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A SOLUTION OF A VISCOSITY CESÀRO MEAN ALGORITHM

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Abstract. Based on the viscosity approximation method, we introduce a new cesàro mean approximation method for finding a common solution of split generalized equilibrium problem in real Hilbert spaces. Under certain conditions control on parameters, we prove a strong convergence theorem for the sequences generated by the proposed iterative scheme. Some numerical examples are presented to illustrate the convergence results. Our results can be viewed as a generalization and improvement of various existing results in the current literature.

1. Introduction

Let $\mathbb{R}$ denote the set of all real number, H_1 and H_2 be real Hilbert spaces and C and Q be nonempty closed convex subset of H_1 and H_2, respectively. A mapping $T : C \to C$ said to be a k-strictly pseudocontractive if there exists a constant $0 \leq k < 1$ such that

$$\|T(x) - T(y)\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \qquad \forall x, y \in C.$$

When $k = 1$, T is said to be pseudocontractive if

$$\|T(x) - T(y)\|^2 \leq \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2, \qquad \forall x, y \in C.$$

If $k = 0$, T is called nonexpansive on C.

The fixed point problem (FPP) for a nonexpansive mapping T is: Find $x \in C$ such that $x \in Fix(T)$, where $Fix(T)$ is the fixed point set of the nonexpansive mapping T.

The class of k-strictly pseudocontractive falls into the one between classes of nonexpansive mapping and pseudocontractive mapping.

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A set-valued $M : H \to 2^H$ is called monotone if for all $x, y \in H, u \in M(x)$ and $v \in M(y)$ such that $\langle x-y, u-v\rangle \geq 0$. A monotone mapping $M : H \to 2^H$ is maximal if the $Graph(M)$ is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping M is maximal if and only if for $(x,u) \in H \times H, \langle x-y, u-v\rangle \geq 0$, for every $(y,v) \in Graph(M)$ implies that $u \in M(x)$.

Let $E : H \to H$ be a single-valued nonlinear mapping, and let $M : H \to 2^H$ be a set-valued mapping. We consider the following variational inclusion problem (VIP), which is: Find $x \in H$ such that

$$\theta \in E(x) + M(x),$$

where θ is the zero vector in H. The solution set of (VIP) is denoted by $I(E, M)$.

Let the set-valued mapping $M : H \to 2^H$ be a maximal monotone. We define the resolvent operator $J_{M,\lambda}$ associate with M and λ as follows:

$$J_{M,\lambda}(x) = (I + \lambda M)^{-1}(x), \qquad x \in H$$

where λ is a positive number. It is worth mentioning that the resolvent operator $J_{M,\lambda}(x)$ is single-valued, nonexpansive and 1-inverse strongly monotone [2, 22].

In 1994 Blum and Oettli [1] introduced and studied the following equilibrium problem (EP): Find $x \in C$ such that $F(x,y) \geq 0$, $\forall y \in C$, where $F : C \times C \to \mathbb{R}$ is a bifunction.

Kumam et al. [11] considered an iterative algorithm in a Hilbert space:

$$\begin{aligned} t_n &= T_{r_n}^{(F_1,\varphi_1)}(x_n - r_n A x_n), \\ u_n &= T_{q_n}^{(F_2,\varphi_2)}(t_n - q_n B t_n), \\ v_n &= J_{M_1,\lambda_1}(u_n - \lambda_1 E_1 u_n), \\ w_n &= J_{M_2,\lambda_2}(v_n - \lambda_2 E_2 v_n), \end{aligned}$$

$$y_{n,i} = \alpha_{n,i} x_0 + (1-\alpha_{n,i}) \tfrac{1}{t_n} \int_0^{t_n} S(s) W_n w_n ds,$$

$$C_{n+1,i} = \{z \in C_{n,i} : \|y_{n,i} - z\|^2 \leq \|x_n - z\|^2 + \alpha_{n,i}(\|x_0\|^2 + 2\langle x_n - x_0, z\rangle)\},$$

$$C_{n+1} = \textstyle\bigcap_{i=1}^{\infty} C_{n+1,i},$$

$$x_{n+1} = P_{C_{n+1}} x_0.$$

Moudafi [15] introduced the following split equilibrium problem (SEP):
Let $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ be nonlinear bimappings and let $A : H_1 \to H_2$ be a bounded linear operator, then the SEP is to find $x^* \in C$ such that

$$F_1(x^*, x) \geq 0, \ \forall x \in C$$

and such that

$$y^* = Ax^* \in Q \ solves \ F_2(y^*, y) \geq 0, \ \forall y \in Q$$

The solution set of (SEP) is denoted by $\Omega = \{p \in EP(F_1) : Ap \in EP(F_2)\}$. (SEP) includes the split variational inequality problem, split zero problem, and split feasibility problem (see, for instance, [3–6, 14, 15]).

Recently, Kazmi and Rizvi [10] introduced a split generalized equilibrium problem $(SGEP)$: Find $x^* \in C$ such that

$$F_1(x^*, x) + \psi_1(x^*, x) \geq 0, \ \forall x \in C$$

and such that

$$y^* = Ax^* \in Q \ solves \ F_2(y^*, y) + \psi_2(y^*, y) \geq 0, \ \forall y \in Q$$

where $F_1, \psi_1 : C \times C \to \mathbb{R}$ and $F_2, \psi_2 : C \times C \to \mathbb{R}$ be nonlinear bi functions and $A : H_1 \to H_2$ is bounded linear operator. The solution set of $(SGEP)$ is denoted by $\Gamma = \{p \in GEP(F_1, \psi_1) : Ap \in GEP(F_2, \psi_2)\}$. They considered the following iterative method:

$$\begin{aligned}
&u_n = T_{r_n}^{(F_1,\psi_1)}(x_n + \delta A^*(T_{r_n}^{(F_2,\psi_2)} - I)Ax_n);\\
&x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n B)\tfrac{1}{s_n}\textstyle\int_0^{s_n} T(s)u_n ds.
\end{aligned}$$

In 2015 Wang [19] introduced and studied the following iterative method to prove a strong convergence theorem for $F(T)$ and VIP in real Hilbert space:

$$\begin{aligned}
y_n &= \alpha_n u + (1-\alpha_n)x_n,\\
x_{n+1} &= \beta_n x_n + (1-\beta_n)TJ_{r_n}(y_n - r_n A y_n), \qquad \forall n \geq 1,
\end{aligned}$$

where u is fixed element and $J_{r_n} = (1 + r_n B)^{-1}$.

In 2017 Zhang and Gui [21] introduced an iterative algorithm in a Hilbert space as follows:

$$\begin{aligned}
u_n &= T_{r_n}^{F_1}(x_n + \delta A^*(T_{s_n}^{F_2} - I)Ax_n)\\
x_{n+1} &= \alpha_n f(x_n) + \tfrac{(1-\alpha_n)}{l}\textstyle\sum_{i=0}^{l} T_i^n u_n,
\end{aligned}$$

where $T_i : C \to C$ is an asymptotically nonexpansive mapping for $i = 0, 1, \ldots, n$.

Motivated by the works of Kumam et al. [11], Kazmi and Rizvi [10], Zhang and Gui [21], Wang [19] and by the ongoing research in direction, we introduce and study an iterative method for approximating a common solution of $SGEP, VIP$ and FPP for a nonexpansive semigroup in real Hilbert spaces.

2. **Preliminaries**

Let H be a Hilbert space and C be a nonempty closed and convex subset of H. For each point $x \in H$, there exists a unique nearest point of C, denoted by $P_C x$, such that $\|x - P_C x\| \leq \|x - y\|$ for all $y \in C$. P_C is called the metric projection of H onto C. It is well known that P_C is nonexpansive mapping and is characterized by the following property:

$$\langle x - P_C x, y - P_C y\rangle \leq 0. \tag{2.1}$$

Further, it is well known that every nonexpansive operator $T : H \to H$ satisfies, for all $(x, y) \in H \times H$, inequality

$$\langle (x - T(x)) - (y - T(y)), T(y) - T(x) \rangle \leq (\frac{1}{2}) \|(T(x) - x) - (T(y) - y)\|^2, \quad (2.2)$$

and therefore, we get, for all $(x, y) \in H \times Fix(T)$,

$$\langle (x - T(x)), (y - T(y)) \rangle \leq (\frac{1}{2}) \|(T(x) - x)\|^2, \quad (2.3)$$

see, e.g. [9].It is also known that H satisfies Opial's condition [16], i.e., for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$ the inequality

$$\liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|x_n - y\| \quad (2.4)$$

holds for every $y \in H$ with $y \neq x$.

