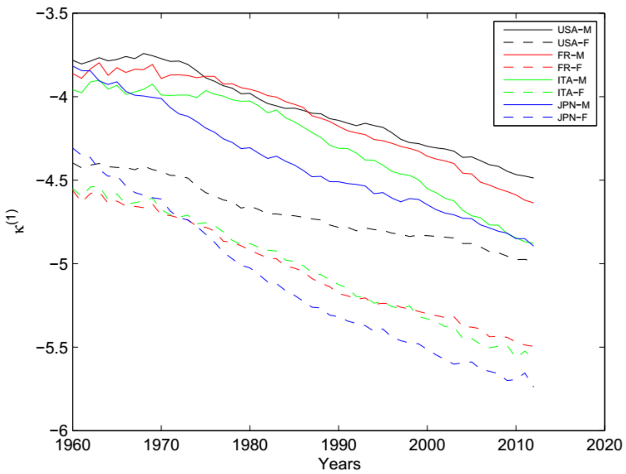

FIGURE 1. Estimation of general tendency parameter ($\kappa^{(1)}$)

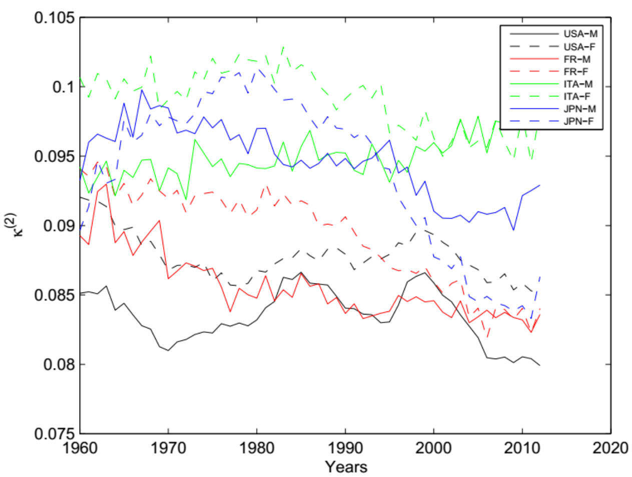

FIGURE 2. Estimation of slope in the logit parameter ($\kappa^{(2)}$)

3.3. **Projection of future mortality rates.** A future time frame of 2013-2022 is achieved by estimating $\kappa^{(\cdot)}$. A validation period 2005-2012 is taken into account to determine the estimation power of M-MTM. Projection of $\kappa^{(1)}$ is performed based on WLS as presented in earlier sections. Whereas, for the projections of $\kappa^{(2)}$, $\kappa^{(3)}$, $\kappa^{(4)}$ and $\kappa^{(5)}$, a stochastic modeling approach, RWWD is employed.

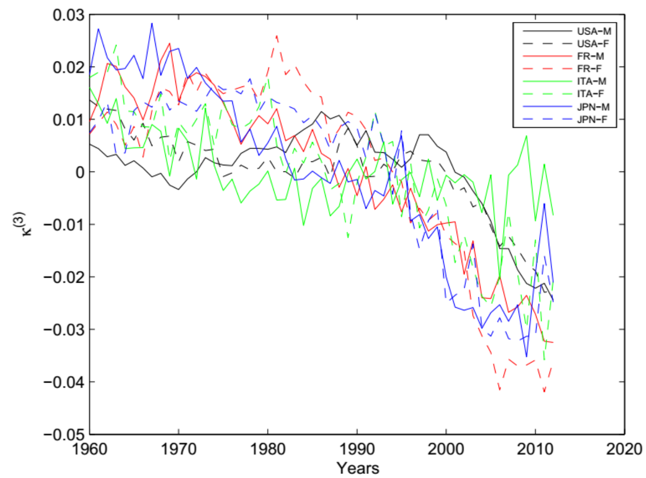

FIGURE 3. Estimation of childhood parameter ($\kappa^{(3)}$)

