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New Relations Concerning a Mean Value of Some Hardy Sums and Ramanujan Sum

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Abstract: Dedekind sum first occured naturally in Dedekind's transformation law of his eta-function. In analogy, Hardy sums are encountered in the transformation formula of the theta function. Up to now, they have many of remarkable applications in analytic number theory (Dedekind's η-function), algebraic number theory (class numbers), lattice point problems, topology and algebraic geometry. Miscellaneous arithmetical properties of these sums have been analyzed by many scholars. Recently, considering the characteristics of Hardy sums and other celebrated sums such as Ramanujan sum and Kloosterman sum, interesting and meaningful identities have been obtained. In this paper, we continue to focus on arithmetical aspects of Hardy sums and Ramanujan sum. More precisely, we consider a mean value problem of these sums and Ramanujan sum with the help of the features of Dirichlet $L-$functions and present some computational formulas for them.

***Key words**:* Mean value, Dedekind sum, Hardy sum, Ramanujan sum.

Bazı Hardy Toplamları ve Ramanujan Toplamının Ortalama Değeri Hakkında Yeni Bağıntılar

Özet: Dedekind toplamı, Dedekind eta-fonksiyonunun dönüşüm formülünde doğal olarak ortaya çıkmıştır. Benzer şekilde, Hardy toplamlarına ise theta fonksiyonunun dönüşüm formüllerinde karşılaşılmıştır. Günümüze kadar bu toplamlar, analitik sayılar teorisi (Dedekind η-fonksiyonu), cebirsel sayılar teorisi (sınıf sayıları), latis noktası problemleri, topoloji ve cebirsel geometri alanlarında çok sayıda önemli uygulamalara sahiptir. Bu toplamların çeşitli aritmetik özellikleri birçok bilim adamı tarafından analiz edilmiştir. Son zamanlarda ise, Hardy toplamları, Ramanujan toplamı ve Kloosterman toplamı gibi önemli toplamların sağladığı özellikler göz önüne alınarak, ilginç ve anlamlı özdeşlikler elde edilmiştir. Bu makalede, Hardy toplamları ve Ramanujan toplamına aritmetik açıdan odaklanmaya devam edeceğiz. Daha açık olarak, Dirichlet L-fonksiyonlarının özellikleri yardımıyla Hardy toplamları ve Ramanujan toplamının ortalama değer problemini ele alacağız ve onlar için bazı hesaplama formülleri sunacağız.

***Anahtar kelimeler**:* Ortalama değer, Dedekind toplamı, Hardy toplamı, Ramanujan toplamı.

1. Introduction

The classical Dedekind sum is defined by

$$S(d,c)=\sum_{j=1}^{c}\left(\left(\frac{j}{c}\right)\right)\left(\left(\frac{dj}{c}\right)\right),$$

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where

$$((x)) := \begin{cases} x - [x] - 1/2, & \text{if } x \in \mathbb{Z}, \\ 0, & \text{otherwise.} \end{cases}$$

Because of important applications, mainly in number theory, the Dedekind sum has been studied extensively by many authors in a variety of contexts. See Rademacher and Grosswald [15] for a bibliography.

Berndt [3] and Goldberg [10] investigated the Mobius transformation formula $\mathrm{V(z)} = (\mathrm{az}+\mathrm{b})/(\mathrm{cz}+\mathrm{d})$ of the theta function for some special cases of *a*,*b*,*c*,*d*, and then encountered six different arithmetic sums, named as Hardy sums. Here, we will consider three of them, defined by:

$$s_4(d,c) = \sum_{j=1}^{c-1} (-1)^{[dj/c]},$$

$$s_3(d,c) = \sum_{j=1}^{c} (-1)^j \left(\left(\frac{dj}{c}\right)\right)$$

and

$$S_1(d,c) = \sum_{j=1}^{c-1} (-1)^{j+1+[dj/c]}.$$

R. Sitaramachandrarao [16] expressed these sums in terms of classical Dedekind sums as

$$s_3(d,c) = 2S(d,c) - 4S(2d,c), \quad \text{for odd } c, \tag{1}$$

$$s_4(d,c) = -4S(d,c) + 8S(d,2c), \quad \text{for odd } d, \tag{2}$$

$$S_1(d,c) = 8S(d,2c) + 8S(2d,c) - 20S(d,c), \quad \text{for odd } d+c. \tag{3}$$

Various arithmetical properties of these sums have been analyzed by many scholars, (see [2, 4-6, 12, 13, 20]).

