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Local T_3 Constant Filter Convergence Spaces

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Highlights

- We characterized each of local $\overline{T}_3$ (resp. $T_3{}'$, $S\overline{T}_3$, $ST_3{}'$) constant filter convergence spaces.
- We investigated the relationships among these various forms.
- We showed that the categories $\overline{T}_3 ConFCO$ and $S\overline{T}_3 ConFCO$ were isomorphic categories.
- We showed that the categories $T_3{}'ConFCO$ and $ST_3{}'ConFCO$ were isomorphic categories.

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Abstract

In this paper, we characterize each of local T_3(resp. $T_3{}'$, $S\overline{T}_3$, $ST_3{}'$) constant filter convergence spaces and investigate the relationships among these various forms. We show that the full subcategories $\overline{T}_3 ConFCO$ and $S\overline{T}_3 ConFCO$ (resp. $T_3{}'ConFCO$ and $ST_3{}'ConFCO$) of $ConFCO$ are isomorphic categories. Moreover, we show that if a constant filter convergence space (B,K) is $\overline{T}_3$ (resp. T_3' , $S\overline{T}_3$ or $ST_3{}'$) at p and $M \subset B$ with $p \in M$, then M is $\overline{T}_3$ (resp. T_3') at p.

1. INTRODUCTION

Filters are first defined in the papers of Cartan [1,2] and play an important role in defining convergence in a manner similar to the role of sequences in a metric space. In 1978, Schwarz [3] introduced the category of constant filter convergence spaces which is isomorphic to the category of Grill spaces.

In 1991, Baran [4] gave a generalization of local T_0 and T_1 axioms of topology to topological categories. Local T_2 objects are defined in terms of local T_0 objects [4] and local T_1 are used to define the notion of closedness [5] in arbitrary topological categories. Furthermore, local T_1 is used to define the local T_3 and T_4 separation properties in arbitrary topological categories [4].

2. PRELIMINARIES

Let B be a non-empty set and $F(B)$ be the set of filters on B. A filter $\alpha \in F(B)$ is called proper (improper) iff $\emptyset \notin \alpha$ (resp. $\emptyset \in \alpha$).
(B, K) is called a constant filter convergence space if the map $K: B \longrightarrow P(F(B))$ satisfies:

(1) $[x] \in K$, $\forall x \in B$, where for $U \subset B$ and $[U] = \{ V \subset B: U \subset V \}$,

(2) if $\alpha \in K$ and $\alpha \subset \beta$, then $\beta \in K$.
Let (X, K) and (Y, L) be constant filter convergence spaces and $f: X \longrightarrow Y$ be a function. Then f is said to be continuous if for any $\alpha \in K$ implies $f(\alpha) \in L$, where

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$f(\alpha) = \{U \subset X: \exists A \in \alpha$ such that $f(A) \subset U\}$).

Let $\boldsymbol{ConFCO}$ be the category of constant filter convergence spaces and continuous maps [3]. Note that the category $\boldsymbol{ConFCO}$ is a normalized topological category [6].

Definition 2.1. A source $\{f_i: (B,K) \longrightarrow (B_i,K_i), i \in I\}$ in $\boldsymbol{ConFCO}$ is an initial lift if and only if $\alpha \in K$ precisely when $f_i(\alpha) \in K_i$ for all $i \in I$ [7].

Definition 2.2. An epi sink $\{f_i: (B_i,K_i) \longrightarrow (B,K), i \in I\}$ is final if and only if $\alpha \in K$ implies there exists $\beta_i \in K_i$ such that $f_i(\beta_i) \subset \alpha$ [7].

Definition 2.3. Let $(B,K) \in \boldsymbol{ConFCO}$. $K = \{[a],\ P(B) = [\emptyset] : a \in B\}$ is the discrete structure on *B*.

3. LOCAL T_3 CONSTANT FILTER CONVERGENCE SPACES

In this section, we give the characterization of local T_3 constant filter convergence spaces and find out relationships amoung them.