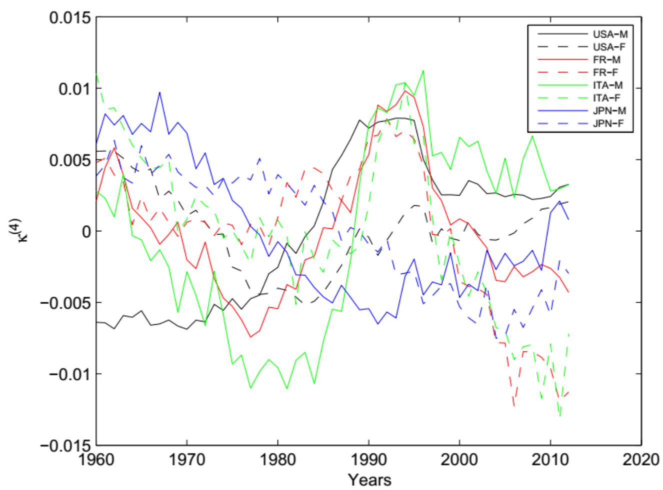

FIGURE 4. Estimation of young age parameter ($\kappa^{(4)}$)

WLS method is applied with five different values of weight factor h in order to detect the best choice. $\kappa^{(1)}$ projection values with the weight factors $h = 1$, $h = 2$, $h = 5$, $h = 10$ and $h = 20$ are shown in Figure 6. Small h values leads to less constribution of previous years. From Figure 6, we see that the projections with $h = 5$, $h = 10$ and $h = 20$ appear to have more stable paths than the projections

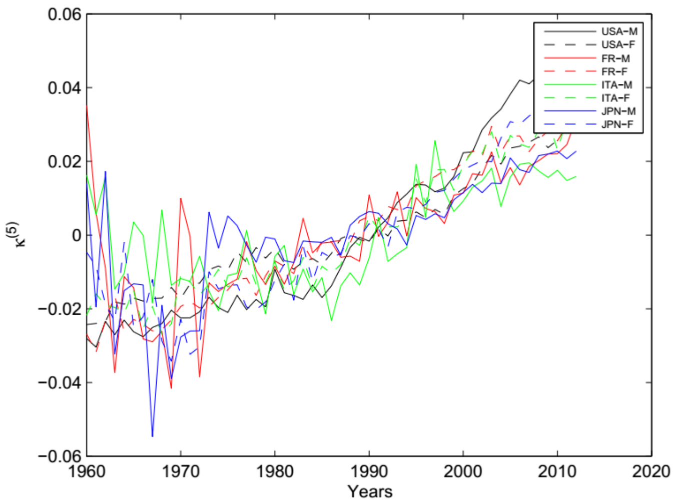

FIGURE 5. Estimation of old age parameter ($\kappa^{(5)}$)

with $h = 1$ and $h = 2$ considering the estimated $\kappa^{(1)}$ values. Hence, it can be inferred that the contribution of past years are important.

The linear model of years and $\kappa^{(1)}$ is significant (p-value < 0.001) and has $R^2 = 98\%$. Residuals also follow normal distribution (p-value of 0.9933).

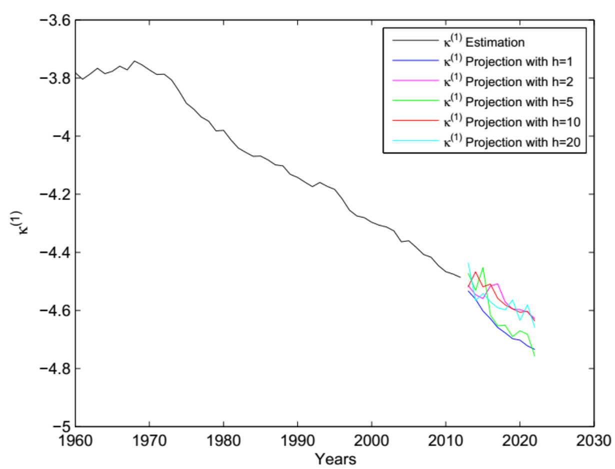

FIGURE 6. Effect of choice of h on $\kappa^{(1)}$ trend

The sequential estimation of $\kappa^{(1)}$ requires the following steps:

