Available online: May 23, 2020

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 69, Number 1, Pages 952–968 (2020)
DOI: 10.31801/cfsuasmas.473090
ISSN 1303–5991 E-ISSN 2618-6470
http://communications.science.ankara.edu.tr

SOME GENERAL INTEGRAL INEQUALITIES FOR LIPSCHITZIAN FUNCTIONS VIA CONFORMABLE FRACTIONAL INTEGRAL

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ABSTRACT. In this paper, the author establishes some Hadamard-type and Bullen-type inequalities for Lipschitzian functions via Riemann Liouville fractional integral.

1. INTRODUCTION

Hermite-Hadamard Inequality. Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1}$$

holds. This double inequality is known in the literature as Hermite-Hadamard integral inequality for convex functions (see [7]). Note that some of the classical inequalities for means can be derived from (1) for appropriate particular selections of the mapping f.

Ostrowski's Inequality. Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a mapping differentiable in I°, the interior of I, and let $a, b \in I^{\circ}$ with $a < b$. If $|f'(x)| \leq M$, $x \in [a, b]$, then we the following inequality holds

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \frac{M}{b-a}\left[\frac{(x-a)^2 + (b-x)^2}{2}\right]$$

for all $x \in [a, b]$ (see [1]).

Received by the editors: October 22, 2018; Accepted: May 02, 2020.
2010 *Mathematics Subject Classification.* 26A51, 26A33, 26D15.
Key words and phrases. Lipschitzian functions, Hadamard inequality, Bullen inequality, conformable fractional integral.

Simpson's Inequality. Let $f:[a,b]\to\mathbb{R}$ be a four times continuously differentiable mapping on (a,b) and $\left\|f^{(4)}\right\|_{\infty}=\sup\limits_{x\in(a,b)}\left|f^{(4)}(x)\right|<\infty$. Then the following inequality holds:

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int\limits_a^b f(x)dx\right|\leq\frac{1}{2880}\left\|f^{(4)}\right\|_{\infty}(b-a)^4$$

(see [3, 11] and therein).

Bullen's inequality. Suppose that $f:[a,b]\to\mathbb{R}$ is a convex function on $[a,b]$. Then we have the inequalities:

$$\frac{1}{b-a}\int\limits_a^b f(x)dx\leq\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]$$

(see [5] and [16]). In what follows we recall the following definition.

Definition 1. *A function $f:I\subseteq\mathbb{R}\to\mathbb{R}$ is called an M -Lipschitzian function on the interval I of real numbers with $M\geq 0$, if*

$$|f(x)-f(y)|\leq M\,|x-y|$$

for all $x,y\in I$.

For some recent results are connected with Hermite-Hadamard type integral inequalities for Lipschitzian functions, see [4, 8, 9, 17, 18]. In [17], Tseng et al. established some Hadamard-type and Bullen-type inequalities for Lipschitzian functions as follows:

Theorem 2. *Let I be an interval in $\mathbb{R}$, $a\leq A\leq B\leq b$ in I, $V=(1-\alpha)a+\alpha b$, $\alpha\in[0,1]$ and let $f:I\to\mathbb{R}$ be an L -Lipschitzian function with $L\geq 0$. Then we have the inequality*

$$\left|\alpha f(A)+(1-\alpha)f(B)-\frac{1}{b-a}\int\limits_a^b f(x)dx\right|\leq\frac{LV_{\alpha}(A,B)}{2\,(b-a)},\tag{2}$$

where

$$V_{\alpha}(A,B)=\left\{\begin{array}{c}(A-a)^2-(A-V)^2+(B-V)^2+(b-B)^2,\\ a\leq V\leq A\leq B\leq b,\\ (A-a)^2+(V-A)^2+(B-V)^2+(b-B)^2,\\ a\leq A\leq V\leq B\leq b,\\ (A-a)^2+(V-A)^2+(b-B)^2-(V-B)^2,\\ a\leq A\leq B\leq V\leq b\end{array}\right..$$

Theorem 3. *Let I be an interval in $\mathbb{R}$, $a \leq A \leq B \leq C \leq b$ in I, $V_1 = (1-\alpha)a+\alpha b$, $V_2 = \gamma a + (\alpha+\beta)\, b$, $\alpha, \beta, \gamma \in [0,1]$, $\alpha+\beta+\gamma = 1$, and let $f : I \to \mathbb{R}$ be an L-Lipschitzian function with $L \geq 0$. Then we have the inequality*

$$\left| \alpha f(A) + \beta f(B) + \gamma f(C) - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{L V_{\alpha,\beta,\gamma}(A,B,C)}{2(b-a)}, \qquad (3)$$

where $V_{\alpha,\beta,\gamma}$ is defined as in [17, Section 3].

We give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used throughout this paper.

Definition 4. *Let $f \in L[a,b]$. The Riemann-Liouville fractional integrals $J_{a+}^{\alpha} f$ and $J_{b-}^{\alpha} f$ of order $\alpha > 0$ are defined by*

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \ x > a$$

and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \ x < b$$

respectively, where $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha) = \int_0^{\infty} e^{-t} t^{\alpha-1} dt$ and $J_{a+}^{0} f(x) = J_{b-}^{0} f(x) = f(x)$ (see [13]*).*

In the case of $\alpha = 1$, the fractional integral reduces to the classical integral. For some recent results connected with fractional integral inequalities, see [2, 10, 14, 15, 19]. In [15], Sarıkaya et. al. represented Hermite–Hadamard's inequalities in fractional integral forms as follows:

Theorem 5. *Let $f : [a,b] \to \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for fractional integrals hold*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a) \right] \leq \frac{f(a)+f(b)}{2} \qquad (4)$$

with $\alpha > 0$.