Let *B* be set with $p \in B$ and $B \vee_p B$ be the wedge at *p* [4]. Define

$S_p: B \vee_p B \longrightarrow B^2$ by $S_p(x_i) = \begin{cases} (x,x), & i = 1 \\ (p,x), & i = 2 \end{cases}$,

$A_p: B \vee_p B \longrightarrow B^2$ by $A_p(x_i) = \begin{cases} (x,p), & i = 1 \\ (p,x), & i = 2 \end{cases}$ and

$\nabla_P: B \vee_p B \longrightarrow B$ by $\nabla p(x_i) = x$ for $i = 1, 2$,

where x_1 (resp. x_2) is in the first (resp. second) component of $B \vee_p B$ [4,5].

Definition 3.1. ([4,5]) Let ***Set*** be the category of sets and functions, *U*: $\boldsymbol{\mathcal{E}} \longrightarrow$***Set*** be a topological functor, and *X* be an object of $\boldsymbol{\mathcal{E}}$ with $p \in U(X) = B$.

(1) If the initial lift of the *U*-source $\{S_p : B \vee_p B \longrightarrow U(X^2) = B^2$ and $\nabla_P: B \vee_p B \longrightarrow UD(B) = B\}$ is discrete, then *X* is called T_1 at *p*, where *D* is discrete functor,

(2) If the initial lift of the *U*-source $S_p: B \vee_p B \longrightarrow U(X^2) = B^2$ and $A_p: B \vee_p B \longrightarrow U(X^2) = B^2$ is agree, then *X* is called $Pre\overline{T}_2$ at *p*,

(3) If the initial lift of the *U*-source $S_p: B \vee_p B \longrightarrow U(X^2) = B^2$ and the final lift of the *U*-sink $i_1, i_2 : U(X) = B \longrightarrow B \vee_p B$ is agree, then *X* is called $PreT_2'$ at *p*, where i_1, i_2 are the canonical injections.

Remark 3.2. Let (B,τ) is a topological space and $p \in B$. $PreT_2'$ and $Pre\overline{T}_2$ at *p* are equivalent and reduces to every $x \in X$ with $x \neq p$, the topological space ($\{x, p\}$, δ) is not indiscrete, then the points *x* and *p* have disjoint neighborhoods [8].

Definition 3.3. ([4]) Let *U*: $\boldsymbol{\mathcal{E}} \longrightarrow$***Set*** be a topological functor, *X* is an object of $\boldsymbol{\mathcal{E}}$ with $p \in U(X)$ and X/F be the final lift of the epi *U*-sink

$$q: U(X) = B \longrightarrow B/F = (B \backslash F) \cup \{*\},$$

where q is the identity on $B \backslash F$ and identifying F with a point $*$ [4].

(1) If X is T_1 at p and X/F is $Pre\overline{T}_2$ at p for every non-empty closed F in *U(X)* missing p, then X is called $\overline{T}_3$ at p,

(2) If X is T_1 at p and X/F is $PreT_2{}'$ at p for $\emptyset \neq F \subset U(X)$ closed with $p \notin F$, then X is called $T_3{}'$ at p,

(3) If X is T_1 at p and X/F is $Pre\overline{T}_2$ at p for $\emptyset \neq F \subset U(X)$ closed with $p \notin F$, then X is called $S\overline{T}_3$ at p,

(4) If X is T_1 at p and X/F is $PreT_2{}'$ at p for $\emptyset \neq F \subset U(X)$ closed with $p \notin F$, then X is called $ST_3{}'$ at p.

Note that if (B,τ) is a topological space and $p \in B$, then by Theorem 2.1 of [8], all of $T_3{}'$ at p, $\overline{T}_3$ at p, $ST_3{}'$ at p, and $S\overline{T}_3$ at p are same.

Remark 3.4. Let α, β∈ $F(A)$ and $f: A \longrightarrow B$ be a function. Then

(1) $f(\alpha \cap \beta) = f(\alpha) \cap f(\beta)$,

(2) $f(\alpha) \cup f(\beta) \subset f(\alpha \cup \beta)$,

(3) $f^{-1} f \alpha \subset \alpha$.

Lemma 3.5. ([9,10]) Let B be a set, $\emptyset \neq F \subset B$, α, β, $\sigma \in F(B)$, and $q: B \longrightarrow B/F$ be identification map defined above.