Regarding the properties of Hardy sums and other celebrated sums such as Ramanujan sum and Kloosterman sum, some authors [7, 8, 11, 17-19] focused on them and obtained meaningful and interesting outcomes.

Han and Zhang [9] studied the general mean value problem for Dedekind sum and derived several sharper mean value formulas by using the properties of Dirichlet L −function. For example, they obtained that

$$\sum_{h=1}^{p^\alpha}{}' S(h,p^\alpha) R_{p^\alpha}^2(h+1)$$
$$= -\frac{1}{12} p^{2\alpha} \phi(p^\alpha) \left[\left(1-\frac{1}{p^\alpha}\right)\left(1-\frac{2}{p^\alpha}\right) - \frac{1}{p}\left(1-\frac{1}{p^2}\right)\right],$$

whenever p is any odd prime and $\alpha \geq 2$ is integer. Here, $\phi(q)$ is Euler function, the summation $\sum_{h=1}^{q}{}'$ is taken over all $1 \leq h \leq q$ with $(h, q) = 1$, and $R_q(c)$ is Ramanujan sum, defined by

$$R_q(c) = \sum_{k=1}^{q}{}' e^{2\pi ikc/q}.$$

In this paper, appying the features of Dirichlet L −function and the analytic method, we deal with Hardy sums and Ramanujan sum with respect to mean value problem above, and consequently derive following conclusions.

Theorem 1 *For any odd prime* p *and integer* $\alpha \geq 2$, *any integer, if* $p^\alpha \equiv 1\ (mod 4)$ *and* $8|(p^{\alpha-1} - 1)$, *then we have*

$$\sum_{h=1}^{p^\alpha}{}' s_3(2h, p^\alpha) R_{p^\alpha}^2(2h+1) = -2\phi(p^\alpha)(p^\alpha - 1),$$

where $\sum_{h=1}^{q}{}'$ indicates that the summation ranges $1 \leq h \leq q$ such that $(h, q) = 1$, $\phi(q)$ is Euler function and $R_q(c)$ is Ramanujan sum.

Theorem 2 *For any odd prime* p *and integer* $\alpha \geq 2$, *any integer, if* $p^\alpha \equiv 1\ (mod 4)$, *we have*

$$\sum_{h=1}^{p^\alpha}{}' s_4(2h, p^\alpha) R_{p^\alpha}^2(2h+1)$$
$$= \frac{1}{12} p^{2\alpha} \phi(p^\alpha) \left[\left(1 - \frac{1}{p^\alpha}\right)\left(1 + \frac{3}{p^\alpha}\right) - \frac{4}{p}\left(1 + \frac{1}{p} - \frac{1}{p^2} + \frac{1}{p^{\alpha-1}}\right)\right].$$

For the case $p^\alpha \equiv 3\ (mod 4)$, further identities can be obtained by the similar way.

2. Several Lemmas

We give some lemmas, needed in the proof of theorems.

Lemma 3 *For an odd prime* p, *an integer* $\alpha \geq 2$ *and any non-principal character* χ *mod* p^α. *The following identity holds for any integer* m *such that* $(m, p^\alpha) = 1$:

$$\sum_{a=1}^{p^\alpha} \chi(a) R_{p^\alpha}^2(ma+1)$$
$$= \begin{cases} \overline{\chi}(-m) p^{2\alpha}\left(1 - \frac{2}{p}\right), & \text{if } \chi \text{ is a primitive character } mod p^\alpha; \\ \chi(-m) p^{2\alpha}\left(1 - \frac{1}{p}\right), & \text{if } \chi \text{ is not a primitive character } mod p^\alpha. \end{cases}$$

