Süleyman Demirel Üniversitesi Fen Edebiyat Fakültesi Fen Dergisi
Süleyman Demirel University Faculty of Arts and Sciences Journal of Science
2020, 15(1): 80-89
DOI: 10.29233/sdufeffd.694376
Atıf için / For Citation: R. Savaş, "Strongly summable bivariate functions of weight g", *Süleyman Demirel Üniversitesi Fen Edebiyat Fakültesi Fen Dergisi*, 15(1), 80-89, 2020.

Strongly Summable Bivariate Measurable Functions of Weight *g*

Rabia SAVAŞ*[1]

[1] *Sakarya Üniversitesi, Matematik Bölümü, Sakarya, Türkiye*

**corresponding author e-mail: rabiasavass@hotmail.com*

(Alınış / Received: 25.02.2020, Kabul / Accepted: 24.03.2020, Yayımlanma / Published: 31.05.2020)

Abstract: In 1965, Borwein presented the concept of strongly summable single valued functions. Using Borwein's results, in 2019 Patterson et all. Introduced the notion of multidimensional linear functions connected with double strongly Cesaro summability theory. The aim of this study is to extend Patterson et all's results to strongly summable bivariate functions with respect to weight of g. To achieve this by considering a real valued non-negative bivariate measurable function defined on the interval $(1,\infty)\times(1,\infty)$ the concepts of double $\left[W^{g}_{\lambda\mu}\right]_f$ – strongly summable and $\left[S^{g}_{\lambda\mu}\right]_f$ – double statistical convergence of weight g will be introduced, where $g : [0,\infty)\times[0,\infty)\to[0,\infty)$ such that $g(x_m, x_n)\to\infty$ as $x_m\to\infty$ and $x_n\to\infty$. Also g is factorable. In addition, the relationship between these two concepts will be examined and some algebraic characterization of real valued lebesgue measurable bivariate functions will be also presented.

***Key words**:* Statistical convergence functions of two variables, real valued Lebesgue measurable function, strong summability, weight g

g Ağırlıklı Kuvvetli Toplanabilir İki Değişkenli Ölçülebilir Fonksiyonlar

Özet: 1965 yılında Borwein tek değişkenli kuvvetli toplanabilir fonksiyonları sunmuştur. Borwein'nin sonuçlarını kullanarak 2019 yılında Patterson ve diğerleri çift kuvvetli Cesaro toplanabilme teorisi ile bağlantılı olarak iki boyutlu lineer fonksiyonları tanımlamıştır. Bu makalenin amacı Patterson ve diğerlerinin sonuçlarının g ağırlığı ile ilişkili olarak kuvvetli toplanabilir iki değişkenli ölçülebilir fonksiyonlara genelleştirmektir. Bunu elde etmek için $(1,\infty)\times(1,\infty)$ aralığında tanımlı negatif olmayan reel değerli iki değişkenli ölçülebilir fonksiyonlar göz önüne alınarak eğer $x_m\to\infty$ and $x_n\to\infty$ iken $g(x_m, x_n)\to\infty$ olacak şekilde $g : [0,\infty)\times[0,\infty)\to[0,\infty)$ ağırlıklı çift $\left[W^{g}_{\lambda\mu}\right]_f$ – kuvvetli toplanabilir ve $\left[S^{g}_{\lambda\mu}\right]_f$ – çift istatistiksel yakınsaklık kavramları sunulacaktır. Ayrıca, g fonksiyonu çarpanlarına ayrılabilir. Buna ek olarak, bu iki kavram arasındaki ilişki incelenecek ve reel değerli Lebesgue anlamında ölçülebilir iki değişkenli fonksiyonların bazı cebirsel özellikleri de verilecektir.

***Anahtar kelimeler**:* İki değişkenli fonksiyonların istatistiksel yakınsaklığı, reel değerli Lebesgue ölçülebilir fonksiyonlar, kuvvetli toplanabilme, g ağırlık.