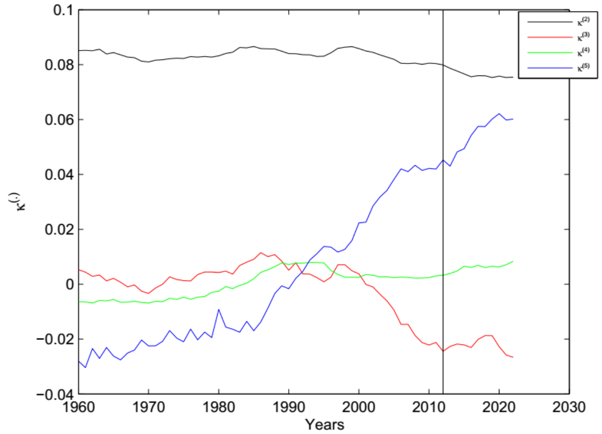

FIGURE 7. Projections of $\kappa^{(2)}$, $\kappa^{(3)}$, $\kappa^{(4)}$ and $\kappa^{(5)}$ for the test and the forecast periods

(i) Apply WLS to n-estimated $\kappa^{(1)}$'s to predict $(n+1)^{th}$ value of $\kappa^{(1)}$,
(ii) Apply WLS to $(n+1)$-estimated $\kappa^{(1)}$'s to predict $(n+2)^{th}$ value of $\kappa^{(1)}$,
(iii) Continue until k projected ($n+k$ in total) $\kappa^{(1)}$ values are obtained.

Since the data of $\kappa^{(2)}$, $\kappa^{(3)}$, $\kappa^{(4)}$ and $\kappa^{(5)}$ are non-stationary processes and have drifts, RWWD method is used to model future values of $\kappa^{(2)}$, $\kappa^{(3)}$, $\kappa^{(4)}$ and $\kappa^{(5)}$. Average and standard deviation of differences between n^{th} and $(n+1)^{th}$ values in each series of $\kappa^{(\cdot)}$'s correspond to drift and volatility, respectively, given by the following equation.

$$\hat{\kappa}_{t+1}^{(\cdot)} = \kappa_t^{(\cdot)} + \mu^{(\cdot)} + \epsilon_t^{(\cdot)}, \tag{9}$$

where $\mu^{(\cdot)}$ is drift, $\sigma^{(\cdot)}$ is standard deviation and it is assumed that $\epsilon_t \sim \mathcal{N}(0, \sigma^{(\cdot)})$.

The trends of $\kappa^{(2)}$, $\kappa^{(3)}$, $\kappa^{(4)}$ and $\kappa^{(5)}$ for a time interval between 1960-2022 are shown in Figure 7. While interpreting the projections of $\kappa^{(4)}$ (additional effect of young ages) and $\kappa^{(5)}$ (additional effect of old ages) parameters, the sustainable decrease in $\kappa^{(1)}$ parameter should be considered as well. Although, an increase in the projections of parameters is observed, it does not necessarily indicate the increase in the mortality rates of specific age groups, such as young and old ages. Since $\kappa^{(1)}$ has a linear decrease as shown in Figure 6, we infer that the rate of reduction in mortality rates of young and old ages decreases over time as values of $\kappa^{(4)}$ and $\kappa^{(5)}$ increase (Figure 7).

Based on the forecasts of $\kappa^{(1)}$, $\kappa^{(2)}$, $\kappa^{(3)}$, $\kappa^{(4)}$ and $\kappa^{(5)}$, mortality rates corresponding to the period of 8 years are projected by employing Monte Carlo simulation with $m = 10^4$ trials for the random components for USA (M) case. To justify the precision and accuracy of the model, MAPE, R^2-values and 95% confidence interval of projected mortality rates are determined. Table 4 shows that MAPE values are smaller than 10%. The range of error is between 4.7% and 8% and the average of all errors is found to be 6.4%. Moreover, R^2-values are considerably high which indicate the accuracy of the proposed M-MTM.