Proof. Let χ be a primitive character $mod\ p^\alpha$. Then, using the notation $e(x) = e^{2\pi ix}$ and the properties of Gauss sums (see for example [1]), one has

$$\sum_{a=1}^{p^\alpha} \chi(a) R_{p^\alpha}^2(ma+1)$$

$$= \sum_{b=1}^{p^\alpha}{}' \sum_{c=1}^{p^\alpha}{}' \sum_{a=1}^{p^\alpha} \chi(a) e\left(\frac{ma(b+c)+b+c}{p^\alpha}\right)$$

$$= \sum_{b=1}^{p^\alpha}{}' \sum_{c=1}^{p^\alpha}{}' \sum_{a=1}^{p^\alpha} \chi(a) e\left(\frac{mac(b+1)+c(b+1)}{p^\alpha}\right)$$

$$= \sum_{b=1}^{p^\alpha}{}' \sum_{c=1}^{p^\alpha}{}' \overline{\chi}(mc) \sum_{a=1}^{p^\alpha} \chi(mac) e\left(\frac{mac(b+1)}{p^\alpha}\right) e\left(\frac{c(b+1)}{p^\alpha}\right)$$

$$= \overline{\chi}(m)\tau(\chi) \sum_{b=1}^{p^\alpha}{}' \overline{\chi}(b+1) \sum_{c=1}^{p^\alpha}{}' \overline{\chi}(c) e\left(\frac{c(b+1)}{p^\alpha}\right)$$

$$= \overline{\chi}(m)\tau(\chi)\tau(\overline{\chi}) \sum_{b=1}^{p^\alpha}{}' \overline{\chi}(b+1)\chi(b+1)$$

$$= \overline{\chi}(-m)\tau(\chi)\overline{\tau(\chi)} \sum_{\substack{b=1 \\ (b(b+1),p)=1}}^{p^\alpha} 1$$

$$= \overline{\chi}(-m)p^\alpha(p^\alpha - 2p^{\alpha-1}) = \overline{\chi}(-m)p^{2\alpha}\left(1 - \frac{2}{p}\right).$$

Now, if χ is not a primitive character $mod p^\alpha$ and $\chi \neq \chi_0$, the principal character $mod p^\alpha$, then χ must be a non-principal character $mod\ p^{\alpha-1}$. Thus, by aid of the well known identity

$$\sum_{b=1}^{k} e\left(\frac{bn}{k}\right) = \begin{pmatrix} 0, & \text{for } (n,k) = 1; \\ k, & \text{for } k|n; \end{pmatrix}$$

and features of the reduced residue system $mod\ p^\alpha$, we have

$$\sum_{a=1}^{p^\alpha} \chi(a) R_{p^\alpha}^2(ma+1)$$

$$= \sum_{b=1}^{p^\alpha}{}' \sum_{c=1}^{p^\alpha}{}' \overline{\chi}(c) \sum_{a=1}^{p^\alpha} \chi(a) e\left(\frac{ma(b+1)+c(b+1)}{p^\alpha}\right)$$

$$= \chi(-m) \sum_{b=1}^{p^\alpha}{}' \left| \sum_{c=1}^{p^\alpha}{}' \overline{\chi}(c) e\left(\frac{c(b+1)}{p^\alpha}\right) \right|^2$$

$$= \chi(-m) \sum_{b=1}^{p^\alpha} \left| \sum_{c=1}^{p^{\alpha-1}}{}' \sum_{u=0}^{p-1} \overline{\chi}(up^{\alpha-1}+c) e\left(\frac{(up^{\alpha-1}+c)(b+1)}{p^\alpha}\right) \right|^2$$

$$-\chi(-m) \sum_{b=1}^{p^{\alpha-1}} \left| \sum_{c=1}^{p^{\alpha-1}}{}' \sum_{u=0}^{p-1} \overline{\chi}(up^{\alpha-1}+c) e\left(\frac{(up^{\alpha-1}+c)(pb+1)}{p^\alpha}\right) \right|^2$$

$$= \chi(-m) \sum_{b=1}^{p^\alpha} \left| \sum_{c=1}^{p^{\alpha-1}}{}' \sum_{u=0}^{p-1} \overline{\chi}(c) e\left(\frac{(up^{\alpha-1}+c)(b+1)}{p^\alpha}\right) \right|^2$$

$$-\chi(-m) \sum_{b=1}^{p^{\alpha-1}} \left| \sum_{c=1}^{p^{\alpha-1}}{}' \sum_{u=0}^{p-1} \overline{\chi}(c) e\left(\frac{up^{\alpha-1}+cpb+c}{p^\alpha}\right) \right|^2$$

$$= \chi(-m)p^{\alpha}\phi(p^{\alpha}) = \chi(-m)p^{2\alpha}\left(1-\frac{1}{p}\right).$$