Definition 2.1. *A mapping $T : H \to H$ is said to be firmly nonexpansive, if*

$$\langle Tx - Ty, x - y \rangle \geq \|Tx - Ty\|^2, \ \forall x, y \in H.$$

Lemma 2.2. *[7] The following inequality holds in real space H:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle, \ \forall x, y \in H.$$

Definition 2.3. *A mapping $T : C \to H$ is said to be monotone, if*

$$\langle Tx - Ty, x - y \rangle \geq 0, \quad \forall x, y \in C.$$

T is called α-inverse-strongly-monotone if there exists a positive real number α such that

$$\langle Tx - Ty, x - y \rangle \geq \alpha \|Tx - Ty\|^2, \quad \forall x, y \in C.$$

Lemma 2.4. *[2] Let $M : H \to 2^H$ be a maximal monotone mapping, and let $E : H \to H$ be a monotone mapping, then the mapping $M + E : H \to 2^H$ is a maximal monotone mapping.*

Lemma 2.5. *[22] Let $x \in H$ be a solution of variational inclusion if and only if $x = J_{M,\lambda}(x - \lambda Ex), \forall \lambda > 0$, that is*

$$I(E, M) = Fix(J_{M,\lambda}(I - \lambda E)), \qquad \forall \lambda > 0.$$

Lemma 2.6. *[13] Assume that B is a strong positive linear bounded self adjoint operator on a Hilbert space H with coefficient $\bar{\gamma} > 0$ and $0 < \rho \leq \|B\|^{-1}$. Then $\|I - \rho B\| \leq 1 - \rho\bar{\gamma}$.*

Lemma 2.7. *[17] Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$, for all integers $n \geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

Lemma 2.8. *[20] Let $\{a_n\}$ be a sequence of nonnegative real numbers such that $a_{n+1} \leq (1-\alpha_n)a_n + \delta_n,\ n \geq 0$ where α_n is a sequence in $(0,1)$ and δ_n is a sequence in $\mathbb{R}$ such that (i) $\Sigma_{n=1}^{\infty}\alpha_n = \infty$; (ii) $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0$ or (iii) $\Sigma_{n=1}^{\infty}\delta_n < \infty$. Then $\lim_{n\to\infty} a_n = 0$.*

Assumption 2.9. *[12] Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying the following assumption:*

(1) $F(x,x) \geq 0,\ \forall x \in C$,
(2) *F is monotone, i.e.,* $F(x,y) + F(y,x) \leq 0,\ \forall x \in C$,
(3) *F is upper hemicontinuous, i.e., for each $x, y, z \in C$,* $\limsup_{t\to 0} F(tz + (1-t)x, y) \leq F(x,y)$,
(4) *For each $x \in C$ fixed, the function $x \to F(x,y)$ is convex and lower semicontinuous;*

let $\psi : C \times C \to \mathbb{R}$ such that

(1) $\psi(x,x) \geq 0,\ \forall x \in C$,
(2) *For each $y \in C$ fixed, the function $x \to \psi(x,y)$ is upper semicontinuous,*
(3) *For each $x \in C$ fixed, the function $y \to \psi(x,y)$ is convex and lower semicontinuous;*

Lemma 2.10. *[10] Assume that $F_1, \psi_1 : C \times C \to \mathbb{R}$ satisfy Assumption 2.9. Let $r > 0$ and $x \in H_1$. Then, there exists $z \in C$ such that*

$$F_1(z,y) + \psi_1(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C.$$

Lemma 2.11. *[4] Assume that the bifunctions $F_1, \psi_1 : C \times C \to \mathbb{R}$ satisfy Assumption 2.9 and ψ_1 is monotone. For $r > 0$ and for all $x \in H_1$, define a mapping $T_r^{(F_1,\psi_1)} : H_1 \to C$ as follows:*

$$T_r^{(F_1,\psi_1)}x = \{z \in C : F_1(z,y) + \psi_1(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0\}, \quad \forall y \in C.$$

Then the followings hold:

$(i) T_r^{(F_1,\psi_1)}\ is\ single-valued.$

$(ii) T_r^{(F_1,\psi_1)}\ is\ firmly\ nonexpansive, i.e.,$

$$\|T_r^{(F_1,\psi_1)}(x) - T_r^{(F_1,\psi_1)}(y)\|^2 \leq \langle T_r^{(F_1,\psi_1)}(x) - T_r^{(F_1,\psi_1)}(y), x - y\rangle, \quad x, y \in H_1.$$

$(iii) Fix(T_r^{(F_1,\psi_1)}) = GEP(F_1, \psi_1).$

$(iv) GEP(F_1, \psi_1)$ *is compact and convex.*

Further, assume that $F_2, \psi_2 : Q \times Q \to \mathbb{R}$ satisfy Assumption 2.9. For $s > 0$ and for all $w \in H_2$, define a mapping $T_s^{(F_2,\psi_2)} : H_2 \to Q$ as follows:

$$T_s^{(F_2,\psi_2)}w = \{d \in Q : F_2(d,e) + \psi_2(d,e) + \frac{1}{s}\langle e - d, d - w\rangle \geq 0\}, \quad \forall e \in Q.$$

Then, we easily observe that $T_a^{(F_2,\psi_2)}$ *satisfies in Lemma 2.11 and* $GEP(F_1,\psi_1)$ *is compact and convex.*

Lemma 2.12. *[8] Let* $F_1 : C \times C \to \mathbb{R}$ *be a bifunction satisfying Assumption 2.9 and let* $T_r^{F_1}$ *be defined as in Lemma 2.11, for* $r > 0$*. Let* $x, y \in H_1$ *and* $r_1, r_2 > 0$*. Then,*

$$\|T_{r_2}^{F_1} y - T_{r_1}^{F_1} x\| \leq \|x - y\| + |\frac{r_2 - r_1}{r_2}| \|T_{r_2}^{F_1} y - y\|.$$

Lemma 2.13. *[18] Let* $F_1 : C \times C \to \mathbb{R}$ *be a bifunction satisfying Assumption 2.9 and let* $T_r^{F_1}$ *be defined as in Lemma 2.11, for* $r > 0$*. Let* $x \in H_1$ *and* $r_1, r_2 > 0$*. Then,*

$$\|T_{r_2}^{F_1} x - T_{r_1}^{F_1} x\|^2 \leq \frac{r_2 - r_1}{r_2} \langle T_{r_2}^{F_1}(x) - T_{r_1}^{F_1}(x), T_{r_2}^{F_1}(x) - x \rangle.$$

Notation. Let $\{x_n\}$ be a sequence in H, then $x_n \to x$ (respectively, $x_n \rightharpoonup x$) denotes strong (respectively, weak) convergence of the sequence $\{x_n\}$ to a point $x \in H$.

3. Viscosity Iterative Algorithm

In this section, we prove a strong convergence theorem based on the explicit iterative for fixed point of nonexpansive semigroup. We firstly present the following unified algorithm.

Let H_1 and H_2 be two real Hilbert spaces; Let $C \subseteq H_1$, $Q \subseteq H_2$ be nonempty, closed and convex subsets; Let $F_1, \psi_1 : C \times C \to \mathbb{R}$ and $F_2, \psi_2 : Q \times Q \to \mathbb{R}$ are nonlinear mappings satisfying Assumption 2.9 and F_2 is upper semicontinuous in first argument. Let $\{V_i : C \to C\}$ be a uniformly k-strict pseudocontractions and $T^i : C \to C$ be a nonexpansive mapping on C for $i = 0, 1, 2, \ldots, n$ defined by $T^i x = tx + (1-t)V_i$ for each $x \in C$, $t \in [k, 1)$. Let $f : H_1 \to H_1$ be a contraction mapping with constant $\alpha \in (0,1)$, $A : H_1 \to H_2$ be a bounded linear operator, B be a strongly positive bounded linear self adjoint operators on H_1 with constant $\bar{\gamma}_1 > 0$, such that $0 < \gamma < \frac{\bar{\gamma}_1}{\alpha} < \gamma + \frac{1}{\alpha}$, E be a $\bar{\gamma}_2$- inverse strongly monotone mapping on H_1 such that $\bar{\gamma}_2 > 0, \lambda \in (0, 2\bar{\gamma}_2)$ and $M : H_1 \to 2^{H_1}$ be a maximal monotone mapping. Suppose that $\Theta = \bigcap_{i=0}^n Fix(T^i) \cap \Gamma \cap I(E,M) \neq \emptyset$.

Algorithm 3.1. *For given* $x_0 \in C$ *arbitrary, let the sequence* $\{x_n\}$ *be generated by the manner:*

$$\begin{cases} u_n = T_{r_n}^{(F_1,\psi_1)}(x_n + \delta A^*(T_{s_n}^{(F_2,\psi_2)} - I)Ax_n) \\ w_n = J_{M,\lambda}(u_n - \lambda E u_n) \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n B)\frac{1}{n+1}\sum_{i=0}^n T^i w_n + \gamma_n e_n, \end{cases} \tag{3.1}$$

where $\{e_n\}$ *is a bounded error sequence in* H_1*,* $\delta \in (0, \frac{1}{L^2})$*,* L *is the spectral radius of the operator* A^*A *and* A^* *is the adjoint of* A*,* $\{\alpha_n\}$*,* $\{\beta_n\}$ *are the sequence in*

$(0,1)$ *and* $\{r_n\} \subset [r,\infty)$ *with* $r > 0$, $\{s_n\} \subset [s,\infty)$ *with* $s > 0$ *satisfying conditions:*

$(C1)\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0,\ \Sigma_{n=1}^{\infty}\alpha_n = \infty;$

$(C2)\lim_{n\to\infty} \frac{\gamma_n}{\alpha_n} = 0$

$(C3)\lim_{n\to\infty} |r_{n+1} - r_n| = 0,\ \liminf_{n\to\infty} r_n > 0,\ \lim_{n\to\infty} |s_{n+1} - s_n| = 0.$