Definition 6. *Let $\alpha \in (n, n+1]$, $n = 0, 1, 2, ...$ and set $\beta = \alpha - n$. Then the left conformable factional integral of any order $\alpha > 0$ is defined by*

$$\left(I_{\alpha}^{a} f\right)(x) = \frac{1}{n!} \int_a^x (x-t)^n (t-a)^{\beta-1} f(t)dt,$$

and analogously, the right conformable fractional integral of any order $\alpha > 0$ is defined by

$$\left({}^{b}I_{\alpha}f\right)(x) = \frac{1}{n!}\int\limits_{x}^{b}(t-x)^{n}(b-t)^{\beta-1}f(t)dt.$$

Notice that, if $\alpha = n+1$ then $\beta = \alpha - n = 1$ and hence $\left(I_{\alpha}^{a}f\right)(x) = J_{a+}^{n+1}f(x)$ and $\left({}^{b}I_{\alpha}f\right)(x) = J_{b-}^{n+1}f(x)$. Also, if $n = 0$ and $\alpha = 1$ then $\beta = 1$ and hence $\left(I_{\alpha}^{a}f\right)(b) = \left({}^{b}I_{\alpha}f\right)(a) = \int\limits_{a}^{b} f(t)dt$.

The Beta function defined as follows:

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int\limits_{0}^{1} t^{a-1}(1-t)^{b-1}dt,\ a,b>0.$$

The Incomplete Beta function is defined by

$$B_{x}(a,b) = \int\limits_{0}^{x} t^{a-1}(1-t)^{b-1}dt,\ x\in[0,1], a,b>0,$$

for $x = 1$, the incomplete beta function coincides with the complete beta function. In [12], Set et. al. represented Hermite–Hadamard's inequalities for conformable fractional integrals as follows:

Theorem 7. *Let $f : [a,b] \to \mathbb{R}$ be a function with $0 \leq a < b$ and $f \in L[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for conformable fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}\Gamma(\alpha-n)}\left[\left(I_{\alpha}^{a}f\right)(b) + \left({}^{b}I_{\alpha}f\right)(a)\right] \leq \frac{f(a)+f(b)}{2}. \tag{5}$$

The aim of this paper is to indicate generalizations of some integral inequalities for Lipschitzian functions via conformable fractional integral. The results are obtained in this study is a generalization of the results which are obtained in Theorem 2 and Theorem 3 by using conformable fractional integrals.

2. A generalization of Hadamard and Ostrowski type inequalities for Lipschitzian functions via fractional integrals

Throughout this section, let I be an interval in $\mathbb{R}$, $a \leq x \leq y \leq b$ in I and let $f : I \to \mathbb{R}$ be an M-Lipschitzian function. In the next theorem, let $\lambda \in [0,1]$, $A = (1-\lambda)a + \lambda b$, and $A_{\alpha,\beta,n}$, $\alpha > 0$, $n = 0,1,2$, $\beta = \alpha - n$, as follows:

(1) If $a \leq A \leq x \leq y \leq b$, then

$$A_{\alpha,\beta,n}(x,y,A) = K_{\alpha,\beta,n}(x,y,A) + L_{\alpha,\beta,n}^{*}(x,y,A).$$

(2) If $a \le x \le A \le y \le b$, then

$$A_{\alpha,\beta,n}(x,y,A) = K^*_{\alpha,\beta,n}(x,y,A) + L^*_{\alpha,\beta,n}(x,y,A).$$

(3) If $a \le x \le y \le A \le b$, then

$$A_{\alpha,\beta,n}(x,y,A) = K^*_{\alpha,\beta,n}(x,y,A) + L_{\alpha,\beta,n}(x,y,A).$$

where

$$\begin{aligned}
K_{\alpha,\beta,n}(x,y,A) &= (A-a)^{\alpha}\left[(x-a)\,B(\beta,n+1)-(A-a)\,B(\beta+1,n+1))\right],\\
K^*_{\alpha,\beta,n}(x,y,A) &= (A-a)^{\alpha}\left\{(x-a)\left[2B_{\frac{x-a}{A-a}}(\beta,n+1)-B(\beta,n+1)\right])\right.\\
&\quad\left.+(A-a)\left[B(\beta+1,n+1)-2B_{\frac{x-a}{A-a}}(\beta+1,n+1)\right]\right\}, A\neq a,\\
K^*_{\alpha,\beta,n}(x,y,a) &= 0,\\
L_{\alpha,\beta,n}(x,y,A) &= (b-A)^{\alpha}\left[(A-y)\,B(n+1,\beta)+(b-A)B(n+2,\beta)\right],\\
L^*_{\alpha,\beta,n}(x,y,A) &= (b-A)^{\alpha}\left\{(y-A)\left[2B_{\frac{y-A}{b-A}}(n+1,\beta)-B(n+1,\beta)\right])\right.\\
&\quad\left.+(b-A)\left[B(n+2,\beta)-2B_{\frac{y-A}{b-A}}(n+2,\beta)\right]\right\}, A\neq b,\\
L^*_{\alpha,\beta,n}(x,y,b) &= 0.
\end{aligned}$$