Lemma 4 *For an integer $p > 2$ and any integer h such that $(h, p) = 1$,*

$$S(h,p) = \frac{1}{\pi^2 p}\sum_{d|p}\frac{d^2}{\phi(d)}\sum_{\substack{\chi\, mod d\\ \chi(-1)=-1}}\chi(h)|L(1,\chi)|^2,$$

in which $L(1,\chi)$ denotes the Dirichlet L −function attached the character $\chi\ mod\ d$ and $\phi(p)$ is Euler function.

Proof. See Lemma 2 in [20].

Lemma 5 *For an odd square-full number $q > 3$,*

$$\sum_{\substack{\chi\, modq\\ \chi(-1)=-1}}{}^{*}|L(1,\chi)|^2 = \frac{\pi^2}{12}\frac{\phi^3(q)}{q^2}\prod_{p|q}\left(1+\frac{1}{p}\right) \quad (4)$$

and

$$\sum_{\substack{\chi\, modq\\ \chi(-1)=-1}}{}^{*}\chi(2)|L(1,\chi)|^2 = \frac{\pi^2}{24}\frac{\phi^3(q)}{q^2}\prod_{p|q}\left(1+\frac{1}{p}\right), \quad (5)$$

where $\sum_{\substack{\chi\, modq\\ \chi(-1)=-1}}{}^{*}$ implies that the summation is over all odd primitive characters χ mod q.

Proof. This is Lemma 6 of [17].

Lemma 6 *Let $q > 3$ be an integer and χ be a Dirichlet character mod q. Then, if $q \equiv 1$ $(mod 4)$,*

$$\sum_{\substack{\chi\, modq\\ \chi(-1)=-1}}\chi(4)|L(1,\chi)|^2 = \frac{\pi^2\phi(q)}{48}\prod_{p|q}\left(1-\frac{1}{p^2}\right) - \frac{3\pi^2\phi^2(q)}{8q^2}, \quad (6)$$

and if $q \equiv 3$ $(mod 4)$,

$$\sum_{\substack{\chi\, modq\\ \chi(-1)=-1}}\chi(4)|L(1,\chi)|^2 = \frac{\pi^2\phi(q)}{48}\prod_{p|q}\left(1-\frac{1}{p^2}\right) - \frac{\pi^2\phi^2(q)}{8q^2},$$

where $\prod_{p|q}$ denotes the product over all distinct prime divisor of q.

Proof. This is Theorem 1.4 of [14].

Lemma 7 *Let $q > 3$ be an integer and χ be a Dirichlet character modulo q. Then, for any integer a such that $a|(q-1)$, the following relation holds:*

$$\sum_{\substack{\chi\, modq\\ \chi(-1)=-1}}\chi(a)|L(1,\chi)|^2 = \frac{\pi^2\phi(q)}{12a}\prod_{p|q}\left(1-\frac{1}{p^2}\right) - \frac{(a^2+2)\pi^2\phi^2(q)}{12aq^2}.$$

Proof. See [14, Theorem 1.5].

3. Proofs

3.1 Proof of Theorem 1

From (1) and Lemma 4, one can write

$$\sum_{h=1}^{q} s_3(2h,q)R_q^2(2h+1)$$

$$= \sum_{h=1}^{q} \{2S(2h,q) - 4S(4h,q)\}R_q^2(2h+1)$$

$$= 2\sum_{h=1}^{q} S(2h,q)R_q^2(2h+1) - 4\sum_{h=1}^{q} S(4h,q)R_q^2(2h+1)$$

$$= \frac{2}{\pi^2 q}\sum_{d|q} \frac{d^2}{\phi(d)}\sum_{\substack{\chi\, mod d\\ \chi\, odd}} \sum_{h=1}^{q} \chi(2h)R_q^2(2h+1)|L(1,\chi)|^2$$

$$-\frac{4}{\pi^2 q}\sum_{d|q} \frac{d^2}{\phi(d)}\sum_{\substack{\chi\, mod d\\ \chi\, odd}} \sum_{h=1}^{q} \chi(4h)R_q^2(2h+1)|L(1,\chi)|^2$$