Lemma 3.2. *Let* $p \in \Theta$. *Then the sequence* $\{x_n\}$ *generated by Algorithm 3.1 is bounded.*

Proof. By Lemma 2.11 (ii), using the similar argument in Remark 3.1 [21], for $\delta \in (0, \frac{1}{2L^2})$, $I + \delta A^*(T_{s_n}^{(F_2,\psi_2)} - I)A$ is a nonexpansive mapping and $A^*(T_{s_n}^{(F_2,\psi_2)} - I)A$ is a $\frac{1}{2L^2}$-inverse strongly monotone mapping. Take $p \in \Theta$.
And similar to Theorem 3.1 [21], we have

$$\|u_n - p\|^2 \le \|x_n - p\|^2 + \delta(\delta - \tfrac{1}{L^2})\|A^*(T_{s_n}^{(F_2,\psi_2)} - I)Ax_n\|^2. \tag{3.2}$$

Since $\delta \in (0, \frac{1}{2L^2})$, we obtain

$$\|u_n - p\|^2 \le \|x_n - p\|^2. \tag{3.3}$$

Now, we show that $I - \lambda E$ is a nonexpansive mapping. Indeed for $x, y \in C$ and $\lambda \in (0, 2\bar{\gamma})$, we have

$$\begin{aligned} \|(I-\lambda E)x - (I-\lambda E)y\|^2 &= \|x - y - \lambda(Ex - Ey)\|^2 \\ &= \|x-y\|^2 - 2\lambda\langle x-y, Ex-Ey\rangle + \lambda^2\|Ex-Ey\|^2 \\ &\le \|x-y\|^2 - 2\lambda\bar{\gamma}_2\|Ex-Ey\|^2 + \lambda^2\|Ex-Ey\|^2 \\ &\le \|x-y\|^2 + \lambda(\lambda - 2\bar{\gamma}_2)\|Ex-Ey\|^2 \\ &\le \|x-y\|^2, \end{aligned} \tag{3.4}$$

then $I - \lambda E$ is a nonexpansive mapping.
Since $J_{M,\lambda}(u_n - \lambda E u_n)$ is a nonexpansive mapping, we have

$$\begin{aligned} \|w_n - p\|^2 &= \|J_{M,\lambda}(u_n - \lambda E u_n) - J_{M,\lambda}(p - \lambda E p)\|^2 \\ &\le \|(u_n - \lambda E u_n) - (p - \lambda E p)\|^2 \\ &\le \|u_n - p\|^2, \end{aligned} \tag{3.5}$$

then

$$\|w_n - p\| \le \|u_n - p\|. \tag{3.6}$$

Then

$$\|w_n - p\| \leq \|x_n - p\|. \tag{3.7}$$

From Theorem 1 [10], we obtain $(1-\beta_n)I-\alpha_n B$ is positive and $\|(1-\beta_n)I-\alpha_n B\| \leq 1-\beta_n-\alpha_n\bar{\gamma}_1$, for any $x, y \in C$.
Now, on setting $t^n := \frac{1}{n+1}\sum_{i=0}^{n} T^i$, we can easily observe that the mapping t^n is nonexpansive. Since $p \in \Theta$, we have

$$t^n p = \frac{1}{n+1}\sum_{i=0}^{n} T^i p = \frac{1}{n+1}\sum_{i=0}^{n} p = p.$$

Since $\{e_n\}$ is bounded, using condition $(C2)$, we obtain that $\{\frac{\gamma_n\|e_n\|}{\alpha_n}\}$ is bounded. Then, there exists a nonnegative real number K such that

$$\|\gamma f(p) - Bp\| + \frac{\gamma_n\|e_n\|}{\alpha_n} \leq K, \qquad \forall n \geq 0, \tag{3.8}$$

therefore

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n\gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n B)t^n w_n + \gamma_n e_n - p\| \\
&\leq \alpha_n\|\gamma f(x_n) - Bp\| + \beta_n\|x_n - p\| \\
&\quad + \|((1-\beta_n)I - \alpha_n B)\|\|t^n w_n - t^n p\| + \gamma_n\|e_n\| \\
&\leq \alpha_n(\|\gamma f(x_n) - \gamma f(p)\| + \|\gamma f(p) - Bp\|) + \beta_n\|x_n - p\| \\
&\quad + (1-\beta_n-\alpha_n\bar{\gamma})\|w_n - p\| + \gamma_n\|e_n\| \\
&\leq \alpha_n\gamma\alpha\|x_n - p\| + \alpha_n\|\gamma f(p) - Bp\| + \beta_n\|x_n - p\| \\
&\quad + (1-\beta_n-\alpha_n\bar{\gamma}_1)\|x_n - p\| + \gamma_n\|e_n\| \\
&\leq (1-(\bar{\gamma}_1 - \gamma\alpha)\alpha_n)\|x_n - p\| + \alpha_n K \\
&\leq \max\{\|x_n - p\|, \tfrac{K}{\bar{\gamma}_1 - \gamma\alpha}\} \\
&\vdots \\
&\leq \max\{\|x_0 - p\|, \tfrac{K}{\bar{\gamma}_1 - \gamma\alpha}\}.
\end{aligned} \tag{3.9}$$

Hence $\{x_n\}$ is bounded. □

We deduce that $\{u_n\}$, $\{w_n\}$, $\{t^n\}$ and $\{f(x_n)\}$ are bounded.

Lemma 3.3. *The following properties are satisfied for the Algorithm 3.1*

$P1.$ $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0.$

$P2.$ $\lim_{n\to\infty}\|x_n-t^n w_n\|=0.$

$P3.$ $\lim_{n\to\infty}\|(T_{s_n}^{(F_2,\psi_2)}-I)Ax_n\|^2=0,\quad \lim_{n\to\infty}\|Eu_n-Ep\|=0.$

$P4.$ $\lim_{n\to\infty}\|u_n-x_n\|=0,\quad \lim_{n\to\infty}\|w_n-u_n\|=0,\quad \lim_{n\to\infty}\|t^n w_n-w_n\|=0.$

Proof. P1: Similar to Theorem 3.1 [21], we obtain

$$\|u_{n+1}-u_n\|\le\|x_{n+1}-x_n\|+\delta\|A\|(\frac{|s_{n+1}-s_n|}{s_{n+1}}\eta_n)^{\frac{1}{2}}+\frac{|r_{n+1}-r_n|}{r_{n+1}}\sigma_{n+1} \tag{3.10}$$

where

$\sigma_{n+1}=\sup_{n\in\mathbb{N}}\|T_{r_{n+1}}^{(F_1,\psi_1)}(x_{n+1}+\delta A^*(T_{s_{n+1}}^{(F_2,\psi_2)}-I)Ax_{n+1})-(x_{n+1}+\delta A^*(T_{s_{n+1}}^{(F_2,\psi_2)}-I)Ax_{n+1})\|,$

$\eta_n=\sup_{n\in\mathbb{N}}\langle T_{s_{n+1}}^{(F_2,\psi_2)}Ax_n-T_{s_n}^{(F_2,\psi_2)}Ax_n, T_{s_{n+1}}^{(F_2,\psi_2)}Ax_n-Ax_n\rangle.$

Since $J_{M,\lambda}(u_n-\lambda Eu_n)$ is a nonexpansive mapping, we have

$$\begin{aligned}\|w_{n+1}-w_n\| &= \|J_{M,\lambda}(u_{n+1}-\lambda Eu_{n+1})-J_{M,\lambda}(u_n-\lambda Eu_n)\|\\ &\le\|(u_{n+1}-\lambda Eu_{n+1})-(u_n-\lambda Eu_n)\|\\ &\le\|u_{n+1}-u_n\|.\end{aligned} \tag{3.11}$$

Next we easily estimate that

$$\|t^{n+1}w_{n+1}-t^n w_n\|\le\|w_{n+1}-w_n\|+\frac{2}{n+2}\|w_n-p\|+\frac{2}{n+2}\|p\|$$

By (3.10) and (3.11) we can write

$$\begin{aligned}\|t^{n+1}w_{n+1}-t^n w_n\| &\le\|x_{n+1}-x_n\|+\delta\|A\|(\tfrac{|s_{n+1}-s_n|}{s_{n+1}}\eta_n)^{\frac{1}{2}}\\ &\quad+\tfrac{|r_{n+1}-r_n|}{r_{n+1}}\sigma_{n+1}+\tfrac{2}{n+2}(\|x_n-p\|+\|p\|),\end{aligned} \tag{3.12}$$

Setting $x_{n+1} = \beta_n x_n + (1-\beta_n)y_n$, then we have

$$\begin{aligned} y_{n+1} - y_n &= \tfrac{\alpha_{n+1}}{1-\beta_{n+1}}(\gamma f(x_{n+1}) - Bt^{n+1}w_{n+1} + \tfrac{\gamma_{n+1}e_{n+1}}{\alpha_{n+1}}) \\ &\quad + t^{n+1}w_{n+1} - t^n w_n + \tfrac{\alpha_n}{1-\beta_n}(Bt_n - \gamma f(x_n) - \tfrac{\gamma_n e_n}{\alpha_n}). \end{aligned}$$