(1) For $a \notin F$, $q\alpha \subset [a]$ iff $\alpha \subset [a]$,

(2) $q\alpha \subset [*]$ iff $\alpha \cup [F]$ is proper,

(3) $\alpha \cup [F]$ is not proper, then $q\sigma \subset q\alpha$ iff $\sigma \subset \alpha$,

(4) $\alpha \cup [F]$ is proper, then $q\sigma \subset q\alpha$ iff $\sigma \cup [F]$ is proper and $\sigma \cap [F] \subset \alpha$,

(5) $q\alpha \cup q\beta$ is proper iff $\alpha \cup \beta$ is proper or $\alpha \cup [F]$ and $\sigma \cup [F]$ are proper.

Theorem 3.6. Let (B, K) be a constant filter convergence space with $p \in B$.

(1) (B,K) is T_1 at p iff $[x] \cap [p] \notin K$, $\forall\, x \in X$ with $x \neq p$,

(2) (B,K) is $pre\overline{T}_2$ at p iff the conditions (i) and (ii) are satisfied, where

(i) If $\alpha, \beta \in K_p$, then $\alpha \cap \beta \in K_p$, where $K_p = \{\, \alpha : \alpha \subset [p]$ and $\alpha \in K \,\}$,

(ii) For any $\alpha \in K_p$ and $\beta \in K$ if $\alpha \cup \beta$ is proper, then $\beta \cap [p] \in K$,

(3) (B,K) is $preT_2{}'$ at p if and only if $K_p = \{[p]\}$.

Proof. (1) (resp. **(2)**) is proved in [5] (resp. [11]).

(3) Suppose (B,K) is $preT_2{}'$ at p and $\alpha \in K$ with $\alpha \subset [p]$. In Theorem 3.15 of [10], let $\alpha_1 = \alpha = \alpha_2$. Note that $\alpha_1 \cup \alpha_3 = \alpha$ is proper,

$$\alpha_1 = \alpha \supset \alpha_3 \cap [p] = \alpha.$$

Hence by Theorem 3.15 of [9], we have a proper filter σ on $B\vee_p B$ so that $\pi_1 S_p\sigma = \alpha = \pi_2 S_p\sigma$. Since (*B*,*K*) is $preT_2{}'$ at *p*, by Definition 3.1, $\sigma \supset i_1\sigma_1$ or $\sigma \supset i_2\sigma_1$ for some $\sigma_1 \in K$.

If $\sigma \supset i_1\sigma_1$, then $\pi_2 S_p\sigma = \alpha \supset \pi_2 S_p i_1\sigma_1 = [p]$ and consequently $\alpha = [p]$.

If $\sigma \supset i_2\sigma_1$, then $\pi_1 S_p\sigma = \alpha \supset \pi_1 S_p i_1\sigma_1 = [p]$ and consequently $\alpha = [p]$. Hence, $K_p = \{[p]\}$.

Conversely, suppose $K_p = \{[p]\}$ and σ is a filter on $B\vee_p B$, and K_{S_p} be the constant filter structure on $B\vee_p B$ induced by S_p and K_W be structure on $B\vee_p B$ induced by the maps i_1, i_2: (*B*,*K*)⟶ $B \vee_p B$. We show that K_{S_p}=K_W.

Suppose $\sigma \in K_{S_p}$. By Definition 2.1, $\pi_1 S_p\sigma \in K$ and $\pi_2 S_p\sigma \in K$. In Theorem 3.15 of [9], let $\alpha_1 = \pi_1 S_p\sigma$ and $\alpha_3 = \pi_2 S_p\sigma$.

In case of (1) of Theorem 3.15 of [9], we have

$$\pi_1 S_p\sigma=[p] \text{ and } (\pi_1 S_p\sigma)\cup(\pi_2 S_p\sigma)$$

is improper. It follows easily that $\sigma \supset i_2\pi_2 S_p\sigma$. Indeed, if $U \in i_2\pi_2 S_p\sigma$, then $U \supset i_2\pi_2 S_p(W)$ for some $W= U_1 \vee_p U_2 \in \sigma$. Since $\pi_1 S_p\sigma=[p]$ and $\pi_2 S_p\sigma \not\subset [p]$, we may assume $U_1 = \emptyset$. Hence,

$$W=U_2 = i_2\pi_2 S_p(W) \subset U$$

and consequently, $U \in \sigma$ and $\sigma \supset i_2\pi_2 S_p\sigma$.