TABLE 4. MAPE and R^2-values for USA (M) between 2005-2012

%	**2005**	**2006**	**2007**	**2008**	**2009**	**2010**	**2011**	**2012**	**Average**
MAPE	4.75	5.43	5.60	5.63	7.47	8.00	6.89	7.65	6.43
R^2	99.99	99.98	99.95	99.97	99.93	99.94	99.98	99.94	

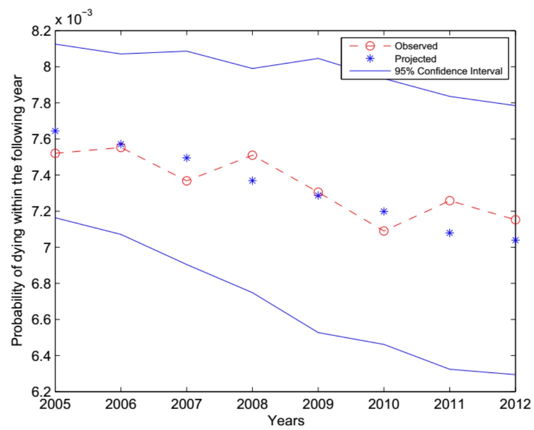

FIGURE 8. In-sample estimates of USA (M) mortality rates ($\alpha =$ 5%, M, x=55)

For the illustration purposes and space limitations, the predicted mortality rates of age 55 for USA (M) are exibited with 95% confidence interval based on the mean and standard deviation of estimates obtained from Monte Carlo simulation for years 2005-2012. As it is seen in Figure 8, all the observed mortality rates remain within the confidence interval of in-sample estimates.

The high values of R^2 (>99%) are also the indication to a systematic risk which is presumed to arise from the high proportion of variance in the dependent variable explained by independent variables in the M-MTM.

Henceforth, we predict future 10-years (2013-2022) mortality rates with 95% confidence band using M-MTM whose outcomes are plotted in Figure 9 and 10 for M and F, respectively. We see that the pattern of mortality rates in USA at age 55 is consistent for both genders.

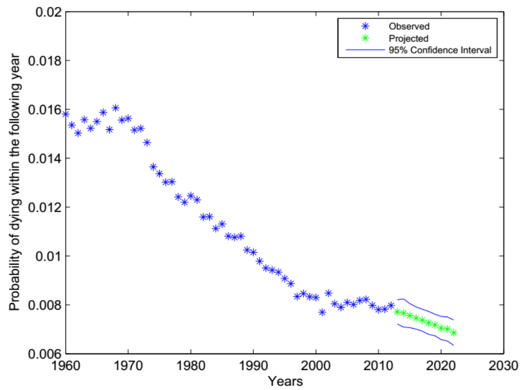

FIGURE 9. Projection for USA mortality rates and 95% CI (x=55, M)

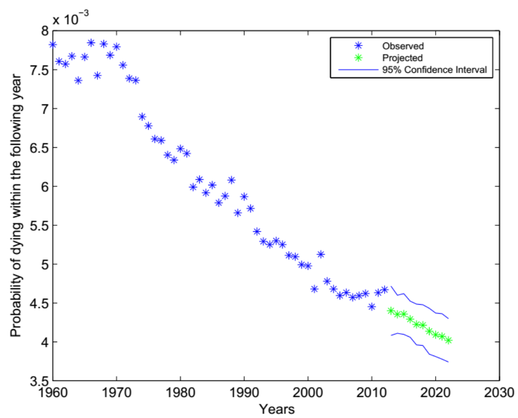

FIGURE 10. Projection for USA mortality rates and 95% CI (x=55, F)