Using (3.12), we have

$$\begin{aligned} \|y_{n+1} - y_n\| &\le \tfrac{\alpha_{n+1}}{1-\beta_{n+1}}(\|\gamma f(x_{n+1}) - Bt^{n+1}w_{n+1}\| + \tfrac{\gamma_{n+1}\|e_{n+1}\|}{\alpha_{n+1}}\|) \\ &\quad + \|t^{n+1}w_{n+1} - t^n w_n\| + \tfrac{\alpha_n}{1-\beta_n}(\|\gamma f(x_n) - Bt^n w_n\| + \tfrac{\gamma_n\|e_n\|}{\alpha_n}\|) \\ &\le \tfrac{\alpha_{n+1}}{1-\beta_{n+1}}(\|\gamma f(x_{n+1}) - Bt^{n+1}w_{n+1}\| + \tfrac{\gamma_{n+1}\|e_{n+1}\|}{\alpha_{n+1}}) + \|x_{n+1} - x_n\| \\ &\quad + \delta\|A\|(\tfrac{|s_{n+1}-s_n|}{s_{n+1}}\eta_n)^{\frac{1}{2}} + \tfrac{|r_{n+1}-r_n|}{r_{n+1}}\sigma_{n+1} + \tfrac{2}{n+2}(\|x_n - p\| + \|p\|) \\ &\quad + \tfrac{\alpha_n}{1-\beta_n}(\|\gamma f(x_n) - Bt^n w_n\| + \tfrac{\gamma_n\|e_n\|}{\alpha_n}\|) \end{aligned}$$

which implies that

$$\begin{aligned} &\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\| \\ &\le \tfrac{\alpha_{n+1}}{1-\beta_{n+1}}(\|\gamma f(x_{n+1}) - Bt^{n+1}w_{n+1}\| + \tfrac{\gamma_{n+1}\|e_{n+1}\|}{\alpha_{n+1}}) + \delta\|A\|(\tfrac{|s_{n+1}-s_n|}{s_{n+1}}\eta_n)^{\frac{1}{2}} \\ &\quad + \tfrac{|r_{n+1}-r_n|}{r_{n+1}}\sigma_{n+1} + \tfrac{2}{n+2}(\|x_n - p\| + \|p\|) + \tfrac{\alpha_n}{1-\beta_n}(\|\gamma f(x_n) - Bt^n w_n\| + \tfrac{\gamma_n\|e_n\|}{\alpha_n}\|). \end{aligned}$$

Hence, it follows by conditions $(C1) - (C3)$ that

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \le 0. \tag{3.13}$$

From Lemma 2.7 and (3.13), we get $\lim_{n\to\infty}\|y_{n+1} - x_n\| = 0$, and

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}(1-\beta_n)\|y_{n+1} - x_n\| = 0. \tag{3.14}$$

Then $\lim_{n\to\infty}\|t^{n+1}w_{n+1} - t^n w_n\| = 0$.

P2: We can write

$$\begin{aligned} \|x_n - t^n w_n\| &\le \|x_{n+1} - x_n\| \\ &\quad + \|\alpha_n\gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n B)t^n w_n + \gamma_n e_n - t^n w_n\| \\ &\le \|x_{n+1} - x_n\| + \alpha_n\|\gamma f(x_n) - Bt^n w_n\| + \beta_n\|x_n - t^n w_n\| + \gamma_n\|e_n\|. \end{aligned}$$

Then

$$(1-\beta_n)\|x_n - t^n w_n\| \le \|x_{n+1} - x_n\| + \alpha_n\|\gamma f(x_n) - Bt^n w_n\| + \gamma_n\|e_n\|.$$

Therefore we have

$$\|x_n - t^n w_n\| \quad \le \tfrac{1}{1-\beta_n}\|x_{n+1} - x_n\| + \tfrac{\alpha_n}{1-\beta_n}(\|(\gamma f(x_n) - Bt^n w_n\| + \tfrac{\gamma_n\|e_n\|}{\alpha_n}).$$

Since $\alpha_n \to 0$ and $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$ we obtain

$$\lim_{n\to\infty} \|x_n - t^n w_n\| = 0. \tag{3.15}$$

P3: Since $\{x_n\}$ is bounded, we may assume a nonnegative real number N such that $\|x_n - p\| \le N$. From (3.5) and (3.2), we have

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&= \|\alpha_n\gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n B)t^n w_n + \gamma_n e_n - p\|^2 \\
&= \|\alpha_n(\gamma f(x_n) - Bp) + \beta_n(x_n - t^n w_n) + (1-\alpha_n B)(t^n w_n - p) + \gamma_n e_n\|^2 \\
&\le \|(1-\alpha_n B)(t^n w_n - p) + \beta_n(x_n - t^n w_n)\|^2 + 2\langle\alpha_n(\gamma f(x_n) - Bp) + \gamma_n e_n, x_{n+1} - p\rangle \\
&\le (\|(1-\alpha_n B)(t^n w_n - p)\| + \beta_n\|x_n - t^n w_n\|)^2 + 2\alpha_n\langle\gamma f(x_n) - Bp, x_{n+1} - p\rangle \\
&\quad +2\langle\gamma_n e_n, x_{n+1} - p\rangle \\
&\le ((1-\alpha_n\bar{\gamma}_1)\|w_n - p\| + \beta_n\|x_n - t^n w_n\|)^2 + 2\alpha_n\langle\gamma f(x_n) - Bp, x_{n+1} - p\rangle \\
&\quad +2\gamma_n\|e_n\|N \\
&= (1-\alpha_n\bar{\gamma}_1)^2\|w_n - p\|^2 + \beta_n^2\|x_n - t^n w_n\|^2 + 2(1-\alpha_n\bar{\gamma}_1)\beta_n\|w_n - p\|\|x_n - t^n w_n\| \\
&\quad +2\alpha_n\langle\gamma f(x_n) - Bp, x_{n+1} - p\rangle + 2\gamma_n\|e_n\|N
\end{aligned} \tag{3.16}$$

$$\leq (1-\alpha_n\bar{\gamma}_1)^2\|u_n-p\|^2+\beta_n^2\|x_n-t^nw_n\|^2+2(1-\alpha_n\bar{\gamma}_1)\beta_n\|w_n-p\|\|x_n-t^nw_n\|$$
$$+2\alpha_n\langle\gamma f(x_n)-Bp,x_{n+1}-p\rangle+2\gamma_n\|e_n\|N$$
$$=(1-\alpha_n\bar{\gamma}_1)^2(\|x_n-p\|^2+\delta(\delta-\tfrac{1}{L^2})\|A^*(T_{s_n}^{(F_2,\psi_2)}-I)Ax_n\|^2)+(\beta_n)^2\|x_n-t^nw_n\|^2$$
$$+2(1-\alpha_n\bar{\gamma}_1)\beta_n\|w_n-p\|\|x_n-t^nw_n\|+2\alpha_n\langle\gamma f(x_n)-Bp,x_{n+1}-p\rangle+2\gamma_n\|e_n\|N$$
$$\leq\|x_n-p\|^2+(\alpha_n\bar{\gamma}_1)^2\|x_n-p\|^2+(1-\alpha_n\bar{\gamma}_1)^2\delta(\delta-\tfrac{1}{L^2})\|A^*(T_{s_n}^{(F_2,\psi_2)}-I)Ax_n\|^2$$
$$+\beta_n^2\|x_n-t^nw_n\|^2+2(1-\alpha_n\bar{\gamma}_1)\beta_n\|w_n-p\|\|x_n-t^nw_n\|$$
$$+2\alpha_n(\langle\gamma f(x_n)-Bp,x_{n+1}-p\rangle+\tfrac{\gamma_n\|e_n\|}{\alpha_n}N).$$

Therefore

$$(1-\alpha_n\bar{\gamma}_1)^2\delta(\tfrac{1}{L^2}-\delta)\|A^*(T_{s_n}^{(F_2,\psi_2)}-I)Ax_n\|^2$$
$$\leq\|x_n-p\|^2-\|x_{n+1}-p\|^2+(\alpha_n\bar{\gamma}_1)^2\|x_n-p\|^2+\beta_n^2\|x_n-t^nw_n\|^2$$
$$+2(1-\alpha_n\bar{\gamma}_1)\beta_n\|w_n-p\|\|x_n-t^nw_n\|+2\alpha_n(\langle\gamma f(x_n)-Bp,x_{n+1}-p\rangle+\tfrac{\gamma_n\|e_n\|}{\alpha_n}N)$$
$$\leq(\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\|+(\alpha_n\bar{\gamma}_1)^2\|x_n-p\|^2+\beta_n^2\|x_n-t^nw_n\|^2$$
$$+2(1-\alpha_n\bar{\gamma}_1)\beta_n\|w_n-p\|\|x_n-t^nw_n\|+2\alpha_n(\gamma\|f(x_n)\|+\|Bp\|+\tfrac{\gamma_n\|e_n\|}{\alpha_n}N).$$