Theorem 8. *Let $x, y, \alpha, \lambda, A, A_{\alpha,\beta,n}$ and the function f be defined as above. Then we have the inequality for fractional integrals*

$$\begin{aligned}
&\left|\lambda^{\alpha}f(x)+(1-\lambda)^{\alpha}f(y)-\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}\,\Gamma(\alpha-n)}\left[\left(I^{a}_{\alpha}f\right)(A)+\left({}^{b}I_{\alpha}f\right)(A)\right]\right|\\
&\le \frac{\Gamma(\alpha+1)A_{\alpha,\beta,n}(x,y,A)}{n!\,(b-a)^{\alpha}\,\Gamma(\alpha-n)}M. \qquad (6)
\end{aligned}$$

Proof. Using the hypothesis of f, we have the following inequality

$$\begin{aligned}
&\left|\lambda^{\alpha}f(x)+(1-\lambda)^{\alpha}f(y)-\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}\,\Gamma(\alpha-n)}\left[\left(I^{a}_{\alpha}f\right)(A)+\left({}^{b}I_{\alpha}f\right)(A)\right]\right|\\
&= \frac{\Gamma(\alpha+1)}{n!\,(b-a)^{\alpha}\,\Gamma(\alpha-n)}\left|\int_a^A\left[f(x)-f(t)\right](A-t)^n\,(t-a)^{\beta-1}\,dt\right.\\
&\quad\left.+\int_A^b\left[f(y)-f(t)\right](t-A)^n\,(b-t)^{\beta-1}\,dt\right|\\
&\le \frac{\Gamma(\alpha+1)}{n!\,(b-a)^{\alpha}\,\Gamma(\alpha-n)}\left[\int_a^A\left|f(x)-f(t)\right|(A-t)^n\,(t-a)^{\beta-1}\,dt\right.
\end{aligned}$$

$$+\int_A^b |f(y)-f(t)|\,(t-A)^n\,(b-t)^{\beta-1}\,dt\Bigg]$$

$$\leq \frac{\Gamma(\alpha+1)M}{n!\,(b-a)^{\alpha}\,\Gamma(\alpha-n)}\left[\int_a^A |x-t|\,(A-t)^n\,(t-a)^{\beta-1}\,dt\right.$$

$$\left.+\int_A^b |y-t|\,(t-A)^n\,(b-t)^{\beta-1}\,dt\right]. \tag{7}$$

Now using simple calculations, we obtain the following identities

$$\int_a^A |x-t|\,(A-t)^n\,(t-a)^{\beta-1}\,dt \quad\text{and}\quad \int_A^b |y-t|\,(t-A)^n\,(b-t)^{\beta-1}\,dt.$$

1. If $a \leq A \leq x \leq y \leq b$, then

$$\int_a^A |x-t|\,(A-t)^n\,(t-a)^{\beta-1}\,dt$$

$$= (A-a)^{\alpha}\left[(x-a)\,B(\beta,n+1)-(A-a)\,B(\beta+1,n+1))\right]$$
$$= K_{\alpha,\beta,n}(x,y,A).$$

and

$$\int_A^b |y-t|\,(t-A)^n\,(b-t)^{\beta-1}\,dt$$

$$= (b-A)^{\alpha}\left\{(y-A)\left[2B_{\frac{y-A}{b-A}}(n+1,\beta)-B(n+1,\beta)\right]\right)$$
$$+(b-A)\left[B(n+2,\beta)-2B_{\frac{y-A}{b-A}}(n+2,\beta)\right]\Big\}$$
$$= L^{*}_{\alpha,\beta,n}(x,y,A).$$

2. If $a \leq x \leq A \leq y \leq b$, then

$$\int_a^A |x-t|\,(A-t)^n\,(t-a)^{\beta-1}\,dt$$

$$= (A-a)^{\alpha}\left\{(x-a)\left[2B_{\frac{x-a}{A-a}}(\beta,n+1)-B(\beta,n+1)\right]\right)$$
$$+(A-a)\left[B(\beta+1,n+1)-2B_{\frac{x-a}{A-a}}(\beta+1,n+1)\right]\Big\}$$
$$= K^{*}_{\alpha,\beta,n}(x,y,A).$$

and

$$\begin{aligned}
&\int_A^b |y-t|\,(t-A)^n\,(b-t)^{\beta-1}\,dt\\
=\;& (b-A)^\alpha \Big\{(y-A)\Big[2B_{\frac{y-A}{b-A}}(n+1,\beta)-B(n+1,\beta)\Big])\\
& +(b-A)\Big[B(n+2,\beta)-2B_{\frac{y-A}{b-A}}(n+2,\beta)\Big]\Big\}\\
=\;& L^{*}_{\alpha,\beta,n}(x,y,A).
\end{aligned}$$

3. If $a\le x\le y\le A\le b$, then

$$\begin{aligned}
&\int_a^A |x-t|\,(A-t)^n\,(t-a)^{\beta-1}\,dt\\
&= (A-a)^\alpha \Big\{(x-a)\Big[2B_{\frac{x-a}{A-a}}(\beta,n+1)-B(\beta,n+1)\Big])\\
&\quad +(A-a)\Big[B(\beta+1,n+1)-2B_{\frac{x-a}{A-a}}(\beta+1,n+1)\Big]\Big\}\\
&= K^{*}_{\alpha,\beta,n}(x,y,A),
\end{aligned}$$

and

$$\begin{aligned}
&\int_A^b |y-t|\,(t-A)^n\,(b-t)^{\beta-1}\,dt\\
&= (b-A)^\alpha\left[(A-y)\,B(n+1,\beta)+(b-A)B(n+2,\beta)\right]=L_{\alpha,\beta,n}(x,y,A).
\end{aligned}$$