$$:= T_1 + T_2,$$

where

$$T_1 = \frac{2}{\pi^2 q}\sum_{d|q} \frac{d^2}{\phi(d)}\sum_{\substack{\chi\, mod\, d\\ \chi\, odd}} \sum_{h=1}^{q} \chi(2h)R_q^2(2h+1)|L(1,\chi)|^2$$

and

$$T_2 = -\frac{4}{\pi^2 q}\sum_{d|q} \frac{d^2}{\phi(d)}\sum_{\substack{\chi\, mod\, d\\ \chi\, odd}} \sum_{h=1}^{q} \chi(4h)R_q^2(2h+1)|L(1,\chi)|^2.$$

Now, in view of Lemma 3 and equations (4) and (6), T_1 can be evaluated as

$$T_1 = \frac{2}{\pi^2 p^\alpha}\sum_{i=1}^{\alpha} \frac{p^{2i}}{\phi(p^i)}\sum_{\substack{\chi\, mod\, p^i\\ \chi\, odd}} \sum_{h=1}^{p^\alpha} \chi(2h)R_{p^\alpha}^2(2h+1)|L(1,\chi)|^2$$

$$= \frac{2p^\alpha}{\pi^2\phi(p^\alpha)}\sum_{\substack{\chi\, mod\, p^\alpha\\ \chi\, odd}} \sum_{h=1}^{p^\alpha} \chi(2h)R_{p^\alpha}^2(2h+1)|L(1,\chi)|^2$$

$$+\frac{2}{\pi^2 p^\alpha}\sum_{i=1}^{\alpha-1} \frac{p^{2i}}{\phi(p^i)}\sum_{\substack{\chi\, mod\, p^i\\ \chi\, odd}} \sum_{h=1}^{p^\alpha} \chi(2h)R_{p^\alpha}^2(2h+1)|L(1,\chi)|^2$$

$$= -\frac{2p^\alpha}{\pi^2\phi(p^\alpha)}\left(\sum_{\substack{\chi\, mod\, p^\alpha\\ \chi\, odd}}{}^{*} p^{2\alpha}\left(1-\frac{2}{p}\right)|L(1,\chi)|^2 + \sum_{\substack{\chi\, mod\, p^{\alpha-1}\\ \chi\, odd}} \chi(4)p^\alpha\phi(p^\alpha)|L(1,\chi)|^2\right)$$

$$-\frac{2\phi(p^\alpha)}{\pi^2}\sum_{i=1}^{\alpha-1} \frac{p^{2i}}{\phi(p^i)}\sum_{\substack{\chi\, mod\, p^i\\ \chi\, odd}} \chi(4)|L(1,\chi)|^2$$

$$= -\frac{2p^{\alpha}}{\pi^2\phi(p^{\alpha})}p^{2\alpha}\left(1-\frac{2}{p}\right)\frac{\pi^2}{12}\frac{\phi^3(p^{\alpha})}{p^{2\alpha}}\left(1+\frac{1}{p}\right)$$

$$-\frac{2p^{\alpha}}{\pi^2\phi(p^{\alpha})}p^{\alpha}\phi(p^{\alpha})\left(\frac{\pi^2\phi(p^{\alpha})}{48}\left(1-\frac{1}{p^2}\right)-\frac{3\pi^2\phi^2(p^{\alpha})}{8p^{2\alpha}}\right)$$

$$-\frac{2\phi(p^{\alpha})}{\pi^2}\sum_{i=1}^{\alpha-1}\frac{p^{2i}}{\phi(p^i)}\left(\frac{\pi^2\phi(p^i)}{48}\left(1-\frac{1}{p^2}\right)-\frac{3\pi^2\phi^2(p^i)}{8p^{2i}}\right)$$

$$= -\frac{1}{6}p^{\alpha}\phi^2(p^{\alpha})\left(1-\frac{1}{p}-\frac{2}{p^2}\right)-\frac{1}{24}p^{2\alpha}\phi(p^{\alpha})\left(1-\frac{1}{p^2}\right)+\frac{3}{4}\phi^2(p^{\alpha})$$