In case of (2) of Theorem 3.15 of [9], $\pi_1 S_p\sigma \not\subset [p]$ and $\pi_1 S_p\sigma= \pi_2 S_p\sigma$. By using similar argument above it is easy that $\sigma \supset i_1\pi_1 S_p\sigma$.

In case of (3) of Theorem 3.15 of [9], we have

$$[p] \supset \pi_1 S_p\sigma,\ (\pi_1 S_p\sigma)\cup(\pi_2 S_p\sigma)$$

is proper and

$$\pi_1 S_p\sigma \supset (\pi_2 S_p\sigma) \cap [p].$$

Note that $\pi_1 S_p\sigma \in K$, $[p] \supset \pi_1 S_p\sigma$ and by assumption, $\pi_1 S_p\sigma=[p]$ and consequently, $\pi_2 S_p\sigma=[p]$ since

$$(\pi_1 S_p\sigma)\cup(\pi_2 S_p\sigma) = [p]\cup(\pi_2 S_p\sigma)$$

is proper iff

$$[p] \supset \pi_2 S_p\sigma$$

and

$$\pi_2 S_p\sigma \in K.$$

Hence, $\sigma = [p_1] = i_1[p]$, where $p_1 \in B\vee_p B$. Consequently, $\sigma \in K_W$ which shows that $K_{S_p} \subset K_W$. Suppose $\sigma \in K_W$. By Definition 2.2, there exists $\sigma_1 \in K$ such that $\sigma \supset i_1\sigma_1$ or $\sigma \supset i_2\sigma_1$.

If $\sigma \supset i_1\sigma_1$, then

$$\pi_1 S_p\sigma \supset \pi_1 i_1\sigma_1 = \sigma_1$$

and

$$\pi_2 S_p\sigma \supset \pi_2 S_p i_2\sigma_1 = \sigma_1$$

and consequently $\pi_i S_p\sigma \in K$, *i*=1, 2. By Definition 2.1, $\sigma \in K_{S_p}$.

If $\sigma \supset i_2\sigma_1$, then

$$\pi_1 S_p \sigma \supset \pi_1 S_p i_2 \sigma_1 = [p]$$

and

$$\pi_2 S_p \sigma \supset \pi_2 S_p i_2 \sigma_1 = \sigma_1,$$

and consequently $\pi_i S_p \sigma \in K$, i=1,2, i.e., $\sigma \in K_{S_p}$. Hence, $K_W \subset K_{S_p}$ and consequently $K_W = K_{S_p}$. By Definition 3.1, (B,K) is $preT_2{}'$ at p.

Lemma 3.7 Let (B,K) be a constant filter convergence space and $\emptyset \neq F \subset B$. The following are equivalent:

(1) F is strongly closed,

(2) F is closed,

(3) $\alpha \not\subset [a]$ or $\alpha \cup [F]$ is improper for any $a \in B$ with $a \notin F$ and $\forall\ \alpha \in K$.

Proof. It is proved in [5].

Theorem 3.8 Let (B,K) be a constant filter convergence space with $p \in B$. The following are equivalent:

(1) (B,K) is $S\overline{T}_3$ at p,

(2) (B,K) is $\overline{T}_3$ at p,

(3) Conditions (i)-(iii) are satisfied, where

(i) For any $x \in B$ with $x \neq p$, $[x] \cap [p] \notin K$,

(ii) If $\alpha,\beta \in K_p$, then $\alpha \cap \beta \in K_p$, where $K_p = \{\ \alpha : \alpha \subset [p]$ and $\alpha \in K\ \}$,

(iii) For any $\alpha \in K_p$, $\beta \in K$ and $\emptyset \neq F \subset B$ closed with $p \notin F$, if $\alpha \cup \beta$ is proper or $\beta \cup [F]$ and $\alpha \cup [F]$ are proper, then $\beta \cap [p] \in K$.