Using the inequality (7) and the above identities $\int_a^A |x-t|\,(A-t)^n\,(t-a)^{\beta-1}\,dt$ and $\int_A^b |y-t|\,(t-A)^n\,(b-t)^{\beta-1}\,dt$, we derive the inequality (6). This completes the proof. □

Under the assumptions of Theorem 8, we have the following corollaries and remarks as follows:

Remark 9. *In Theorem 8, if we take $\alpha=\beta=1$ and $n=0$, then the inequality (6) reduces the inequality (2) in Theorem 2 under the appropriate symbols.*

Corollary 10. *In Theorem 8, let $\delta\in\left[\frac{1}{2},1\right]$, $x=\delta a+(1-\delta)b$ and $y=(1-\delta)a+\delta b$. Then, we have the inequality*

$$\begin{aligned}
&\Big|\lambda^\alpha f(\delta a+(1-\delta)b)+(1-\lambda)^\alpha f((1-\delta)a+\delta b)\\
&\quad-\frac{\Gamma(\alpha+1)}{(b-a)^\alpha\,\Gamma(\alpha-n)}\left[\left(I_\alpha^a f\right)(A)+\left({}^bI_\alpha f\right)(A)\right]\Big|\\
\le\;& \frac{\Gamma(\alpha+1)A_{\alpha,\beta,n}(\delta a+(1-\delta)b,(1-\delta)a+\delta b,A)}{n!\,(b-a)^\alpha\,\Gamma(\alpha-n)}M. \qquad (8)
\end{aligned}$$

Specially if we choose, if we take $x=y=A$, then we have Ostrowski-type inequality as follows:

$$\left|[\lambda^\alpha+(1-\lambda)^\alpha]f(x)-\frac{\Gamma(\alpha+1)}{(b-a)^\alpha\Gamma(\alpha-n)}\left[\left(I_\alpha^a f\right)(A)+\left({}^bI_\alpha f\right)(A)\right]\right| \tag{9}$$
$$\leq \frac{\Gamma(\alpha+1)A_{\alpha,\beta,n}(x,y,A)}{n!\,(b-a)^\alpha\Gamma(\alpha-n)}M,$$

where

$$A_{\alpha,\beta,n}(x,y,A)=(x-a)^{\alpha+1}\left(B(\beta,n+1)-B(\beta+1,n+1)\right)+(b-x)^{\alpha+1}B(n+2,\beta).$$

Remark 11. *In the inequality (9), if we take $\alpha=n+1$, then the inequality (9) reduces the inequality (2.4) obtained via Riemann-Liouville fractional integrals in* [10, Corollary 2.1].

Corollary 12. *We have the following weighted Hadamard-type inequalities for Lipschitzian functions via conformable fractional integrals as follows:*

In the inequality (8), if we take $\delta=1$, then we have

$$\left|\lambda^\alpha f(a)+(1-\lambda)^\alpha f(b)-\frac{\Gamma(\alpha+1)}{(b-a)^\alpha\Gamma(\alpha-n)}\left[\left(I_\alpha^a f\right)(A)+\left({}^bI_\alpha f\right)(A)\right]\right|$$
$$\leq \frac{\Gamma(\alpha+1)A_{\alpha,\beta,n}(a,b,A)}{n!\,(b-a)^\alpha\Gamma(\alpha-n)}M,$$

where

$$\begin{aligned}A_{\alpha,\beta,n}(a,b,A) &= (A-a)^{\alpha+1}\left[B(\beta+1,n+1)-B(\beta,n+1)\right]\\ &\quad+(b-A)^{\alpha+1}\left[B(n+1,\beta)-B(n+2,\beta)\right],\end{aligned}$$

in this inequality, specially if we choose $\lambda=\frac{x-a}{b-a}$ for $x\in[a,b]$, then

$$\left|\frac{(x-a)^\alpha f(a)+(b-x)^\alpha f(b)}{(b-a)^\alpha}-\frac{\Gamma(\alpha+1)}{(b-a)^\alpha\Gamma(\alpha-n)}\left[\left(I_\alpha^a f\right)(x)+\left({}^bI_\alpha f\right)(x)\right]\right|$$
$$\leq \frac{\Gamma(\alpha+1)A_{\alpha,\beta,n}(a,b,x)}{n!\,(b-a)^\alpha\Gamma(\alpha-n)}M,$$

Corollary 13. *In the inequality (9),*

(i) if we choose $\lambda=\frac{1}{2}$, then

$$\left|f\left(\frac{a+b}{2}\right)-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha\Gamma(\alpha-n)}\left[\left(I_\alpha^a f\right)\left(\frac{a+b}{2}\right)+\left({}^bI_\alpha f\right)\left(\frac{a+b}{2}\right)\right]\right|$$
$$\leq \frac{2^{\alpha-1}\Gamma(\alpha+1)A_{\alpha,\beta,n}(\frac{a+b}{2},\frac{a+b}{2},\frac{a+b}{2})}{n!\,(b-a)^\alpha\Gamma(\alpha-n)}M,$$

where

$$A_{\alpha,\beta,n}(\frac{a+b}{2},\frac{a+b}{2},\frac{a+b}{2})$$

$$= \left(\frac{b-a}{2}\right)^{\alpha+1}\left[B(\beta,n+1)-B(\beta+1,n+1)+B(n+2,\beta)\right].$$