$$-\frac{1}{24}\phi(p^{\alpha})\sum_{i=1}^{\alpha-1}\left(p^{2i}-p^{2i-2}\right)+\frac{3}{4}\phi(p^{\alpha})\sum_{i=1}^{\alpha-1}\left(p^i-p^{i-1}\right)$$

$$= -\frac{1}{6}p^{\alpha}\phi^2(p^{\alpha})\left(1-\frac{1}{p}-\frac{2}{p^2}\right)-\frac{1}{24}p^{2\alpha}\phi(p^{\alpha})\left(1-\frac{1}{p^2}\right)+\frac{3}{4}\phi^2(p^{\alpha})$$

$$-\frac{1}{24}\phi(p^{\alpha})(p^{2\alpha-2}-1)+\frac{3}{4}\phi(p^{\alpha})(p^{\alpha-1}-1). \qquad (7)$$

Similarly, from Lemmas 3 and 7 and equation (5), one obtains that

$$T_2 = -\frac{4}{\pi^2 p^{\alpha}}\sum_{i=1}^{\alpha}\frac{p^{2i}}{\phi(p^i)}\sum_{\substack{\chi \bmod p^i \\ \chi\ odd}}\sum_{h=1}^{q}\chi(4h)R_{p^{\alpha}}^2(2h+1)|L(1,\chi)|^2$$

$$= \frac{-4p^{\alpha}}{\pi^2\phi(p^{\alpha})}\sum_{\substack{\chi \bmod p^{\alpha} \\ \chi\ odd}}\sum_{h=1}^{p^{\alpha}}\chi(4h)R_{p^{\alpha}}^2(2h+1)|L(1,\chi)|^2$$

$$-\frac{4}{\pi^2 p^{\alpha}}\sum_{i=1}^{\alpha-1}\frac{p^{2i}}{\phi(p^i)}\sum_{\substack{\chi \bmod p^i \\ \chi\ odd}}\sum_{h=1}^{p^{\alpha}}\chi(4h)R_{p^{\alpha}}^2(2h+1)|L(1,\chi)|^2$$

$$= \frac{4p^{\alpha}}{\pi^2\phi(p^{\alpha})}\left(\sum_{\substack{\chi \bmod p^{\alpha} \\ \chi\ odd}}{}^{*}\, p^{2\alpha}\left(1-\frac{2}{p}\right)\chi(2)|L(1,\chi)|^2 + \right.$$

$$\left.\sum_{\substack{\chi \bmod p^{\alpha-1} \\ \chi\ odd}}\chi(8)p^{\alpha}\phi(p^{\alpha})|L(1,\chi)|^2\right)$$

$$+\frac{4\phi(p^{\alpha})}{\pi^2}\sum_{i=1}^{\alpha-1}\frac{p^{2i}}{\phi(p^i)}\sum_{\substack{\chi \bmod p^i \\ \chi\ odd}}\chi(8)|L(1,\chi)|^2$$

$$= \frac{4p^{\alpha}}{\pi^2\phi(p^{\alpha})}p^{2\alpha}\left(1-\frac{2}{p}\right)\frac{\pi^2}{24}\frac{\phi^3(p^{\alpha})}{p^{2\alpha}}\left(1+\frac{1}{p}\right)$$

$$+\frac{4p^{\alpha}}{\pi^2\phi(p^{\alpha})}p^{\alpha}\phi(p^{\alpha})\left(\frac{\pi^2\phi(p^{\alpha})}{96}\left(1-\frac{1}{p^2}\right)-\frac{66\pi^2\phi^2(p^{\alpha})}{96p^{2\alpha}}\right)$$

$$+\frac{4\phi(p^{\alpha})}{\pi^2}\sum_{i=1}^{\alpha-1}\frac{p^{2i}}{\phi(p^i)}\left(\frac{\pi^2\phi(p^i)}{96}\left(1-\frac{1}{p^2}\right)-\frac{66\pi^2\phi^2(p^i)}{96p^{2i}}\right)$$

$$= \frac{1}{6}p^{\alpha}\phi^2(p^{\alpha})\left(1-\frac{1}{p}-\frac{2}{p^2}\right)+\frac{1}{24}p^{2\alpha}\phi(p^{\alpha})\left(1-\frac{1}{p^2}\right)-\frac{33}{12}\phi^2(p^{\alpha})$$

$$+\frac{1}{24}\phi(p^{\alpha})(p^{2\alpha-2}-1)-\frac{33}{12}\phi(p^{\alpha})(p^{\alpha-1}-1). \qquad (8)$$

So, combining (7) and (8) gives the desired result.