Proof. By Lemma 3.7 and by Definition 3.3, a constant convergence space (B,K) is $\overline{T}_3$ at p iff (B,K) is $S\overline{T}_3$ at p. Hence, (1)⟺(2).

We need to show that (2)⟺(3). Suppose (B,K) is $\overline{T}_3$ at p. By Definition 3.3, in particular, (B,K) is T_1 at p and by Theorem 3.7(1), $[x] \cap [p] \notin K$, $\forall\ x \in B$ with $x \neq p$.

Suppose $\alpha,\beta \in K_p$. Then $q\alpha, q\beta \in K'$ and $q\alpha \subset [p]$, $q\beta \subset [p]$, where K' is the final constant filter structure on B/F. Since (B,K) is $\overline{T}_3$ at p, by Definition 3.3, $(B/F, K')$ is $pre\overline{T}_2$ at p for $\emptyset \neq F \subset B$ closed with $p \notin F$ and by Theorem 3.6(2),

$$q(\alpha \cap \beta) = q(\alpha) \cap q(\beta) \in K_p{}'.$$

By Definition 2.2, there exists $\delta \in K$ such that $q(\delta) \subset q(\alpha \cap \beta)$. Since $\alpha \cap \beta \subset [p]$ and F is closed,. by Lemma 3.7,

$$(\alpha \cap \beta) \cup [F]$$

is improper and Lemma 3.5(3), $\delta \subset \alpha \cap \beta$ which shows that $\alpha \cap \beta \in K$ and consequently,

$$\alpha \cap \beta \in K_P.$$

Suppose that for any $\alpha \in K_p$ and $\beta \in K$, $\alpha \cup \beta$ is proper or $\beta \cup [F]$ and $\alpha \cup [F]$ are proper for $\emptyset \neq F \subset B$ closed with $p \notin F$. Note that $q\alpha, q\beta \in K'$ and by Lemma 3.5 (5), $q\alpha \cup q\beta$ is proper and $q\alpha \subset [q(p)] = [p]$. Since $(B/F, K')$ is $pre\overline{T}_2$ at p, by Theorem 3.6(2), $q(\alpha) \cap [p] \in K'$. By Definition 2.2, there exists $\delta \in K$ such that

$$q(\delta) \subset q(\beta) \cap [p] = q(\beta \cap [p]).$$

Since $\beta \cap [p] \subset [p]$ and F is closed by Lemma 3.7, $(\beta \cap [p]) \cup [F]$ is improper and by Lemma 3.5(3), $\delta \subset \beta \cap [p]$ and consequently, $\beta \cap [p] \in K$. As a result, (iii) is proved.

Conversely, suppose that the conditions (i)-(iii) hold. By the condition (i) and Theorem 3.6(1), (*B*,*K*) is T_1 at *p*. By Definition 3.3, we need to show that $(B/F, K')$ is $pre\overline{T}_2$ at p for $\emptyset \neq F \subset B$ closed with $p \notin F$, where K' is a structure on *B/F*. Suppose that $\alpha, \beta \in K'$ with $\alpha \subset [p]$ and $\beta \subset [p]$. By Definition 2.2, $\alpha_1, \beta_1 \in K$ such that

$$q\alpha_1 \subset \alpha \subset [p] = q[p]$$

and

$$q\beta_1 \subset \beta \subset [p] = q[p].$$

Since $p \notin F$, $q\alpha_1 \subset [p]$ and $q\beta_1 \subset [p]$, by Lemma 3.5(1), we get $\alpha_1 \subset [p]$ and $\beta_1 \subset [p]$. By the condition (ii), $\alpha_1 \cap \beta_1 \in K_p$ and consequently, $\alpha \cap \beta \in K_p{}'$.

Now suppose that $\alpha \in K_p{}'$ and $\beta \in K'$ with $\alpha \cup \beta$ is proper. By Definition 2.2, there exists $\alpha_1, \alpha_2 \in K$ such that $q\alpha_1 \subset \alpha$, $q\alpha_2 \subset \beta$ and $q\alpha_1 \subset [p] = [q(p)]$.