(ii) In the inequality (9), if we take $\lambda=\frac{1}{2}$ *and* $\delta=\frac{3}{4}$ *then*

$$\left|\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]\right.$$
$$\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}\Gamma(\alpha-n)}\left[\left(I_{\alpha}^{a}f\right)\left(\frac{a+b}{2}\right)+\left({}^{b}I_{\alpha}f\right)\left(\frac{a+b}{2}\right)\right]\right|$$
$$\leq\frac{2^{\alpha-1}\Gamma(\alpha+1)A_{\alpha,\beta,n}(\frac{3a+b}{4},\frac{a+3b}{4},\frac{a+b}{2})}{n!\,(b-a)^{\alpha}\Gamma(\alpha-n)}M,$$

where

$$A_{\alpha,\beta,n}(\frac{3a+b}{4},\frac{a+3b}{4},\frac{a+b}{2})$$
$$=\left(\frac{b-a}{2}\right)^{\alpha+1}\left[B_{1/2}(\beta,n+1)+B_{1/2}(n+1,\beta)-2B_{1/2}(\beta+1,n+1)\right.$$
$$\left.-2B_{1/2}(n+2,\beta)+B(\beta+1,n+1)+B(n+2,\beta)-B(\beta,n+1)\right].$$

3. A generalization of Bullen and Simpson type inequalities for Lipschitzian functions via fractional integrals

Throughout this section, let I be an interval in $\mathbb{R}$, $a\leq x\leq y\leq z\leq b$ in I and $f:I\to\mathbb{R}$ be an M-lipschitzian function. In the next theorem, let $\lambda+\eta+\mu=1$, $\lambda,\eta,\mu\in[0,1]$, $A=(1-\lambda)a+\lambda b$, $C=\mu a+(\lambda+\eta)b$, and define $I_{\alpha,\lambda,\eta,\mu}$, $\alpha>0$, as follows:

(1) If $A\leq C\leq x\leq y\leq z$ or $A\leq x\leq C\leq y\leq z$, then
$$I_{\alpha,\lambda,\eta,\mu}(x,y,z)=M_{\alpha,\lambda,\eta,\mu}(x,y,z)+N_{\alpha,\lambda,\eta,\mu}(x,y,z)+O^{*}_{\alpha,\lambda,\eta,\mu}(x,y,z).$$
(2) If $A\leq x\leq y\leq C\leq z$, then
$$I_{\alpha,\lambda,\eta,\mu}(x,y,z)=M_{\alpha,\lambda,\eta,\mu}(x,y,z)+N^{*}_{\alpha,\lambda,\eta,\mu}(x,y,z)+O^{*}_{\alpha,\lambda,\eta,\mu}(x,y,z).$$
(3) If $A\leq x\leq y\leq z\leq C$, then
$$I_{\alpha,\lambda,\eta,\mu}(x,y,z)=M_{\alpha,\lambda,\eta,\mu}(x,y,z)+N^{*}_{\alpha,\lambda,\eta,\mu}(x,y,z)+O_{\alpha,\lambda,\eta,\mu}(x,y,z).$$
(4) If $x\leq A\leq C\leq y\leq z$, then
$$I_{\alpha,\lambda,\eta,\mu}(x,y,z)=M^{*}_{\alpha,\lambda,\eta,\mu}(x,y,z)+N_{\alpha,\lambda,\eta,\mu}(x,y,z)+O^{*}_{\alpha,\lambda,\eta,\mu}(x,y,z).$$
(5) If $x\leq A\leq y\leq C\leq z$, then
$$I_{\alpha,\lambda,\eta,\mu}(x,y,z)=M^{*}_{\alpha,\lambda,\eta,\mu}(x,y,z)+N^{*}_{\alpha,\lambda,\eta,\mu}(x,y,z)+O^{*}_{\alpha,\lambda,\eta,\mu}(x,y,z).$$
(6) If $x\leq A\leq y\leq z\leq C$, then
$$I_{\alpha,\lambda,\eta,\mu}(x,y,z)=M^{*}_{\alpha,\lambda,\eta,\mu}(x,y,z)+N^{*}_{\alpha,\lambda,\eta,\mu}(x,y,z)+O_{\alpha,\lambda,\eta,\mu}(x,y,z).$$

(7) If $x \le y \le A \le C \le z$, then

$I_{\alpha,\lambda,\eta,\mu}(x,y,z) = M^*_{\alpha,\lambda,\eta,\mu}(x,y,z) - N_{\alpha,\lambda,\eta,\mu}(x,y,z) + O^*_{\alpha,\lambda,\eta,\mu}(x,y,z)$.

(8) If $x \le y \le A \le z \le C$ or $x \le y \le z \le A \le C$, then

$I_{\alpha,\lambda,\eta,\mu}(x,y,z) = M^*_{\alpha,\lambda,\eta,\mu}(x,y,z) - N_{\alpha,\lambda,\eta,\mu}(x,y,z) + O_{\alpha,\lambda,\eta,\mu}(x,y,z)$.