Rabia SAVAŞ, rabiasavass@hotmail.com, ORCID: https://orcid.org/ 0000-0002-4911-9067

1. Introduction

Both Fast [11] and Steinhaus [31] presented the study of statistical convergence in 1951 in the same journal and volume. Since their presentation the notion of statistical convergence has spawned an impressive volume of research articles. Three years later Buck in [3] presentation an example of statistical convergence via convergence of density. In the current iteration of statistical convergence Fridy in [13] began the analysis of these definitions by means of sequence spaces and linked it to summability theory. Following Fridy's work Connor [4], Moricz [18], Mursaleen [23] and countless other authors continue the analysis along this vain. In a similar manner Zygmund in [32] examined the relationship between statistical convergence and strongly summability. Moreover, a lot of mathematicians applied the notion of statistical convergence to different fields. For instance, probability theory [14], Banach spaces [5], optimization [25], number theory [9], measure theory [17], approximation theory [8], and trigonometric series [32].

In 1981, Freedman and Sember [12] presented the concept of asymptote density. By considering the definition of this notion, Schoenberg [30] extended the concept of convergence of real sequences to statistical convergence. Following Fridy's [13] examination Mursaleen in [19] gave a notion of $\lambda-$statistically convergence sequences, and this concept was also studied in [15], [6], [7] and [10].

Additionally, Pringsheim presented the concept of convergence for double sequences [26]. A double sequence $y=(y_{m,n})$ has Pringsheim limit $L\in \mathrm{R}$ (denoted by $P-\lim\limits_{m,n\to\infty} y_{m,n}=L$) if $\varepsilon>0$ there exists $N\in \mathrm{N}$ such that $\left|y_{m,n}-L\right|<\varepsilon,$ whenever $m,n>N.$ The double sequence y will shortly describe as "$P-convergent$".

In recent times, Mursaleen and Edely [21] have defined the following definition for double sequences $y=(y_{m,n})$. Let $R\subseteq \mathrm{N}\times\mathrm{N}$ be a multidimensional set of positive integers and let $R(p,s)$ be the numbers of (m,n) in R such that $m\le p$ and $n\le s$. Then Mursaleen and Belen [22] defined the multidimensional analogue of natural density as follows.

Definition 1.1. ([22]). *Let $y=(y_{m,n})$ be a double sequence. $y=(y_{m,n})$ is defined to be statistically convergent to the number ℓ provided that $\varepsilon>0$, the set*

$$\left\{(m,n):\ m\le p \text{ and } n\le s\ :\ \left|y_{m,n}-\ell\right|\ge\varepsilon\right\}$$

has double natural density zero. Whenever this occurs we write $st_2-\lim\limits_{m,n} y_{m,n}=\ell$.

By using Liendler's definition [16], Mursaleen et al. [20] defined the following definitions.

Definition 1.2. ([20]). *Let $\lambda=(\lambda_p)$ and $\mu=(\mu_s)$ are two non-descreasing sequences of positive real numbers tending to ∞ and $\lambda_{p+1}\le\lambda_p+1,\ \lambda_1=1$ and $\mu_{s+1}\le\mu_s+1, \mu_1=1.$ The collection of such sequence (λ,μ) will be denoted by Δ. Let $R\subseteq \mathrm{N}\times\mathrm{N}$ be a two-dimensional set of positive integers. Then the (λ,μ) density of R is defined as*

$$\delta_{\lambda,\mu}=P-\lim_{p,s\to\infty}\frac{1}{\lambda_p\mu_s}\left|\left\{p-\lambda_p+1\le m\le p; s-\mu_s+1\le n\le s\ :\ (m,n)\in R\right\}\right|.$$

If the limit exists, where $I_p=\left[p-\lambda_p+1,p\right]$ *and* $J_s=\left[s-\mu_s+1,s\right]$. *During this article we shall represent* $\left(m\in I_p, n\in J_s\right)$ *by* $(m,n)\in I_{p,s}$.

Definition 1.3. ([20]) *A real double sequence* $y=\left(y_{m,n}\right)$ *is said to be* $(\lambda,\mu)-$ *statistically convergent to the number* ℓ *if for every* $\varepsilon>0$,

$$P-\lim_{p,s\to\infty}\frac{1}{\lambda_p\mu_s}\left|\left\{(m,n)\in I_{p,s}\ :\ \left|y_{m,n}-\ell\right|\ge\varepsilon\right\}\right|=0.$$

This will be denoted by $S_{\lambda,\mu}-\lim y=\ell$.