Because of $\delta(\frac{1}{L^2}-\delta)>0$, $\|x_n-x_{n+1}\|\to 0$ and $\|x_n-t^nw_n\|\to 0$ as $n\to\infty$ and $(C1)$ we obtain $\lim_{n\to\infty}\|A^*(T_{s_n}^{(F_2,\psi_2)}-I)Ax_n\|^2=0$

which implies that

$$\lim_{n\to\infty}\|(T_{s_n}^{(F_2,\psi_2)}-I)Ax_n\|^2=0. \tag{3.17}$$

It follows from (3.16)

$$\begin{aligned}\|x_{n+1}-p\|^2 &= (1-\alpha_n\bar{\gamma}_1)^2\|w_n-p\|^2+\beta_n^2\|x_n-t^nw_n\|^2\\ &\quad +2(1-\alpha_n\bar{\gamma}_1)\beta_n\|w_n-p\|\|x_n-t^nw_n\|\\ &\quad +2\alpha_n\langle\gamma f(x_n)-Bp,x_{n+1}-p\rangle+2\gamma_n\|e_n\|N\\ &\le (1-\alpha_n\bar{\gamma}_1)^2(\|u_n-p\|^2+\lambda(\lambda-2\bar{\gamma}_2)\|Eu_n-Ep\|^2)\\ &\quad +\beta_n^2\|x_n-t^nw_n\|^2+2(1-\alpha_n\bar{\gamma}_1)\beta_n\|w_n-p\|\|x_n-t^nw_n\|\\ &\quad +2\alpha_n\langle\gamma f(x_n)-Bp,x_{n+1}-p\rangle+2\gamma_n\|e_n\|N.\end{aligned}$$

Therefore

$$\begin{aligned}&(1-\alpha_n\bar{\gamma}_1)^2\lambda(2\bar{\gamma}_2-\lambda)\|Eu_n-Ep\|^2\\ &\le (1-\alpha_n\bar{\gamma}_1)^2\|u_n-p\|^2-\|x_{n+1}-p\|^2+\beta_n^2\|x_n-t^nw_n\|^2\\ &\quad +2(1-\alpha_n\bar{\gamma}_1)\beta_n\|w_n-p\|\|x_n-t^nw_n\|+2\alpha_n\langle\gamma f(x_n)-Bp,x_{n+1}-p\rangle+2\gamma_n\|e_n\|N\\ &\le \|x_n-p\|^2-\|x_{n+1}-p\|^2+(\alpha_n\bar{\gamma}_1)^2\|x_n-p\|^2+\beta_n^2\|x_n-t^nw_n\|^2\\ &\quad +2(1-\alpha_n\bar{\gamma}_1)\beta_n\|w_n-p\|\|x_n-t^nw_n\|+2\alpha_n\langle\gamma f(x_n)-Bp,x_{n+1}-p\rangle+2\gamma_n\|e_n\|N\\ &\le (\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\|+(\alpha_n\bar{\gamma}_1)^2\|x_n-p\|^2+\beta_n^2\|x_n-t^nw_n\|^2\\ &\quad +2(1-\alpha_n\bar{\gamma}_1)\beta_n\|w_n-p\|\|x_n-t^nw_n\|+2\alpha_n(\gamma\|f(x_n)\|+\|Bp\|+\tfrac{\gamma_n\|e_n\|}{\alpha_n})N.\end{aligned}$$

Because of $\lambda(2\bar{\gamma}_2-\lambda)>0, \|x_n-x_{n+1}\|\to 0$ and $\|x_n-t^nw_n\|\to 0$ as $n\to\infty$ and $(C1)$ we obtain

$$\lim_{n\to\infty}\|Eu_n-Ep\|=0. \tag{3.18}$$

P4: Since $p\in\Theta$, we can obtain

$$\|u_n-p\|^2\le\|x_n-p\|^2-\|u_n-x_n\|^2+2\delta\|u_n-x_n\|\|A^*(T_{a_n}^{(F_2,\psi_2)}-I)Ax_n\|,$$

see [21]. It follows from (3.16) that

$$\begin{aligned}
&\|x_{n+1}-p\|^2\\
&\le (1-\alpha_n\bar{\gamma}_1)^2\|w_n-p\|^2+\beta_n^2\|x_n-t^nw_n\|^2+2(1-\alpha_n\bar{\gamma}_1)\beta_n\|w_n-p\|\|x_n-t^nw_n\|\\
&\quad +2\alpha_n\langle\gamma f(x_n)-Bp,x_{n+1}-p\rangle+2\gamma_n\|e_n\|N\\
&\le (1-\alpha_n\bar{\gamma}_1)^2\|u_n-p\|^2+\beta_n^2\|x_n-t^nw_n\|^2+2(1-\alpha_n\bar{\gamma}_1)\beta_n\|w_n-p\|\|x_n-t^nw_n\|\\
&\quad +2\alpha_n\langle\gamma f(x_n)-Bp,x_{n+1}-p\rangle+2\gamma_n\|e_n\|N\\
&\le (1-\alpha_n\bar{\gamma}_1)^2(\|x_n-p\|^2-\|u_n-x_n\|^2+2\delta\|A(u_n-x_n)\|\|(T_{s_n}^{(F_2,\psi_2)}-I)Ax_n\|)\\
&\quad +\beta_n^2\|x_n-t^nw_n\|^2+2(1-\alpha_n\bar{\gamma}_1)\beta_n\|w_n-p\|\|x_n-t^nw_n\|\\
&\quad +2\alpha_n\langle\gamma f(x_n)-Bp,x_{n+1}-p\rangle+2\gamma_n\|e_n\|N\\
&\le \|x_n-p\|^2+(\alpha_n\bar{\gamma}_1)^2\|x_n-p\|^2-(1-\alpha_n\bar{\gamma}_1)^2\|u_n-x_n\|^2\\
&\quad +2(1-\alpha_n\bar{\gamma}_1)^2\delta\|A(u_n-x_n)\|\|(T_{s_n}^{(F_2,\psi_2)}-I)Ax_n\|+\beta_n^2\|x_n-t^nw_n\|^2\\
&\quad +2(1-\alpha_n\bar{\gamma}_1)\beta_n\|w_n-p\|\|x_n-t^nw_n\|+2(\alpha_n\langle\gamma f(x_n)-Bp,x_{n+1}-p\rangle+\tfrac{\gamma_n\|e_n\|}{\alpha_n}N).
\end{aligned}$$

Therefore we have

$$\begin{aligned}
&(1-\alpha_n\bar{\gamma}_1)^2\|u_n-x_n\|^2\\
&\le \|x_n-p\|^2-\|x_{n+1}-p\|^2+(\alpha_n\bar{\gamma}_1)^2\|x_n-p\|^2\\
&\quad +2(1-\alpha_n\bar{\gamma}_1)^2\delta\|A(u_n-x_n)\|\|(T_{s_n}^{(F_2,\psi_2)}-I)Ax_n\|+\beta_n^2\|x_n-t^nw_n\|^2\\
&\quad +2(1-\alpha_n\bar{\gamma}_1)\beta_n\|w_n-p\|\|x_n-t^nw_n\|+2\alpha_n(\langle\gamma f(x_n)-Bp,x_{n+1}-p\rangle+\tfrac{\gamma_n\|e_n\|}{\alpha_n}N)\\
&\le (\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\|+(\alpha_n\bar{\gamma}_1)^2\|x_n-p\|^2\\
&\quad +2(1-\alpha_n\bar{\gamma}_1)^2\delta\|A(u_n-x_n)\|\|(T_{s_n}^{(F_2,\psi_2)}-I)Ax_n\|+\beta_n^2\|x_n-t^nw_n\|^2\\
&\quad +2(1-\alpha_n\bar{\gamma}_1)\beta_n\|w_n-p\|\|x_n-t^nw_n\|+2\alpha_n(\gamma\|f(x_n)\|+\|Bp\|+\tfrac{\gamma_n\|e_n\|}{\alpha_n})N.
\end{aligned}$$

Since $\alpha_n\to 0$, $\beta_n\to 0$, $\|(T_{s_n}^{(F_2,\psi_2)}-I)Ax_n\|\to 0$ and $\|x_n-t^nw_n\|\to 0$ as $n\to\infty$ and from $(C1)$, we obtain

$$\lim_{n\to\infty}\|x_n-u_n\|=0. \tag{3.19}$$

Since $p \in \Theta$ and $J_{M,\lambda}$ is 1-inverse strongly monotone [2, 22], we can obtain

$$\begin{aligned}
\|w_n - p\|^2 &= \|J_{M,\lambda}(u_n - \lambda E u_n) - J_{M,\lambda}(p - \lambda E p)\|^2 \\
&\le \langle (u_n - \lambda E u_n) - (p - \lambda E p), w_n - p\rangle \\
&= \tfrac{1}{2}(\|(u_n - \lambda E u_n) - (p - \lambda E p)\|^2 + \|w_n - p\|^2 \\
&\quad - \|(u_n - \lambda E u_n) - (p - \lambda E p) - (w_n - p)\|^2) \\
&\le \tfrac{1}{2}(\|u_n - p\|^2 + \|w_n - p\|^2 - \|(u_n - \lambda E u_n) - (p - \lambda E p) - (w_n - p)\|^2) \\
&\le \tfrac{1}{2}(\|x_n - p\|^2 + \|w_n - p\|^2 - \|w_n - u_n\|^2 + 2\lambda\langle u_n - w_n, E u_n - E p\rangle \\
&\quad - \lambda^2\|E u_n - E p\|^2),
\end{aligned}$$

and hence,

$$\|w_n - p\|^2 \le \|x_n - p\|^2 - \|w_n - u_n\|^2 + 2\lambda\|u_n - w_n\|\|E u_n - E p\|. \tag{3.20}$$