Where

$$\begin{aligned}
M_{\alpha,\lambda,\eta,\mu}(x,y,z) &= (A-a)^{\alpha}\left[(x-a)\,B(\beta,n+1)-(A-a)\,B(\beta+1,n+1)\right],\\
N_{\alpha,\lambda,\eta,\mu}(x,y,z) &= (C-A)^{\alpha}\left[(y-A)\,B(n+1,\beta)-(C-A)\,B(n+2,\beta)\right],\\
O_{\alpha,\lambda,\eta,\mu}(x,y,z) &= (b-C)^{\alpha}\left[(C-z)\,B(n+1,\beta)+(b-C)\,B(n+2,\beta)\right],
\end{aligned}$$

$$\begin{aligned}
&M^*_{\alpha,\lambda,\eta,\mu}(x,y,z) = (A-a)^{\alpha}\left\{(x-a)\left[2B_{\frac{x-a}{A-a}}(\beta,n+1)-B(\beta,n+1)\right]\right)\\
&+(A-a)\left[B(\beta+1,n+1)-2B_{\frac{x-a}{A-a}}(\beta+1,n+1)\right]\Big\},\ A\neq a\ (or\ \lambda\neq 0),\\
&M^*_{\alpha,0,\eta,\mu}(x,y,z)=0,\\
&N^*_{\alpha,\lambda,\eta,\mu}(x,y,z) = (C-A)^{\alpha}\left\{(y-A)\left[2B_{\frac{y-A}{C-A}}(n+1,\beta)-B(n+1,\beta)\right]\right)\\
&+(C-A)\left[B(n+2,\beta)-2B_{\frac{y-A}{C-A}}(n+2,\beta)\right]\Big\},\ A\neq C\ (or\ \eta\neq 0),\\
&N^*_{\alpha,\lambda,0,\mu}(x,y,z)=0,\\
&O^*_{\alpha,\lambda,\eta,\mu}(x,y,z) = (b-C)^{\alpha}\left\{(z-C)\left[2B_{\frac{z-C}{b-C}}(n+1,\beta)-B(n+1,\beta)\right]\right)\\
&+(b-C)\left[B(n+2,\beta)-2B_{\frac{z-C}{b-C}}(n+2,\beta)\right]\Big\},\ C\neq b\ (or\ \mu\neq 0),\\
&O^*_{\alpha,\lambda,\eta,0}(x,y,z)=0.
\end{aligned}$$

Theorem 14. *Let $x, y, z, \lambda, \eta, \mu, A_1, A_2, A_{\alpha,\lambda,\eta,\mu}$ and the function f be defined as above. Then we have the inequality*

$$\begin{aligned}
&\left|\lambda^{\alpha} f(x)+\eta^{\alpha} f(y)+\mu^{\alpha} f(z)\right.\\
&\left.-\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}\,\Gamma(\alpha-n)}\left[\left(I^{a}_{\alpha} f\right)(A)+\left({}^{C}I_{\alpha} f\right)(A)+\left({}^{b}I_{\alpha} f\right)(C)\right]\right|\\
&\le \frac{\Gamma(\alpha+1) I_{\alpha,\lambda,\eta,\mu}(x,y,z)}{n!\,(b-a)^{\alpha}\,\Gamma(\alpha-n)} M. \qquad (10)
\end{aligned}$$

Proof. Using the hypothesis of f, we have the inequality

$$\begin{aligned}
&\left|\lambda^{\alpha} f(x)+\eta^{\alpha} f(y)+\mu^{\alpha} f(z)\right.\\
&\left.-\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}\,\Gamma(\alpha-n)}\left[\left(I^{a}_{\alpha} f\right)(A)+\left({}^{C}I_{\alpha} f\right)(A)+\left({}^{b}I_{\alpha} f\right)(C)\right]\right|\\
&=\frac{\Gamma(\alpha+1)}{n!\,(b-a)^{\alpha}\,\Gamma(\alpha-n)}\left|\int_a^A \left[f(x)-f(t)\right](A-t)^{n}(t-a)^{\beta-1}\,dt\right.
\end{aligned}$$

$$+\int_A^C [f(y)-f(t)](t-A)^n (C-t)^{\beta-1}\,dt + \int_C^b [f(z)-f(t)](t-C)^n (b-t)^{\beta-1}\,dt\Bigg|$$

$$\le \frac{\Gamma(\alpha+1)}{n!\,(b-a)^\alpha\,\Gamma(\alpha-n)}\Bigg|\int_a^A |f(x)-f(t)|(A-t)^n (t-a)^{\beta-1}\,dt$$

$$+\int_A^C |f(y)-f(t)|(t-A)^n (C-t)^{\beta-1}\,dt + \int_C^b |f(z)-f(t)|(t-C)^n (b-t)^{\beta-1}\,dt\Bigg|$$

$$\le \frac{\Gamma(\alpha+1)M}{n!\,(b-a)^\alpha\,\Gamma(\alpha-n)}\Bigg|\int_a^A |x-t|(A-t)^n (t-a)^{\beta-1}\,dt \tag{11}$$

$$+\int_A^C |y-t|(t-A)^n (C-t)^{\beta-1}\,dt + \int_C^b |z-t|(t-C)^n (b-t)^{\beta-1}\,dt\Bigg|.$$

Now, using simple calculations, we obtain the following identities $\int_a^A |x-t|(A-t)^n (t-a)^{\beta-1}\,dt$, $\int_A^C |y-t|(t-A)^n (C-t)^{\beta-1}\,dt$ and $\int_C^b |z-t|(t-C)^n (b-t)^{\beta-1}\,dt$.