Lately, it has been displayed in [1] that one can further extend the notion of asymptotic density by considering natural density of weight g^* when $g^*:\mathrm{N}\to[0,\infty)$ is a function with $\lim_{r\to\infty}g^*(r)=\infty$ and $\frac{r}{g^*(r)}\not\to 0$ as $r\to\infty$.

Definition 1.4. ([1]) *Let* $R\subseteq \mathrm{N}\times\mathrm{N}$. *As before if* $p,s\in\mathrm{N}\times\mathrm{N}$, *by* $R_{p,s}$ *where we stand for the cardinality of the set* $\left\{(m,n)\in R\ :\ 1\le m\le r \text{ and } 1\le n\le s\right\}$. *Let* $g\in G$ *and define* (m,n) *in* R *such that* $m\le p$ *and* $n\le s$. *In addition, let us define* $\underline{\delta_2^g}(R)=P-\liminf_{p,s}\frac{R_{p,s}}{g(p,s)}$ *and* $\overline{\delta_2^g}(R)=P-\limsup_{p,s}\frac{R_{p,s}}{g(p,s)}$ *which are the lower and upper double density of weight* g *of the set* R, *separately. If* $P-\lim_{p,s}\frac{R_{p,s}}{g(p,s)}$ *exists in Pringheim's sense then the double density of weight* g *of the set* R *will exist and it will be symbolized by* $\delta_2^g(R)$.

Later, strongly summable single valued functions introduced and studied by Borwein in [2]. Following Borwein's definition, in [24] Nuray introduced $\lambda-$strongly summable and $\lambda-$statistically convergent functions by taking nonnegative real-valued Lebesgue measurable function on $(1,\infty)$ instead of sequences. Later R. F. Patterson et al. presented the concept of multidimensional linear functions connected with double strongly Cesáro summability. Using R.F. Patterson et all's results, R. Savas [28] introduced definition of $\lambda\mu-$strongly double summable and $\lambda\mu-$double statistical convergence by taking nonnegative real valued Lebesgue measurable function of two variables in the interval $(1,\infty)\times(1,\infty)$, and R. Savas and R. F. Patterson [29] presented the notion of ideal lacunary strongly summability for multidimensional measurable functions.

Definition 1.5. ([27]) $\lambda,\mu\in\Delta$, *and real valued non-negative function of two variables* $f(\xi,\eta)$ *measurable on* $(1,\infty)\times(1,\infty)$ *is defined to be* $\lambda\mu-$*double statistically convergent to* ℓ, *if for every* $\varepsilon>0$,

$$P-\lim_{p,s\to\infty}\frac{1}{\lambda_p\mu_s}\left|\left\{(\xi,\eta)\in I_{p,s}\ :\ \left|f(\xi,\eta)-\ell\right|\ge\varepsilon\right\}\right|=0,$$

where the vertical bars show the Lebesgue measure of the enclosed set. Under this condition, we can write $f(\xi,\eta) \to \ell\left([S_{\lambda\mu}]_f\right)$, *and*

$$\left[S,\lambda,\mu\right]_f \ := \left\{ f(\xi,\eta)) : \exists\ell, [S_{\lambda\mu}]_f - \lim f(\xi,\eta) = \ell \right\}.$$

If we take $\lambda_p = p$ *and* $\mu_s = s$, *then* $\left[S,\lambda,\mu\right]_f$ *is the same as* S_f, *the set of double statistically convergent functions.*

In this paper using by considering a factorable function $g : [0,\infty)\times[0,\infty)\to[0,\infty)$ such that $P-\lim g\left(x_m, x_n\right) = \infty$ as $x_m \to \infty$ and $x_n \to \infty$, the notions of $\left[W,\lambda,\mu\right]_f -$strongly double summability and $\lambda\mu$-double statistical convergence, the concepts of $\left[W^g_{\lambda\mu}\right]_f$-strongly double summable and $\left[S^g_{\lambda\mu}\right]_f -$double statistical convergence of weight g of real-valued function of bivariate $f(\xi,\eta)$ measurable on $(1,\infty)\times(1,\infty)$ are presented. Please note that the class of all such functions will be represented by $\mathbf{G}_2$.

2. Material and Method

In this section, we will give new definitions. By using these definitions we examine a series of basic results, and inclusions of theorem, extensions as well as variations are proved. Throughout by function $f(\xi,\eta)$ we shall mean a non-negative real-valued Lebesgue bivariate measurable function defined on $(1,\infty)\times(1,\infty)$.