3.4. **Comparison of models.** The comparison of three models, MTM, M-MTM and LC, is carried out for each country on gender base for the years 2005-2012 except for Israel whose time frame is taken as 2007-2014. MAPE values are shown in Table 5. The average of the values are taken as indicators of performance of the models. The MAPE values for years between 2005 and 2012 yield an average of value ranging between 4.69% and 20.78% for old and middle age countries. However, MAPE averages for young age country (Israel) are around 17% and 20%. The best performance is marked by bold font for each case in Table 5. It is observed that the minimum average error is achieved by M-MTM, except Italy (M) where LC model outperforms compare to the others. It is also interesting to note that Italy (F) results do not show any superiority in any model. It can be concluded that M-MTM estimation carries almost the majority in good performance compared to the other two. To determine if the MAPE values of each model are statistically

TABLE 5. MAPE values for old and middle-aged countries (in %)

GROUP	DATA	GND	MODEL	2005	2006	2007	2008	2009	2010	2011	2012	Mean
			MTM	16.56	19.59	18.33	17.37	20.54	26.00	23.24	27.38	21.13
OLD	**ITA**	**M**	**M-MTM**	15.18	17.33	16.96	14.29	19.65	24.76	22.68	25.83	19.59
			LC	8.36	9.21	10.14	9.40	13.37	15.45	16.36	18.45	**12.59**
			MTM	8.42	8.42	8.84	8.45	8.54	9.74	8.06	7.66	8.52
OLD	**ITA**	**F**	**M-MTM**	7.48	7.14	8.58	7.63	7.61	9.75	8.23	9.25	**8.21**
			LC	8.64	5.64	8.38	8.45	9.62	10.34	10.94	11.68	9.21
			MTM	11.04	9.03	9.28	10.04	9.53	12.09	17.24	11.98	11.28
OLD	**JPN**	**M**	**M-MTM**	8.01	5.70	6.13	7.15	6.92	9.44	14.01	8.99	**8.30**
			LC	9.13	7.54	8.24	8.67	9.13	10.81	14.34	11.06	9.87
			MTM	13.21	14.06	14.12	16.42	17.15	19.25	29.18	21.90	18.16
OLD	**JPN**	**F**	**M-MTM**	10.77	11.59	11.79	13.85	14.83	16.70	27.15	19.32	**15.75**
			LC	16.49	17.39	17.37	19.10	20.19	21.45	30.63	23.62	20.78
			MTM	8.85	10.92	12.86	13.50	12.47	13.82	15.57	18.50	13.31
MIDDLE	**FR**	**M**	**M-MTM**	8.59	11.15	12.92	13.10	12.27	13.53	14.63	17.53	12.96
			LC	9.10	9.72	11.48	12.28	12.29	13.85	14.74	17.52	12.62
			MTM	8.51	7.53	6.51	7.85	8.88	8.65	7.58	10.16	**8.21**
MIDDLE	**FR**	**F**	**M-MTM**	6.72	6.93	7.14	7.52	8.31	9.94	9.23	11.64	**8.43**
			LC	10.30	10.86	11.18	11.59	11.39	13.05	13.15	14.32	11.98
			MTM	5.43	6.37	6.96	7.88	9.41	10.78	9.60	10.01	8.31
MIDDLE	**USA**	**M**	**M-MTM**	4.76	5.47	5.50	5.99	7.49	8.48	7.35	7.74	**6.60**
			LC	6.04	6.85	7.43	7.54	9.01	9.55	9.86	10.29	8.32
			MTM	7.42	7.61	7.22	8.06	7.88	8.56	9.07	9.37	8.15
MIDDLE	**USA**	**F**	**M-MTM**	4.70	3.69	3.78	4.29	5.08	5.03	5.30	5.69	**4.69**
			LC	6.60	7.62	7.60	8.76	9.54	10.59	11.07	11.37	9.14
				2007	**2008**	**2009**	**2010**	**2011**	**2012**	**2013**	**2014**	
			MTM	15.69	15.26	19.21	19.10	22.13	26.42	28.15	24.69	21.33
YOUNG	**ISR**	**M**	**MMTM**	14.55	13.07	14.70	16.08	18.05	22.73	23.60	20.01	**17.85**
			LC	16.48	15.86	17.45	17.83	23.60	28.49	25.03	24.39	21.14
			MTM	14.86	18.50	17.51	17.44	17.91	17.49	22.07	18.78	**18.07**
YOUNG	**ISR**	**F**	**MMTM**	13.72	18.88	17.06	19.13	20.06	20.53	24.92	21.93	19.53
			LC	16.44	19.29	19.48	19.26	18.45	17.88	25.52	20.38	19.59