3.2 Proof of Theorem 2

By considering the fact that if k is positive integer then $S(kh,kq)=S(h,q)$ and using (2) and Lemma 4, one has

$$\sum_{h=1}^{q} s_4(2h,q)R_q^2(2h+1)$$

$$=\sum_{h=1}^{q}\{-4S(2h,q)+8S(h,q)\}R_q^2(2h+1)$$

$$=-4\sum_{h=1}^{q} S(2h,q)R_q^2(2h+1)+8\sum_{h=1}^{q} S(h,q)R_q^2(2h+1)$$

$$=-2T_1+\frac{8}{\pi^2 q}\sum_{d|q}\frac{d^2}{\phi(d)}\sum_{\substack{\chi \bmod d\\ \chi\ odd}}\sum_{h=1}^{q}\chi(h)R_q^2(2h+1)|L(1,\chi)|^2$$

$$:=-2T_1+T_3, \tag{9}$$

where T_3 can be computed as follows:

$$T_3=\frac{8}{\pi^2 p^\alpha}\sum_{i=1}^{\alpha}\frac{p^{2i}}{\phi(p^i)}\sum_{\substack{\chi \bmod p^i\\ \chi\ odd}}\sum_{h=1}^{p^\alpha}\chi(h)R_{p^\alpha}^2(2h+1)|L(1,\chi)|^2$$

$$=\frac{8p^\alpha}{\pi^2\phi(p^\alpha)}\sum_{\substack{\chi \bmod p^\alpha\\ \chi\ odd}}\sum_{h=1}^{p^\alpha}\chi(h)R_{p^\alpha}^2(2h+1)|L(1,\chi)|^2$$

$$+\frac{2}{\pi^2 p^\alpha}\sum_{i=1}^{\alpha-1}\frac{p^{2i}}{\phi(p^i)}\sum_{\substack{\chi \bmod p^i\\ \chi\ odd}}\sum_{h=1}^{p^\alpha}\chi(h)R_{p^\alpha}^2(2h+1)|L(1,\chi)|^2$$

$$=-\frac{8p^\alpha}{\pi^2\phi(p^\alpha)}\left(\sum_{\substack{\chi \bmod p^\alpha\\ \chi\ odd}}{}^{*}p^{2\alpha}\left(1-\frac{2}{p}\right)\overline{\chi}(2)|L(1,\chi)|^2+\right.$$

$$\left.\sum_{\substack{\chi \bmod p^{\alpha-1}\\ \chi\ odd}}\chi(2)p^\alpha\phi(p^\alpha)|L(1,\chi)|^2\right)$$

$$-\frac{8\phi(p^\alpha)}{\pi^2}\sum_{i=1}^{\alpha-1}\frac{p^{2i}}{\phi(p^i)}\sum_{\substack{\chi \bmod p^i\\ \chi\ odd}}\chi(2)|L(1,\chi)|^2. \tag{10}$$

Now, let us apply [17, Eq. 2.7] and (5) to find the right hand side of (10) as

$$-\frac{1}{3}p^\alpha\phi(p^\alpha)\left(1-\frac{1}{p}-\frac{2}{p^2}\right)-\frac{1}{3}\phi^2(p^\alpha)(p^{\alpha-1}+p^{\alpha-2}-6) \tag{11}$$

$$-\frac{1}{3}\phi(p^\alpha)(p^{2\alpha-2}-1)+2\phi(p^\alpha)(p^{\alpha-1}-1).$$

Therefore, gathering (7), (9) and (11) completes the proof.

Remark 8 *Using (3), we can write*

$$\sum_{h=1}^{q} S_1(2h,q)R_q^2(2h+1)$$

$$= \sum_{h=1}^{q} \{8S(h,q) + 8S(4h,q) - 20S(2h,q)\}R_q^2(2h+1).$$

Observe that the right hand side of the last identity is equal to $-10T_1 - 2T_2 + T_3$, mentioned above. So, from this, the counterpart result for $S_1(2h,q)$ can be verified easily by the similar way.