Since $\alpha \cup \beta$ is proper, then $q\alpha_1 \cup q\alpha_2$ is proper and by Lemma 3.5(5), we have either $\alpha_1 \cup \alpha_2$ is proper or $\alpha_1 \cup [F]$ and $\alpha_2 \cup [F]$ are proper. Note that $\alpha_1 \subset [p]$ and $\alpha_2 \in K$. If $\alpha_1 \cup \alpha_2$ is proper, the by the condition (iii), we have $\alpha_2 \cap [p] \in K$. So, $q(\alpha_2 \cap [p]) \in K'$, and $\beta \cap [p] \in K'$.

Suppose $\alpha_1 \cup [F]$ and $\alpha_2 \cup [F]$ are proper. Since F is closed, by Lemma 3.7, $\alpha_1 \subset [p]$ and $\alpha_2 \subset [p]$. By the condition (ii), $\alpha_1 \cap \alpha_2 \in K_p$ and consequently, $\beta \cap [p] \in K$.

Theorem 3.9 Let (*B*, *K*) be a constant filter convergence spaces with $p \in B$. The following are equivalent:

(1) (*B*,*K*) is $ST_3{}'$ at p,

(2) (*B*,*K*) is $T_3{}'$ at p,

(3) $[x] \cap [p] \notin K$ for $x \in B, p \in F$ with $x \neq p$ and $K_p = \{ [p] \}$, where $\emptyset \neq F \subset B$ is closed with $p \notin F$ and $K_p = \{ \alpha : \alpha \subset [p] \text{ and } \alpha \in K \}$.

Proof. By Lemma 3.7 and by Definition 3.3, (*B*,*K*) is $ST_3{}'$ at p iff (*B*, *K*) is $T_3{}'$ at p. Hence (1)⟺(2).

Suppose (*B*,*K*) is $T_3{}'$ at *p*. By Definition 3.3, in particular, (*B*, *K*) is T_1 at p and by Theorem 3.6(1), $[x] \cap [p] \notin K$, $\forall x \in B$ with $x \neq p$.

Suppose $\alpha \in K$ with $\alpha \subset [p]$ and $\emptyset \neq F \subset B$ is closed with $p \notin F$, then it follows that $q\alpha \in K'$ and $q\alpha \subset [q(p)] = [p]$. Since (*B*, *K*) is $T_3{}'$ at *p*, $(B/F, K')$ is $preT_2{}'$ at p for $\emptyset \neq F \subset B$ closed with $p \notin F$, by Theorem 3.6(3), $q\alpha = [p]$ and consequently, by Remark 3.4(3),

$$\alpha \supset q^{-1}q\alpha = [q^{-1}(p)] = [p].$$

Hence, $\alpha = [p]$, i.e., $K_p = \{ [p] \}$.

Suppose (3) holds. We show that (*B*,*K*) is $T_3{}'$ at p. Suppose $x \in B$ with $x \neq p$. If $p \in F$, then by assumption, $[x] \cap [p] \notin K$. If $p \notin F$ and $[x] \cap [p] \in K$, then $[x] \cap [p] \in K_p$ and by assumption, $[x] \cap [p] = [p]$ which means $x = p$, a contradiction. Thus, $[x] \cap [p] \notin K$, $\forall x \in B$ with $x \neq p$. By Theorem 3.6(1), (*B*,*K*) is T_1 at p.

Next, we show that $(B/F, K')$ is $preT_2{}'$ at p for $\emptyset \neq F \subset B$ closed with $p \notin F$. Suppose $\alpha \in K_p{}'$. By Definition 2.2, there exists $\beta \in K$ such that

$$q\beta \subset \alpha \subset [p]$$

and by Lemma 3.5(1), $\beta \subset [p]$ (since $p \notin F$). Hence, $\beta \in K_p$ and by assumption, $\beta = [p]$. It follows that $q(\beta) = [p] \subset \alpha$ and consequently, $\alpha = [p]$. Hence, $K_p{}' = \{[p]\}$ and by Theorem 3.6(3), $(B/F, K')$ is $preT_2{}'$ at *p*. Hence, by Definition 3.3, (*B*,*K*) is $T_3{}'$ at *p*.