different from each other, t-test is performed for each population considered (Table 6). The results of comparison tests indicate that the M-MTM generally has a better precision than MTM and LC. For Italy (M) which is in old group, LC has a better precision compared with M-MTM and MTM. It can be generalized that M-MTM performs better than Lee Carter for the rest of the cases. MTM is found to be preferable for some cases such as Israel (F) mortality rates. This can be distinguished in pairwise comparisons which are also differentiated by colors (Table 6). As blue color refers to M-MTM is preferable, green to MTM and red color stands for superiority of LC to the others. Black color shows no favoration on the model choice.

Table 6. MAPE comparisons via p-values of t-tests

DATA	GND	MMTM&MTM	MMTM&LC	MTM&LC
USA	**M**	0,00012*	0.00005*	0.94370
	F	$<$0.00001*	0.00002*	0.03590*
FR	**M**	0.05530	0.25270	0.01760*
	F	0.62840	$<$0.00001*	0.00005*
ITA	**M**	0.00091	$<$0.00001*	$<$0.00001*
	F	0.37610	0.08410	0.37480
JPN	**M**	$<$0.00001*	0.00020*	0.00100*
	F	$<$0.00001*	$<$0.00001*	0.00002*
ISR	**M**	0.00016*	0.00018*	0.11270
	F	0.00310*	0.8754	0.02430*

*statistically significant at 5% level

4. Performance of the Models on the Net Single Premium Calculations

In order to illustrate the impact of the models on the valuation of NSP, we assume a hypothetical term life insurance scenario with the following assumptions: (i) a constant annual interest rate of 10%, (ii) an 8-year term life insurance for ages 25, 35, 45 and 55. The risk premium corresponding to one unit benefit of life insurance payment for aged x, which covers next n years [13].

$$A^{1}_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^{k+1} {}_{k}p_{x} q_{x+k}; \tag{10}$$

where v denotes discount factor, ${}_kp_x$ is the probability of living k years at age x and q_{x+k} stands for the probability of death in one year between age $(x+k)$ and $(x+k+1)$.

Table 7 demonstrates the values of NSP under three models and the original mortality rates (between 2005-2012) and depicts the best model yielding the closest NSP to the one obtained with respect to the original. Moreover, Table 7 shows that MTM gives the closest values to original NSP for age 25, whereas, M-MTM outperforms the other two models for the ages, 35, 45, 55 for the term life insurance of 8 years.

5. Conclusion

The prediction of future mortality rates using proposed model (M-MTM) incorporates the childhood effect into the mortality trend model (MTM) [2] and the stochastic approach to estimate its parameters are found to yield remarkable results. Since the mortality trend of young people has a different slope than the rest of the population, forecasts including younger ages, such as 5-20 and 10-20, have