(1) If $A\le C\le x\le y\le z$ or $A\le x\le C\le y\le z$, then we have

$$\begin{aligned}&\int_a^A |x-t|(A-t)^n (t-a)^{\beta-1}\,dt\\ &= (A-a)^\alpha\left[(x-a)\,B(\beta,n+1)-(A-a)\,B(\beta+1,n+1)\right]\\ &= M_{\alpha,\lambda,\eta,\mu}(x,y,z),\\ &\int_A^C |y-t|(t-A)^n (C-t)^{\beta-1}\,dt\\ &= (C-A)^\alpha\left[(y-A)\,B(n+1,\beta)-(C-A)\,B(n+2,\beta)\right]\\ &= N_{\alpha,\lambda,\eta,\mu}(x,y,z),\end{aligned}$$

and

$$\begin{aligned}&\int_C^b |z-t|(t-C)^n (b-t)^{\beta-1}\,dt\\ &= (b-C)^\alpha\Big\{(z-C)\Big[2B_{\frac{z-C}{b-C}}(n+1,\beta)-B(n+1,\beta)\Big])\\ &+(b-C)\Big[B(n+2,\beta)-2B_{\frac{z-C}{b-C}}(n+2,\beta)\Big]\Big\}\end{aligned}$$

$$= O^*_{\alpha,\lambda,\eta,\mu}(x,y,z).$$

(2) If $A \leq x \leq y \leq C \leq z$, then we have

$$\int_a^A |x-t|\,(A-t)^n\,(t-a)^{\beta-1}\,dt = M_{\alpha,\lambda,\eta,\mu}(x,y,z),$$

$$\int_A^C |y-t|\,(t-A)^n\,(C-t)^{\beta-1}\,dt$$
$$= (C-A)^\alpha \left\{(y-A)\left[2B_{\frac{y-A}{C-A}}(n+1,\beta) - B(n+1,\beta)\right])\right.$$
$$\left.+ (C-A)\left[B(n+2,\beta) - 2B_{\frac{y-A}{C-A}}(n+2,\beta)\right]\right\}$$
$$= N^*_{\alpha,\lambda,\eta,\mu}(x,y,z),$$

and

$$\int_C^b |z-t|\,(t-C)^n\,(b-t)^{\beta-1}\,dt = O^*_{\alpha,\lambda,\eta,\mu}(x,y,z).$$

(3) If $A \leq x \leq y \leq z \leq C$, then we have

$$\int_a^A |x-t|\,(A-t)^n\,(t-a)^{\beta-1}\,dt = M_{\alpha,\lambda,\eta,\mu}(x,y,z),$$

$$\int_A^C |y-t|\,(t-A)^n\,(C-t)^{\beta-1}\,dt = N^*_{\alpha,\lambda,\eta,\mu}(x,y,z),$$

and

$$\int_C^b |z-t|\,(t-C)^n\,(b-t)^{\beta-1}\,dt$$
$$= (b-C)^\alpha\left[(C-z)\,B(n+1,\beta) + (b-C)\,B(n+2,\beta)\right]$$
$$= O_{\alpha,\lambda,\eta,\mu}(x,y,z).$$

(4) If $x \leq A \leq C \leq y \leq z$, then we have

$$\int_a^A |x-t|\,(A-t)^n\,(t-a)^{\beta-1}\,dt = M^*_{\alpha,\lambda,\eta,\mu}(x,y,z),$$

$$\int_A^C |y-t|\,(t-A)^n\,(C-t)^{\beta-1}\,dt = N_{\alpha,\lambda,\eta,\mu}(x,y,z),$$

and

$$\int_C^b |z-t|\,(t-C)^n\,(b-t)^{\beta-1}\,dt = O^*_{\alpha,\lambda,\eta,\mu}(x,y,z).$$

(5) If $x \le A \le y \le C \le z$, then we have

$$\int_a^A |x-t|\,(A-t)^n\,(t-a)^{\beta-1}\,dt = M^*_{\alpha,\lambda,\eta,\mu}(x,y,z),$$

$$\int_A^C |y-t|\,(t-A)^n\,(C-t)^{\beta-1}\,dt = N^*_{\alpha,\lambda,\eta,\mu}(x,y,z),$$

and

$$\int_C^b |z-t|\,(t-C)^n\,(b-t)^{\beta-1}\,dt = O^*_{\alpha,\lambda,\eta,\mu}(x,y,z).$$

(6) If $x \le A \le y \le z \le C$, then we have

$$\int_a^A |x-t|\,(A-t)^n\,(t-a)^{\beta-1}\,dt = M^*_{\alpha,\lambda,\eta,\mu}(x,y,z),$$

$$\int_A^C |y-t|\,(t-A)^n\,(C-t)^{\beta-1}\,dt = N^*_{\alpha,\lambda,\eta,\mu}(x,y,z),$$

and

$$\int_C^b |z-t|\,(t-C)^n\,(b-t)^{\beta-1}\,dt = O_{\alpha,\lambda,\eta,\mu}(x,y,z).$$

(7) If $x \le y \le A \le C \le z$, then we have

$$\int_a^A |x-t|\,(A-t)^n\,(t-a)^{\beta-1}\,dt = M^*_{\alpha,\lambda,\eta,\mu}(x,y,z),$$

$$\int_A^C |y-t|\,(t-A)^n\,(C-t)^{\beta-1}\,dt = -N_{\alpha,\lambda,\eta,\mu}(x,y,z)$$

and

$$\int_C^b |z-t|\,(t-C)^n\,(b-t)^{\beta-1}\,dt = O^*_{\alpha,\lambda,\eta,\mu}(x,y,z).$$

(8) If $x \leq y \leq A \leq z \leq C$ or $x \leq y \leq z \leq A \leq C$, then we have

$$\int_a^A |x-t| (A-t)^n (t-a)^{\beta-1} dt = M^*_{\alpha,\lambda,\eta,\mu}(x,y,z),$$

$$\int_A^C |y-t| (t-A)^n (C-t)^{\beta-1} dt = -N_{\alpha,\lambda,\eta,\mu}(x,y,z),$$

and

$$\int_C^b |z-t| (t-C)^n (b-t)^{\beta-1} dt = O_{\alpha,\lambda,\eta,\mu}(x,y,z).$$