Definition 2.1. *A function* $f(\xi,\eta)$ *is defined to be strongly double summable with respect to weight* g *to* L *if* $g \in \mathbf{G}_2$,

$$P-\lim_{p,s\to\infty} \frac{1}{g(p,s)} \int_1^p\int_1^s \left| f(\xi,\eta) - L\right|^q d\xi d\eta = 0, \quad 1 \le q < \infty$$

and $f(\xi,\eta) \to L\left(\left[W_2^g\right]_f^q\right)$. *The set of all bivariate functions that are strongly double summable with respect to weight* g *will be symbolized simply by* $\left[W_2^g\right]_f^q$.

Definition 2.2. *Let* $\lambda,\mu \in \Delta$, *and* $g \in \mathbf{G}_2$. *A function* $f(\xi,\eta)$ *is defined to be* $\lambda\mu-$*strongly double summable with respect to weight* g *to* L *provided that*

$$P-\lim_{p,s\to\infty} \frac{1}{g(\lambda_p,\mu_s)} \int_{p-\lambda_p+1}^{p}\int_{s-\mu_s+1}^{s} \left| f(\xi,\eta) - L\right|^q d\xi d\eta = 0, \ 1 \le q < \infty,$$

under this condition, we write $f(\xi,\eta) \to L\left([W^g_{\lambda\mu}]^q_f\right)$. *The class of all* $\lambda\mu-$*strongly double summable with respect to weight* g *will be represented by* $\left[W^g,\lambda,\mu\right]_f^q$. *For* $\lambda_p = p$ *and* $\mu_s = s$ *for all* $p,s \in \mathrm{N}$, *we shall write* $\left[W_2^g\right]_f^q$ *instead of* $\left[W^g,\lambda,\mu\right]_f^q$.

Definition 2.3. *A function* $f(\xi,\eta)$ *is defined to be double statistically convergent with respect to weight* g *to* L *as long as for* $\varepsilon > 0$, *and* $g \in \mathbf{G}_2$,

$$P-\lim_{p,s\to\infty} \frac{1}{g(p,s)} \left|\{(\xi,\eta) : \ \xi \le p \text{ and } \eta \le s \ :| f(\xi,\eta) - L| \ge \varepsilon\}\right| = 0,$$

where the vertical bars show the Lebesgue measure of the enclosed set. In this case we shall write $f(\xi,\eta) \to L\left([S_2^g]_f\right)$*. The set of all double statistically convergent functions with respect to weight* g *will be symbolized by* $[S_2^g]_f$ *.*

Definition 2.4. *Let* $\lambda, \mu \in \Delta$*, and* $g \in \mathbf{G}_2$*. A function* $f(\xi,\eta)$ *is defined to be* $\lambda\mu-$ *double statistically convergent with respect to weight* g *to* L *if for every* $\varepsilon > 0$ *,*

$$P - \lim_{p,s\to\infty} \frac{1}{g(\lambda_p,\mu_s)} \left| \{(\xi,\eta) \in I_{p,s} \ : \ | f(\xi,\eta) - L | \geq \varepsilon\} \right| = 0,$$

where the vertical bars show the Lebesgue measure of the enclosed set. Under this condition, we write $f(\xi,\eta) \to L\left([S_{\lambda\mu}^g]_f\right)$*. The set of all* $\lambda\mu-$*double statistically convergent functions with respect to weight* g *will be represented by* $[S^g, \lambda, \mu]_f$ *.*

3. Results

The subsequent theorem grants the algebraic characterization of real valued Lebesgue measurable function of two variables.

Theorem 3.1. *Let* $g \in \mathbf{G}_2$*,* $f(\xi,\eta)$ *and* $g(\xi,\eta)$ *be two real-valued non-negative Lebesgue bivariate measurable functions on* $(1,\infty)\times(1,\infty)$*, at that point*

(i) If $[S_{\lambda\mu}^g]_f - \lim f(\xi,\eta) = L$ *and* $c_1 \in \mathrm{R}$*,* $[S_{\lambda\mu}^g]_f - \lim(c_1 f(\xi,\eta)) = c_1 L$