4. Conclusion and Comment

As is well known, Dirichlet L-function, Dedekind sums (and analogues) and Ramanujan sum all play vital roles in the analytic number theory. In this paper, we used the properties of Dirichlet L-functions to cope with Hardy sums and Ramanujan sum with regard to a mean value problem, and then give several explicit formulas. We consider here the prime modulo p case. We remark that the question of whether there exist the formulas

$$\sum_{h=1}^{q}{}' s_3(2h,q)R_q^2(2h+1) \quad \text{and} \quad \sum_{h=1}^{q}{}' s_4(2h,q)R_q^2(2h+1)$$

for an arbitrary square-full number $q > 3$ is an open problem.

Author Statement

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Conflict of Interest

As the author of this study, I declare that I do not have any conflict of interest statement.

Ethics Committee Approval and Informed Consent

As the author of this study, I declare that I do not have any ethics committee approval and/or informed consent statement.

References

[1] T. M. Apostol, *Introduction to Analytic Number Theory*. New York: Springer-Verlag, 1976, pp. 165–173.

[2] T. M. Apostol, *Modular Functions and Dirichlet Series in Number Theory*. New York: Springer-Verlag, 1976, ch. 3.

[3] B. C. Berndt, "Analytic Eisenstein series, theta functions and series relations in the spirit of Ramanujan," *J. Reine Angew. Math.*,303/304, 332–365, 1978.

[4] L. Carlitz, "The reciprocity theorem of Dedekind sums," *Pac. J. Math.*,3, 513–522, 1953.

[5] J. Chaohua, "On the mean value of Dedekind sums," *J. Number Theory*,87, 173–188, 2001.

[6] J. B. Conrey, E. Fransen, R. Klein, and C. Scott, "Mean values of Dedekind sums," *J. Number Theory*,56, 214–226, 1996.

[7] M. C. Dağlı, "On the hybrid mean value of generalized Dedekind sums, generalized Hardy sums and Ramanujan sum," Accepted for publication in *Bull. Math. Soc. Sci. Math. Roumanie*.

[8] M. C. Dağlı, "On some identities involving certain Hardy sums and Kloosterman sum," Accepted for publication in *Ukr. Math. J.*

[9] D. Han, W. Zhang, "Some new identities involving Dedekind sums and the Ramanujan sum," *Ramanujan J.*,35, 253–262, 2014.

[10] L. A. Goldberg, "*Transformations of theta-functions and analogues of Dedekind sums,*" Ph.D. thesis, University of Illinois, Urbana, United States of America, 1981.

[11] H. Y. Liu, W. Zhang, “Some identities involving certain Hardy sums and Ramanujan sum,” *Acta Math. Sin. Engl. Ser.*,21, 109–116, 2005.
[12] W. Liu, “Mean value of Hardy sums over short intervals,” *Acta Math. Acad. Paedagog. Nyházi*,28, 1–11, 2012.
[13] W. Liu, “On the mean values of Dedekind sums over short intervals,” *Notes Number Theory Discrete Math.*, 19 (2), 60–68, 2013.
[14] R. Ma, Y. L. Zhang, and M. Grützmann, “Some notes on identities for Dirichlet $L-$functions,” *Acta Math. Sin. Engl. Ser.*,30, 747–754, 2014.
[15] H. Rademacher, E. Grosswald, *Dedekind sums.* Washington: Math. Assoc. of America, 1972, ch. 6.
[16] R. Sitaramachandrarao, “Dedekind and Hardy sums,” *Acta Arith.*,XLIII, 325–340, 1987.
[17] W. Wang, D. Han, “An identity involving certain Hardy sums and Ramanujan’s sum,” *Adv. Difference Equ.*,261, 1– 8, 2013.
[18] H. Zhang, T. Zhang, “Some identities involving certain Hardy sums and general Kloosterman sums,” *Mathematics*, 8 (1), 1– 13, 2020.
[19] H. Zhang, W. Zhang, “On the identity involving certain Hardy sums and Kloosterman sums,” *J. Inequal. Appl.*,52, 1– 9, 2014.
[20] W. Zhang, “On the mean values of Dedekind sums,” *J. Theor. Nombres Bordeaux*,8, 429–442, 1996.