It follows from (3.16) and (3.20) that

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\le (1 - \alpha_n\bar{\gamma}_1)^2\|w_n - p\|^2 + \beta_n^2\|x_n - t^n w_n\|^2 + 2(1 - \alpha_n\bar{\gamma}_1)\beta_n\|w_n - p\|\|x_n - t^n w_n\| \\
&\quad + 2\alpha_n\langle \gamma f(x_n) - Bp, x_{n+1} - p\rangle + 2\gamma_n\|e_n\|N \\
&\le (1 - \alpha_n\bar{\gamma}_1)^2(\|u_n - p\|^2 - \|w_n - u_n\|^2 + 2\lambda\|u_n - w_n\|\|E u_n - E p\|) \\
&\quad + \beta_n^2\|x_n - t^n w_n\|^2 + 2(1 - \alpha_n\bar{\gamma}_1)\beta_n\|w_n - p\|\|x_n - t^n w_n\| \\
&\quad + 2\alpha_n\langle \gamma f(x_n) - Bp, x_{n+1} - p\rangle + 2\gamma_n\|e_n\|N,
\end{aligned}$$

therefore we have

$$
\begin{aligned}
&(1-\alpha_n\bar{\gamma}_1)^2\|w_n-u_n\|^2\\
&\le \|x_n-p\|^2-\|x_{n+1}-p\|^2+(\alpha_n\bar{\gamma}_1)^2\|x_n-p\|^2+(1-\alpha_n\bar{\gamma}_1)^2 2\lambda\|u_n-w_n\|\|Eu_n-Ep\|\\
&\quad+\beta_n^2\|x_n-t^nw_n\|^2+2(1-\alpha_n\bar{\gamma}_1)\beta_n\|w_n-p\|\|x_n-t^nw_n\|\\
&\quad+2\alpha_n\langle\gamma f(x_n)-Bp,x_{n+1}-p\rangle+2\gamma_n\|e_n\|N\\
&\le (\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\|+(\alpha_n\bar{\gamma}_1)^2\|x_n-p\|^2\\
&\quad+(1-\alpha_n\bar{\gamma}_1)^2 2\lambda\|u_n-w_n\|\|Eu_n-Ep\|+\beta_n^2\|x_n-t^nw_n\|^2\\
&\quad+2(1-\alpha_n\bar{\gamma}_1)\beta_n\|w_n-p\|\|x_n-t^nw_n\|+2\alpha_n(\gamma\|f(x_n)\|+\|Bp\|+\tfrac{\gamma_n\|e_n\|}{\alpha_n})N.
\end{aligned}
$$

Since $\|x_n-x_{n+1}\|\to 0, \|x_n-t^nw_n\|\to 0$ and $\|Eu_n-Ep\|\to 0$ and from $(C1)$, we obtain

$$\lim_{n\to\infty}\|w_n-u_n\|=0. \tag{3.21}$$

Using (3.15), (3.19) and (3.21), we obtain

$$\|t^nw_n-w_n\|\le\|t^nw_n-x_n\|+\|x_n-u_n\|+\|u_n-w_n\|\to 0, \text{ as } n\to\infty$$

which implies

$$\lim_{n\to\infty}\|t^nw_n-w_n\|=0.$$

□

4. Main Result

Theorem 4.1. *The Algorithm defined by (3.1) convergence strongly to* $z\in\bigcap_{i=1}^n Fix(T^i)\cap\Gamma\cap I(E,M)$*, which is a unique solution of the variational inequality* $\langle(\gamma f-B)z,y-z\rangle\le 0,\quad\forall y\in\Theta$.

Proof. Let $s=P_\Theta$. We get

$$
\begin{aligned}
\|s(I-B+\gamma f)(x)-s(I-B+\gamma f)(y)\| &\le \|(I-B+\gamma f)(x)-(I-B+\gamma f)(y)\|\\
&\le \|I-B\|\|x-y\|+\gamma\|f(x)-f(y)\|\\
&\le (1-\bar{\gamma}_1)\|x-y\|+\gamma\alpha\|x-y\|\\
&= (1-(\bar{\gamma}_1-\gamma\alpha))\|x-y\|.
\end{aligned}
$$

Then $s(I-B+\gamma f)$ is a contraction mapping from H_1 into itself. Therefore by Banach contraction principle, there exists $z\in H_1$ such that $z=s(I-B+\gamma f)z=P_\Theta(I-B+\gamma f)z$.

We show that $\limsup_{n\to\infty}\langle(\gamma f - B)z, x_n - z\rangle \le 0$ where $z = P_\Theta(I - B + \gamma f)$. To show this inequality, we choose a subsequence $\{w_{n_i}\}$ of $\{w_n\}$ such that

$$\limsup_{n\to\infty}\langle(\gamma f - B)z, w_n - z\rangle = \lim_{n\to\infty}\langle(\gamma f - B)z, w_{n_i} - z\rangle. \tag{4.1}$$

Since $\{w_{n_i}\}$ is bounded, there exists a subsequence $\{w_{n_{i_j}}\}$ of $\{w_{n_i}\}$ which converges weakly to some $w \in C$. Without loss of generality, we can assume that $w_{n_i} \rightharpoonup w$. From $\|t^n w_n - w_n\| \to 0$, we obtain $t^n w_{n_i} \rightharpoonup w$.

Now, we prove that $w \in \bigcap_{i=0}^n Fix(T^i) \cap \Gamma \cap I(E, M)$. Let us first show that $w \in Fix(t^n) = \frac{1}{n+1}\sum_{i=0}^n Fix(T^i)$. Assume that $w \notin \frac{1}{n+1}\sum_{i=0}^n Fix(T^i)$. Since $w_{n_i} \rightharpoonup w$ and $t^n w \neq w$, from Opial's conditions (2.4) and Lemma 3.3 (P4), we have

$$\begin{aligned}\liminf_{n\to\infty}\|w_{n_i} - w\| &< \liminf_{n\to\infty}\|w_{n_i} - t^n w\| \\ &\le \liminf_{n\to\infty}(\|w_{n_i} - t^n w_{n_i}\| + \|t^n w_{n_i} - t^n w\|) \\ &\le \liminf_{n\to\infty}\|w_{n_i} - w\|,\end{aligned}$$

which is a contradiction. Thus, we obtain $w \in Fix(t^n)$. We show that $w \in \Gamma$. Since $u_n = T_{r_n}^{(F_1,\psi_1)}(x_n + \delta A^*(T_{s_n}^{(F_2,\psi_2)} - I)Ax_n)$, where $d_n = x_n + \delta A^*(T_{s_n}^{(F_2,\psi_2)} - I)Ax_n$, we have

$$F_1(u_n, y) + \psi_1(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - d_n\rangle \ge 0, \qquad \forall y \in C.$$

It follows from the monotonicity of F_1 that

$$\psi_1(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - d_n\rangle \ge F_1(u_n, y), \qquad \forall y \in C$$

which implies that

$$\psi_1(u_n, y) + \langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_n} + \delta A^*(\frac{(T_{s_{n_i}}^{(F_2,\psi_2)} - I)Ax_{n_i}}{r_n})\rangle \ge F_1(y, u_{n_i}), \qquad \forall y \in C.$$

Because of $\|u_n - x_n\| \to 0$, we get $u_{n_i} \rightharpoonup w$ and $\frac{u_{n_i} - x_{n_i}}{r_n} \to 0$.

Since $\lim_{n\to\infty}\|A^*(T_{s_n}^{(F_2,\psi_2)} - I)Ax_n\| = 0$ then $A^*(\frac{(T_{s_{n_i}}^{(F_2,\psi_2)} - I)Ax_{n_i}}{r_n}) \to 0$.

Therefore

$$\psi_1(u_{n_i}, y) \ge F_1(y, u_{n_i}), \qquad h_1(w, y) \ge F_1(y, w).$$

Let $y_t = ty + (1-t)w$ for all $t \in (0,1]$. Since $y \in C$ and $w \in C$, we get $y_t \in C$. It follows from Assumption 2.9 that

$$\begin{aligned} 0 = F_1(y_t, y_t) + \psi_1(y_t, y_t) &\leq tF_1(y_t, y) + (1-t)F_1(y_t, w) \\ &\quad + t\psi_1(y_t, y) + (1-t)\psi_1(y_t, w) \\ &= t(F_1(y_t, y) + \psi_1(y_t, y)) \\ &\quad + (1-t)(F_1(y_t, w) + \psi_1(y_t, w)) \\ &\leq F_1(y_t, y) + \psi_1(y_t, y), \end{aligned}$$

so $0 \leq F_1(y_t, y) + \psi_1(y_t, y)$.