Using the inequality (11) and the above identities $\int_a^A |x-t| (A-t)^n (t-a)^{\beta-1} dt$, $\int_A^C |y-t| (t-A)^n (C-t)^{\beta-1} dt$ and $\int_C^b |z-t| (t-C)^n (b-t)^{\beta-1} dt$, we derive the inequality (10). This completes the proof. □

Under the assumptions of Theorem 14, we have the following corollaries and remarks as follows:

Remark 15. *In Theorem 14, if we take $\alpha = \beta = 1$ and $n = 0$, then then the inequality (10) reduces the inequality (3) in Theorem 3 under the appropriate symbols.*

Corollary 16. *In Theorem 14, let $\delta \in \left[\frac{1}{2}, 1\right]$, $x = \delta a + (1-\delta)b$, $y = \frac{a+b}{2}$ and $z = (1-\delta)a + \delta b$. Then, we have the inequality*

$$\begin{aligned}
&\Bigg|\lambda^\alpha f(\delta a + (1-\delta)b) + \eta^\alpha f(\frac{a+b}{2}) + \mu^\alpha f((1-\delta)a + \delta b) \\
&- \frac{\Gamma(\alpha+1)}{(b-a)^\alpha \Gamma(\alpha-n)} \left[\left(I_\alpha^a f\right)(A) + \left({}^C I_\alpha f\right)(A) + \left({}^b I_\alpha f\right)(C) \right] \Bigg| \\
\leq\ & \frac{\Gamma(\alpha+1) I_{\alpha,\lambda,\eta,\mu}(\delta a + (1-\delta)b, \frac{a+b}{2}, (1-\delta)a + \delta b)}{n!\,(b-a)^\alpha \Gamma(\alpha-n)} M.
\end{aligned}$$

Corollary 17. *In Corollary 16, if we take $\delta = 1$, $\lambda = \mu = \frac{\theta}{2}$ and $\eta = 1-\theta$ with $\theta \in [0,1]$, then we have the following weighted Bullen-type inequality for M-Lipschitzian functions via fractional integrals*

$$\begin{aligned}
&\Bigg| \left(\frac{\theta}{2}\right)^\alpha (f(a) + f(b)) + (1-\theta)^\alpha f\left(\frac{a+b}{2}\right) \\
&- \frac{\Gamma(\alpha+1)}{(b-a)^\alpha \Gamma(\alpha-n)} \left[\left(I_\alpha^a f\right)(A) + \left({}^C I_\alpha f\right)(A) + \left({}^b I_\alpha f\right)(C) \right] \Bigg| \\
\leq\ & \frac{\Gamma(\alpha+1) I_{\alpha,\frac{\theta}{2},1-\theta,\frac{\theta}{2}}(a, \frac{a+b}{2}, b)}{n!\,(b-a)^\alpha \Gamma(\alpha-n)} M, \qquad (12)
\end{aligned}$$

where

$$
\begin{aligned}
& I_{\alpha,\frac{\theta}{2},1-\theta,\frac{\theta}{2}}(a,\frac{a+b}{2},b) \\
= \; & (b-a)^{\alpha+1}\left\{\begin{array}{c} \left(\frac{\theta}{2}\right)^{\alpha+1}\left[B(\beta+1,n+1)+B(n+1,\beta)-B(n+2,\beta)\right] \\ +(1-\theta)^{\alpha+1}\left[\begin{array}{c} B(n+2,\beta)-\frac{1}{2}B(n+1,\beta)+B_{1/2}(n+1,\beta) \\ -2B_{1/2}(n+2,\beta)\end{array}\right]\end{array}\right\}.
\end{aligned}
$$

Specially, in the inequality (12), if we take $n=0$ and $\alpha=\beta=1$, then the inequality (12) reduces to the following general inequality for M-Lipschitzian functions

$$
\begin{aligned}
& \left|\left(\frac{\theta}{2}\right)(f(a)+f(b))+(1-\theta)f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)dt\right| \\
\leq \; & \frac{M}{4}(b-a)\left[2\theta^2+(1-\theta)^2\right]. \qquad (13)
\end{aligned}
$$

Remark 18. *In the inequality (12), if we take $\alpha=n+1$, then the inequality (12) reduces the inequality obtained via Riemann-Liouville fractional integrals in* [10, Corollary 3.2].

Remark 19. *In the inequality (13), if we take $\theta=\frac{1}{3}$, then the inequality (13) reduces to the following Simpson-type inequality for M-Lipschitzian functions*

$$
\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\frac{M}{6}(b-a).
$$

Remark 20. *In the inequality (13), if we take $\theta=\frac{1}{2}$, then the inequality (13) reduces to the following Bullen type inequality for M-Lipschitzian functions*

$$
\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\frac{3M}{16}(b-a).
$$

Remark 21. *In the inequality (13), if we take $\theta=0$, then the inequality (13) reduces to the following Midpoint type inequality for M -Lipschitzian functions*

$$
\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\frac{M}{4}(b-a).
$$

Remark 22. *In the inequality (13), if we take $\theta=1$, then the inequality (13) reduces to the following Trapezoid type inequality for M -Lipschitzian functions*

$$
\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\frac{M}{2}(b-a).
$$

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