Let $\boldsymbol{\mathcal{E}}$ be a topological category, X is an object of $\boldsymbol{\mathcal{E}}$ with $p \in U(X)$. Note that by [3,12] if X is $\overline{T}_0$ at p and $preT_2{}'$ (resp. $pre\overline{T}_2$) at *p*, then *X* is called LT_2 (resp. $\overline{T}_2$) at *p*.

Remark 3.10 (1) Let $\boldsymbol{Top}$ be the category of topological spaces and $(B, \tau) \in \boldsymbol{Top}$ with $p \in B$.

(i) By Remark 3.2, LT_2 at p and $\overline{T}_2$ at p are same and reduces to T_2 at *p*, i.e., every $x \in B$, *x*≠*p*, then the points *x* and *p* have disjoint neighborhoods [8],

(ii) $T_3{}'$ at $p \Leftrightarrow ST_3{}'$ at $p \Leftrightarrow \overline{T}_3$ at $p \Leftrightarrow S\overline{T}_3$ at $p \Longrightarrow LT_2$ at $p \Leftrightarrow \overline{T}_2$ at $p \Longrightarrow T_1$ at $p \Longrightarrow \overline{T}_0$ at *p*,

(iii) Let $\boldsymbol{T_3Top}$ be the full subcategory of $\boldsymbol{Top}$ consisting of all local T_3 topological spaces. By Theorem 2.1 of [8], the categories $\boldsymbol{\overline{T}_3Top}$, $\boldsymbol{T_3{}'Top}$, $\boldsymbol{S\overline{T}_3Top}$, and $\boldsymbol{ST_3{}'Top}$ are isomorphic.

(2) Let $(B, K) \in \boldsymbol{ConFCO}$ with $p \in B$.

(i) By Theorems 3.8 and 3.9,

$$T_3{}' \text{ at } p \Leftrightarrow ST_3{}' \text{ at } p \Longrightarrow \overline{T}_3 \text{ at } p \Leftrightarrow \overline{ST}_3 \text{ at } p,$$

(ii) By Theorems 3.6 and 3.8,

$$ST_3{}' \text{ at } p \Longrightarrow S\overline{T}_3 \text{ at } p \Longrightarrow \overline{T}_2 \text{ at } p \Longrightarrow T_1 \text{ at } p \Leftrightarrow \overline{T}_0 \text{ at } p,$$

(iii) By (ii) and Theorems 3.6 and 3.9,

$$T_3{}' \text{ at } p \Longrightarrow LT_2 \text{ at } p \Leftrightarrow preT_2{}' \text{ at } p \Longrightarrow \overline{T}_2 \text{ at } p \Longrightarrow pre\overline{T}_2 \text{ at } p$$

but converse of each implication is not true. Take *R*, the set of reel numbers and *K*=*F*(*R*). By Theorem 3.6, (*R*, *F*(*R*)) is $pre\overline{T}_2$ at *p* for each *p*∈ *R* but it is not $\overline{T}_2$ at *p*.

Let $B = \{x, y, z\}$ and $K = \{[x], [y], [z], [\emptyset], [x] \cap [y]\}$.

By Theorem 3.4 of [12] and Theorem 3.6, (*B*, *K*) is $\overline{T}_2$ at *z* but (*B*, *K*) is not $preT_2{}'$ at *z*.

(iv) Let $\boldsymbol{T_3ConFCO}$ be the full subcategory of $\boldsymbol{ConFCO}$ whose objects are local T_3 constant filter convergence spaces, where $T_3 = T_3{}', \overline{T}_3, S\overline{T}_3$, and ST_3'. By Theorems 3.8 and 3.9,

(a) $\boldsymbol{\overline{T}_3ConFCO}$ and $\boldsymbol{S\overline{T}_3ConFCO}$ are isomorphic categories,

(b) $\boldsymbol{T_3{}'ConFCO}$ and $\boldsymbol{ST_3{}'ConFCO}$ are isomorphic categories,

(3) Let $\boldsymbol{\mathcal{E}}$ be a normalized topological category and *X* be an object of $\boldsymbol{\mathcal{E}}$ with $p \in U(X)$.
(i) By Theorem 7 of [12], if *X* is LT_2 at p, then *X* is $\overline{T}_2$ at p and by Theorems 2.7 and 2.8 of [10], if *X* is $preT_2'$ at p, then *X* is $pre\overline{T}_2$ at p. Moreover, by Theorem 2.8 of [10], if *X* is $\overline{T}_3$ (resp. $S\overline{T}_3$, T_3', $ST_3{}'$), then *X* is $\overline{T}_3$ at p (resp. $S\overline{T}_3$ at p, T_3' at p, ST_3' at p).