Letting $t \to 0$, we obtain $0 \leq F_1(w, y) + \psi_1(w, y)$. This implies that $w \in GEP(F_1, \psi_1)$. Now we show that $Aw \in GEP(F_2, \psi_2)$. Since $\|u_n - x_n\| \to 0$, $u_n \rightharpoonup w$ as $n \to \infty$ and $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup w$ and since A is bounded linear operator so that $Ax_{n_j} \rightharpoonup Aw$.
Because of $\|(T_{s_n}^{(F_2,\psi_2)} - I)Ax_n\| \to 0$, we have $T_{s_n}^{(F_2,\psi_2)} Ax_{n_j} \rightharpoonup Aw$. Therefore from Lemma 2.11, we have

$$\begin{aligned} &F_2(T_{s_{n_j}}^{(F_2,\psi_2)} Ax_{n_j}, v) + \psi_2(T_{s_{n_j}}^{(F_2,\psi_2)} Ax_{n_j}, v) \\ &\qquad + \frac{1}{s_{n_j}} \langle v - T_{s_{n_j}}^{(F_2,\psi_2)} Ax_{n_j}, T_{s_{n_j}}^{(F_2,\psi_2)} Ax_{n_j} - Aw \rangle \geq 0, \quad \forall v \in Q. \end{aligned}$$

Since F_2 is upper semicontinuous in first argument, taking lim sup to above inequality as $j \to \infty$, we obtain

$$F_2(Aw, v) + \psi_2(Aw, v) \geq 0, \qquad \forall v \in Q,$$

which means that $Aw \in GEP(F_2, \psi_2)$ and hence $w \in \Gamma$.
Now we show that $w \in I(E, M)$. It follows from Lemma 2.4 that $M + E$ is a maximal monotone. Let $(y, g) \in G(M + E)$, that is $g - Ey \in M(y)$.
Since $w_{n_i} = J_{M,\lambda}(u_{n_i} - \lambda E u_{n_i})$, we have $u_{n_i} - \lambda E u_{n_i} \in (I + \lambda M)(w_{n_i})$, then $\frac{1}{\lambda}(u_{n_i} - w_{n_i} - \lambda E u_{n_i}) \in M(w_{n_i})$.
Since $M + E$ is a maximal monotone, we have

$$\langle y - w_{n_i}, g - Ey - \frac{1}{\lambda}(u_{n_i} - w_{n_i} - \lambda E u_{n_i}) \rangle \geq 0,$$

and so

$$\begin{aligned} \langle y - w_{n_i}, g \rangle &\geq \langle y - w_{n_i}, Ey + \tfrac{1}{\lambda}(u_{n_i} - w_{n_i} - \lambda E u_{n_i}) \rangle \\ &= \langle y - w_{n_i}, Ey - Ew_{n_i} + Ew_{n_i} - Eu_{n_i} + \tfrac{1}{\lambda}(u_{n_i} - w_{n_i}) \rangle \\ &\geq 0 + \langle y - w_{n_i}, Ew_{n_i} - Eu_{n_i} \rangle + \langle y - w_{n_i}, \tfrac{1}{\lambda}(u_{n_i} - w_{n_i}) \rangle. \end{aligned}$$

Since E is a $\bar{\gamma}_2$-inverse strongly monotone, we can easily observe that $\|Ew_n -$

$Eu_n\| \to 0$.
It follows from (3.21), $\|Ew_n - Eu_n\| \to 0$ and $w_{n_i} \rightharpoonup w$ that

$$\lim_{n\to\infty} \langle y - w_{n_i}, g\rangle = \langle y - w, g\rangle \geq 0.$$

It follows from the maximal monotonicity of $M + E$ that $0 \in (M + E)(w)$, that is $w \in I(E, M)$.
We claim that $\limsup_{n\to\infty}\langle(\gamma f - B)z, x_n - z\rangle \leq 0$, where $z = P_\Theta(I - B + \gamma f)$. Now from (2.1), we have

$$\begin{aligned} \limsup_{n\to\infty}\langle(\gamma f - B)z, x_n - z\rangle &= \limsup_{n\to\infty}\langle(\gamma f - B)z, t^n w_n - z\rangle \\ &\leq \limsup_{i\to\infty}\langle(\gamma f - B)z, t^n w_{n_i} - z\rangle \\ &= \langle(\gamma f - B)z, w - z\rangle \leq 0. \end{aligned} \tag{4.2}$$

Next, we show that $x_n \to z$. It follows from (3.3) that

$$\begin{aligned} &\|x_{n+1} - z\|^2 \\ &= \alpha_n\langle\gamma f(x_n) - Bz, x_{n+1} - z\rangle + \beta_n\langle x_n - z, x_{n+1} - z\rangle \\ &\quad +\langle((1 - \beta_n)I - \alpha_n B)(t^n w_n - z) + \gamma_n e_n, x_{n+1} - z\rangle \\ &\leq \alpha_n(\gamma\langle f(x_n) - f(z), x_{n+1} - z\rangle + \langle\gamma f(z) - Bz, x_{n+1} - z\rangle) + \beta_n\|x_n - z\|\|x_{n+1} - z\| \\ &\quad +\|(1 - \beta_n)I - \alpha_n B\|\|t^n w_n - z\|\|x_{n+1} - z\| + \gamma_n\|e_n\|N \\ &\leq \alpha_n\alpha\gamma\|x_n - z\|\|x_{n+1} - z\| + \alpha_n\langle\gamma f(z) - Bz, x_{n+1} - z\rangle + \beta_n\|x_n - z\|\|x_{n+1} - z\| \\ &\quad +(1 - \beta_n - \alpha_n\bar{\gamma}_1)\|x_n - z\|\|x_{n+1} - z\| + \gamma_n\|e_n\|N \\ &\leq \tfrac{1-\alpha_n(\bar{\gamma}_1-\alpha\gamma)}{2}(\|x_n - z\|^2 + \|x_{n+1} - z\|^2) + \alpha_n\langle\gamma f(z) - Bz, x_{n+1} - z\rangle + \gamma_n\|e_n\|N \\ &\leq \tfrac{1-\alpha_n(\bar{\gamma}_1-\alpha\gamma)}{2}\|x_n - z\|^2 + \tfrac{1}{2}\|x_{n+1} - z\|^2 + \alpha_n\langle\gamma f(z) - Bz, x_{n+1} - z\rangle + \gamma_n\|e_n\|N. \end{aligned}$$

This implies that

$$\begin{aligned} 2\|x_{n+1} - z\|^2 &\leq (1 - \alpha_n(\bar{\gamma}_1 - \alpha\gamma))\|x_n - z\|^2 + \|x_{n+1} - z\|^2 \\ &\quad +2\alpha_n(\langle\gamma f(z) - Bz, x_{n+1} - z\rangle + \tfrac{\gamma_n\|e_n\|}{\alpha_n}N). \end{aligned}$$

Then we have

$$\|x_{n+1} - z\|^2 \leq (1 - \alpha_n(\bar{\gamma}_1 - \alpha\gamma))\|x_n - z\|^2 + 2\alpha_n M_n, \tag{4.3}$$

where $k_n = \alpha_n(\bar{\gamma}_1 - \alpha\gamma)$ and $M_n = \langle \gamma f(z) - Bz, x_{n+1} - z\rangle + \frac{\gamma_n \|e_n\|}{\alpha_n}N$.
Since $\lim_{n\to\infty}\alpha_n = 0$ and $\Sigma_{n=0}^{\infty}\alpha_n = \infty$, it is easy to see that $\lim_{n\to\infty} k_n = 0$, $\Sigma_{n=0}^{\infty}k_n = \infty$ and $\limsup_{n\to\infty} M_n \le 0$. Hence, from (4.2) and (4.3) and Lemma 2.8, we deduce that $x_n \to z$, where $z = P_\Theta(I - B + \gamma f)z$. □

5. Numerical Examples

In this section, we give some examples and numerical results for supporting our main theorem.

Example 5.1. *Let $H_1 = H_2 = R$, $C = [0,2]$ and $Q = [-4,-2]$; let $F_1, \psi_1 : C \times C \to R$ and $F_2, \psi_2 : Q \times Q \to R$ be defined by $F_1(x,y) = x(y-x), \psi_1(x,y) = 2x(y-x)$, $\forall x, y \in C$ and $F_2(u,v) = -2u(u-v), \psi_2(u,v) = 3u(v-u)$, $\forall u, v \in Q$, and let for each $x \in R$, we define $f(x) = \frac{1}{6}x$, $A(x) = -2x$, $B(x) = \frac{1}{2}x$, $E(x) = 2x - 6$, and*

$$Mx = \begin{cases} \{x\}, & x > 2 \\ \{2\}, & x \le 2 \end{cases}$$

and let, for each $x \in C$, $V_i x = -2\alpha_i x$, where $\alpha_i = i+1$, $i = 0, 1, \cdots, 5$ and $e_n = \sin n$. Then there exist unique sequences $\{w_n\}$, $\{x_n\} \subset R$, $\{u_n\} \subset C$, and $\{z_n\} \subset Q$ generated by the iterative schemes

$$z_n = T_{s_n}^{(F_2,\psi_2)}(Ax_n); \qquad u_n = T_{r_n}^{(F_1,\psi_1)}(x_n + \tfrac{1}{32}A^*(z_n - Ax_n)); \\ w_n = (I+M)^{-1}(u_n - Eu_n); \tag{5.1}$$

$$x_{n+1} = (\frac{1}{3n} + \frac{1}{2(n+1)^2})x_n + ((1 - \frac{1}{2(n+1)^2})I - \frac{1}{n}B)\frac{1}{n+1}\sum_{i=0}^{n} T^i w_n + \gamma_n e_n \tag{5.2}$$