TABLE 7. The impact of models on NSP of 8-year Term Life insurance

Age	Country	Gender	M-MTM	MTM	LC	Original	Best
25-33	**USA**	**M**	0.008287	**0.008302**	0.008182	0.008561	MTM
		F	**0.003286**	0.002875	0.003235	0.003530	**M-MTM**
	FR	**M**	0.005655	**0.005223**	0.006684	0.005179	MTM
		F	0.001987	**0.001790**	0.002190	0.001855	MTM
	ITA	**M**	0.004739	**0.003997**	0.004964	0.003912	MTM
		F	0.001407	**0.001330**	0.001400	0.001280	MTM
	JPN	**M**	0.003565	**0.003613**	0.002969	0.003874	MTM
		F	0.001551	**0.001616**	0.001173	0.001996	MTM
	ISR	**M**	**0.003750**	0.004143	0.004262	0.003676	**M-MTM**
		F	0.001554	**0.001414**	0.001533	0.001399	MTM
35-43	**USA**	**M**	**0.012007**	0.012344	0.012515	0.011400	**M-MTM**
		F	**0.006408**	0.006022	0.006347	0.006580	**M-MTM**
	FR	**M**	0.010960	**0.009764**	0.010964	0.009125	MTM
		F	**0.004422**	0.004099	0.004697	0.004385	**M-MTM**
	ITA	**M**	0.007123	0.006334	**0.006067**	0.005757	LC
		F	0.003176	**0.003003**	0.002924	0.003065	MTM
	JPN	**M**	**0.006319**	0.005884	0.005558	0.006762	**M-MTM**
		F	0.003098	**0.003271**	0.002572	0.003709	MTM
	ISR	**M**	**0.005950**	0.005785	0.007240	0.005901	**M-MTM**
		F	0.003413	0.003030	**0.003306**	0.003256	LC
45-53	**USA**	**M**	**0.026204**	0.027077	0.023213	0.026604	**M-MTM**
		F	0.015079	**0.015142**	0.014813	0.016409	MTM
	FR	**M**	0.026190	0.026482	**0.025168**	0.024169	LC
		F	**0.011161**	0.010772	0.010458	0.011906	**M-MTM**
	ITA	**M**	0.016626	0.017518	**0.013139**	0.014356	LC
		F	0.008301	**0.008529**	0.007575	0.008677	MTM
	JPN	**M**	0.015239	0.014135	**0.015901**	0.016220	LC
		F	**0.007166**	0.006875	0.006604	0.008543	**M-MTM**
	ISR	**M**	0.015268	**0.015095**	0.013805	0.014885	MTM
		F	**0.008291**	0.008404	0.008285	0.008333	**M-MTM**
55-63	**USA**	**M**	**0.060004**	0.061893	0.051921	0.055702	**M-MTM**
		F	**0.033970**	0.035285	0.037536	0.032919	**M-MTM**
	FR	**M**	**0.057042**	0.061561	0.049459	0.053340	**M-MTM**
		F	0.020789	**0.022280**	0.019778	0.023075	MTM
	ITA	**M**	0.045992	0.050541	**0.038009**	0.037771	LC
		F	**0.020042**	0.021421	0.018474	0.019912	**M-MTM**
	JPN	**M**	0.037035	0.034616	**0.038211**	0.040878	LC
		F	**0.015273**	0.013961	0.013958	0.017944	**M-MTM**
	ISR	**M**	**0.042078**	0.043441	0.032790	0.040081	**M-MTM**
		F	**0.021520**	0.023886	0.020389	0.021706	**M-MTM**

always been challenging in the mortality trend modeling. However, we show that M-MTM handles this problem with a better accuracy in predictions. The implementation and illustration of the proposed model are done on the mortality rates of 5 countries in order to determine the effect of demographic structure (old, middle and young age). Monte Carlo simulation is used to generate possible projections and construct confidence interval for these projections. Comparison of the proposed model is done with respect to MTM and LC model where the performance is examined via efficiency indicators. In most of the cases, our proposed model performs more accurate results than the other two models. In other words, in the case that these three models are conducted in a wider range of ages including childhood ages, modified model projects the future mortality rates with less margin of errors. Additionally, projections performed by M-MTM generally have narrower confidence intervals and more precise forecasts compared to MTM and LC model. For this reason, the future mortality projected by M-MTM would be more contributing. For a term life insurance, net single premium (NSP) estimation by M-MTM for the ages over 35 generally gives closest results to realized NSP compared to the estimations by other two models. The outcomes of this study show that M-MTM is advantageous in mortality modeling since the opportunity of changing age boundaries including the childhood makes M-MTM applicable for all types of different age-level populations.

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