(ii) Note that all objects of a set-based arbitrary topological category may be $pre\overline{T}_2$ at p. For example, it is shown, in [13], that all Cauchy spaces [14] are $pre\overline{T}_2$ at p. Also, $preT_2'$ at p objects could be only discrete objects [15].

(iii) Let $\boldsymbol{pre\overline{T}_2(\mathcal{E})}$ be the full subcategory of $\boldsymbol{\mathcal{E}}$ consisting of all $pre\overline{T}_2$ objects. By Theorem 3.4 of [16], $\boldsymbol{pre\overline{T}_2(\mathcal{E})}$ is a topological category.

Theorem 3.11 (1) If a constant filter convergence space (*B*,*K*) is $\overline{T}_3$ (resp. T_3') at p and $M \subset B$ with $p \in M$, then *M* is $\overline{T}_3$ (resp. T_3') at p,

(2) For all $i \in I$ and $p_i \in B_i$, (B_i, K_i) is $\overline{T}_3$ at p_i if $(B = \prod_{i \in I} B_i, K)$ is $\overline{T}_3$ at $p = (p_1, p_2, \ldots)$, where *K* is the product structure on *B*.

Proof. **(1)** Let $i: M \subset B$ be the inclusion map, K_M be a structure on *M* induced from *i*, and $[x] \cap [p] \in K_M$ for $x \in M$ with $x \neq p$. By Definition 2.1,

$$i([x] \cap [p]) = i([x]) \cap i([p]) = [x] \cap [p] \in K$$

for $x \in X$ with $x \neq p$, a contradiction since (*B*,*K*) is $\overline{T}_3$ (resp. T_3') at p. Thus, $[x] \cap [p] \notin K_M$ for $x \in M$. with $x \neq p$.

Suppose $\alpha, \beta \in (K_M)_p$. Then $i(\alpha), i(\beta) \in K$, $i(\alpha) \subset [p]$, $i(\beta) \subset [p]$ and by Theorem 3.8, $i(\alpha \cap \beta) \in K_p$. By Definition 2.1, $\alpha \cap \beta \in (K_M)_p$.

Suppose $\alpha \in (K_M)_p$, $\beta \in K_M$ and for $\emptyset \neq F \subset M$ closed with $p \notin F$ such that $\alpha \cup \beta$ is proper or $\beta \cup [F]$ and $\alpha \cup [F]$ are proper.

By Definition 2.1, $i(\alpha), i(\beta) \in K_p$, $i(\alpha) \cup i(\beta) = i(\alpha \cup \beta)$ is proper or

$$i(\alpha) \cup [i(F) = F]$$

and

$$i(\beta) \cup [i(F) = F]$$

are proper. By Theorem 3.8, $i(\beta \cap [p]) \in K$ and by Definition 2.1, $\beta \cap [p] \in K_M$. Hence, (M, K_M) is $\overline{T}_3$ at p. It remains to show that $(K_M)_p = \{[p]\}$.

Let $\alpha \in (K_M)_p$ and for $\emptyset \neq F \subset M$ closed with $p \notin F$. By Definition 2.1, $i(\alpha) \in K_p$ and by Theorem 3.9, $K_p = \{[p]\}$.

Thus, $i(\alpha) = [p]$ and Definition 2.1, $\alpha = [p]$. Hence, (M, K_M) is T_3' at p.

(2) Suppose that $(B = \prod_{i \in I} B_i, K)$ is $\overline{T}_3$ at p. Since each (B_i, K_i) is isomorphic to a subspace of (*B*,*K*), by Part (1), $\forall i \in I$, (B_i, K_i) is $\overline{T}_3$ at p_i.

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CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

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