*where $\alpha_n = \frac{1}{n}$, $\beta_n = \frac{1}{2(n+1)^2}$, $\gamma_n = \frac{1}{n^3}$, $r_n = 1 + \frac{2}{n}$ and $s_n = \frac{n}{2n+1}$.
It is easy to prove that the bifunctions F_1, ψ_1 and F_2, ψ_2 satisfy the Assumption 2.9 and F_2 is upper semicontinuous, A is a bounded linear operator on R with adjoint operator A^* and $\|A\| = \|A^*\| = 1$. Hence $\delta \in (0,1)$, so we can choose $\delta = \frac{1}{32}$. Further, f is contraction mapping with constant $\alpha = \frac{1}{5}$ and B is a strongly positive bounded linear operator with constant $\bar{\gamma}_1 = \frac{1}{4}$ on R. Therefore, we can choose $\gamma = 2$ which satisfies $0 < \gamma < \frac{\bar{\gamma}_1}{\alpha} < \gamma + \frac{1}{\alpha}$. And E is a inverse strongly monotone mapping on R with $\bar{\gamma}_2 \in (0,1]$, then $\lambda \in (0,2)$. We can choose $\lambda = 1$. Furthermore, it is easy to observe that $2 \in I(E,M)$, $2 \in EP(F_1,\psi_1)$, $-4 \in EP(F_2,\psi_2)$. Hence $\Theta = \{2\} \neq \emptyset$. After simplification, schemes (5.5) and (5.6) reduce to*

$$z_n = -\tfrac{16n + (4n+2)x_n}{6n+1}; \\ u_n = \tfrac{592n^2 + 1248n + 192 + (88n^2 + 16n)x_n}{32(2n+3)(6n+1)}; \\ w_n = -u_n + 6 \tag{5.3}$$

$$x_{n+1} = (\frac{1}{3n} + \frac{1}{2(n+1)^2})x_n + \frac{1}{6}(1 - \frac{1}{2(n+1)^2} - \frac{1}{2n})(24t - 20)w_n + \frac{1}{n^3}\sin n, \quad (5.4)$$

where $t \in [\frac{7}{9}, 1)$. *Following the proof of Theorem 4.1, we obtain that* $\{z_n\}$ *converges strongly to* $\{-4\} \in GEP(F_2, \psi_2)$ *and* $\{x_n\}$, $\{u_n\}$, $\{w_n\}$ *converges strongly to* $w = \{2\} \in \bigcap_{i=0}^{3} Fix(T^i) \cap \Omega \cap I(E, M) \neq \emptyset$ *as* $n \to \infty$. *Figure* 1 *indicates the behavior of* x_n *for algorithm (5.4).*

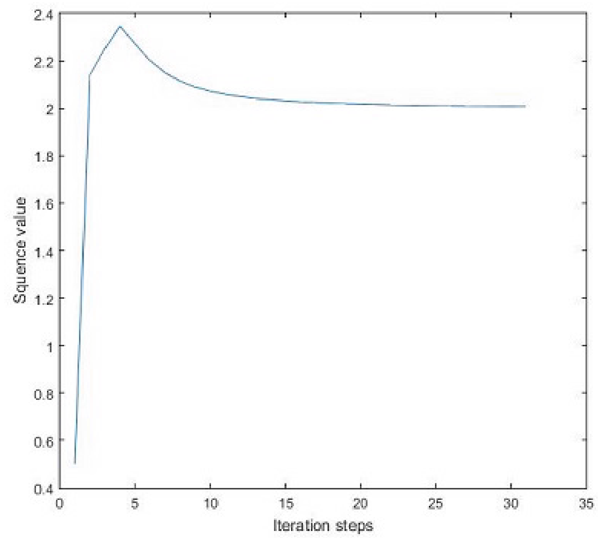

FIGURE 1. The graph of $\{x_n\}$ with initial value $x_1 = 0.5$.

Example 5.2. *Let* $H_1 = H_2 = R$, $C = [0, 4]$ *and* $Q = [0, 2]$; *let* $F_1 : C \times C \to R$ *and* $F_2 : Q \times Q \to R$ *be defined by* $F_1(x, y) = x(y - x)$, $\forall x, y \in C$ *and* $F_2(u, v) = -2u(u - v)$, $\forall u, v \in Q$, *and let for each* $x \in R$, *we define* $f(x) = \frac{1}{8}x$, $A(x) = -x$, $B(x) = x$, $E(x) = 2x$, *and*

$$Mx = \begin{cases} \{2x\}, & x > 0 \\ \{0\}, & x \leq 0 \end{cases}$$

and let, for each $x \in C$, $V_i x = -\alpha_i x$, *where* $\alpha_i = \frac{2}{i+1}$, $i = 0, 1, \cdots, 5$ *and* $e_n = \cos n$. *Then there exist unique sequences* $\{w_n\}$, $\{x_n\} \subset R$, $\{u_n\} \subset C$, *and* $\{z_n\} \subset Q$ *generated by the iterative schemes*

$$\begin{aligned} z_n = T_{s_n}^{F_2}(Ax_n); \qquad u_n = T_{r_n}^{F_1}(x_n + \tfrac{1}{4}A^*(z_n - Ax_n)); \\ w_n = (I + \tfrac{3}{2}M)^{-1}(u_n - \tfrac{3}{2}Eu_n); \end{aligned} \quad (5.5)$$

$$x_{n+1} = (\frac{1}{4\sqrt{n}} + \frac{1}{n+1})x_n + ((1 - \frac{1}{n+1})I - \frac{1}{\sqrt{n}}B)\frac{1}{n+1}\sum_{i=0}^{n} T^i w_n + \gamma_n e_n \quad (5.6)$$

where $\alpha_n = \frac{1}{\sqrt{n}},\ \beta_n = \frac{1}{n+1},\ \gamma_n = \frac{1}{n^2},\ r_n = 1 + \frac{1}{n}$ *and* $s_n = \frac{2n}{3n-1}$.
It is easy to prove that the bifunctions F_1 *and* F_2 *satisfy the Assumption 2.9 and* F_2 *is upper semicontinuous,* A *is a bounded linear operator on* R *with adjoint operator* A^* *and* $\|A\| = \|A^*\| = 1$. *Hence* $\delta \in (0, 1)$, *so we can choose* $\delta = \frac{1}{4}$. *Further,* f *is contraction mapping with constant* $\alpha = \frac{1}{7}$ *and* B *is a strongly positive bounded linear operator with constant* $\bar{\gamma}_1 = 1$ *on* R. *Therefore, we can choose* $\gamma = 2$ *which satisfies* $0 < \gamma < \frac{\bar{\gamma}_1}{\alpha} < \gamma + \frac{1}{\alpha}$. *And* E *is a inverse strongly monotone mapping on* R *with* $\bar{\gamma}_2 \in (0, 1]$, *then* $\lambda \in (0, 2)$. *We can choose* $\lambda = \frac{3}{2}$. *Furthermore, it is easy to observe that* $0 \in I(E, M)$, $0 \in EP(F_1)$, $0 \in EP(F_2)$. *Hence* $\Theta = \{0\} \neq \emptyset$. *After simplification, schemes (5.5) and (5.6) reduce to*

$$z_n = \frac{(3n-1)x_n}{7n-1}; \qquad u_n = \frac{(18n-2)x_n}{4(7n-1)}; \qquad w_n = -\frac{1}{8}u_n \quad (5.7)$$

$$x_{n+1} = (\frac{1}{4\sqrt{n}} + \frac{1}{n+1})x_n + \frac{1}{1080}(1 - \frac{1}{n+1} - \frac{1}{\sqrt{n}})(227t - 67)w_n + \frac{1}{n^2}\cos n, \quad (5.8)$$

where $t \in [\frac{1}{3}, 1)$. *Following the proof of Theorem 4.1, we obtain that* $\{z_n\}$ *converges strongly to* $\{0\} \in EP(F_2)$ *and* $\{x_n\}$, $\{u_n\}$, $\{w_n\}$ *converges strongly to* $w = \{0\} \in \bigcap_{i=0}^{5} Fix(T^i) \cap \Omega \cap I(E, M) \neq \emptyset$ *as* $n \to \infty$. *Figure* 2 *indicates the behavior of* x_n *for algorithm (5.8).*

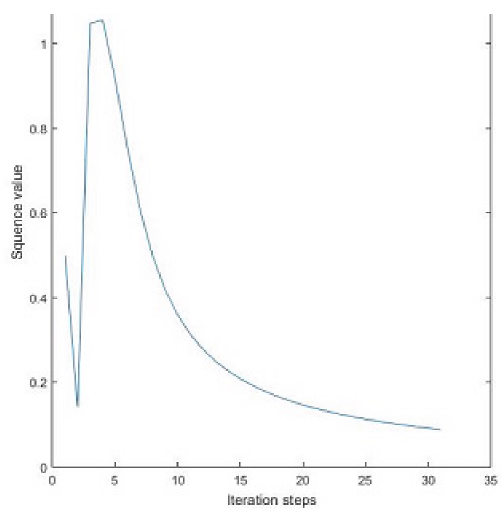

FIGURE 2. The graph of $\{x_n\}$ with initial value $x_1 = 0.45$.

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