(ii) If $[S_{\lambda\mu}^g]_f - \lim f(\xi,\eta) = L_1$ *and* $[S_{\lambda\mu}^g]_f - \lim g(\xi,\eta) = L_2$ *, then*

$$[S_{\lambda\mu}^g]_f - \lim(f(\xi,\eta) + g(\xi,\eta)) = L_1 + L_2$$

Proof. (i) For $c_1 = 0$, the consequence holds easily. Let $c_1 \neq 0$. We shall obtain

$$\frac{1}{g(\lambda_p,\mu_s)} \left| \{(\xi,\eta) \in I_{p,s} \ : \ \left| c_1 f(\xi,\eta) - c_1 L \right| \geq \varepsilon\} \right|$$

$$= \frac{1}{g(\lambda_p,\mu_s)} \left| \left\{ (\xi,\eta) \in I_{p,s} \ : \ \left| f(\xi,\eta) - L \right| \geq \frac{\varepsilon}{|c_1|} \right\} \right|.$$

(ii) The result will be granted as follows

$$\frac{1}{g(\lambda_p,\mu_s)} \left| \{(\xi,\eta) \in I_{p,s} \ : \ \left| f(\xi,\eta) + g(\xi,\eta) - (L_1 + L_2) \right| \geq \varepsilon\} \right|$$

$$\leq \frac{1}{g(\lambda_p,\mu_s)} \left| \{(\xi,\eta) \in I_{p,s} \ : \ \left| f(\xi,\eta) - L_1 \right| \geq \frac{\varepsilon}{2}\} \right|$$

$$+ \frac{1}{g(\lambda_p,\mu_s)} \left| \{(\xi,\eta) \in I_{p,s} \ : \ \left| g(\xi,\eta) - L_2 \right| \geq \frac{\varepsilon}{2}\} \right|.$$

Theorem 3.2. *Let* $\lambda, \mu \in \Delta$ *and* $g_1, g_2 \in \mathbf{G}_2$ *be such that there exists* $R > 0$ *and* $(r,t) \in \mathbb{N} \times \mathbb{N}$ *such that* $\frac{g_1(\lambda_p,\mu_s)}{g_2(\lambda_p,\mu_s)} \leq R$ *for all* $p \geq r$ *and* $s \geq t$ *. Then* $[S_{\lambda\mu}^{g_1}]_f \subset [S_{\lambda\mu}^{g_2}]_f$ *.*

Proof. Observe that,

$$\frac{1}{g_2(\lambda_p,\mu_s)}\left|\{(\xi,\eta)\in I_{p,s} \ : \ \left|f(\xi,\eta)-L\right|\geq\varepsilon\}\right|$$
$$=\frac{g_1(\lambda_p,\mu_s)}{g_2(\lambda_p,\mu_s)}\cdot\frac{1}{g_1(\lambda_p,\mu_s)}\left|\{(\xi,\eta)\in I_{p,s} \ : \ \left|f(\xi,\eta)-L\right|\geq\varepsilon\}\right|$$
$$\leq M\cdot\frac{1}{g_1(\lambda_p,\mu_s)}\left|\{(\xi,\eta)\in I_{p,s} \ : \ \left|f(\xi,\eta)-L\right|\geq\varepsilon\}\right|$$

for all $p\geq r$ and $s\geq t$. If $f(\xi,\eta)\in S_{\lambda\mu}^{g_1}$ then the right hand side leans towards to 0 for every $\varepsilon>0$,

$$\frac{1}{g_2(\lambda_p,\mu_s)}\left|\{(\xi,\eta)\in I_{p,s} \ : \ \left|f(\xi,\eta)-L\right|\geq\varepsilon\}\right|=0$$

and $f(\xi,\eta)\in S_{\lambda\mu}^{g_2}$. Therefore, $\left[S_{\lambda\mu}^{g_1}\right]_f\subseteq\left[S_{\lambda\mu}^{g_2}\right]_f$.

The following consequence is obtained from Theorem 3.2.

Corollary 3.1. *Let $\lambda,\mu\in\Delta$ and $f(\xi,\eta)$ be a nonnegative real-valued Lebesgue measurable function of two variables on $(1,\infty)\times(1,\infty)$. Particularly if $g\in\mathbf{G}_2$ be such that there exiss $R>0$, and $(r,t)\in\mathrm{N}\times\mathrm{N}$ such that $\frac{ps}{g(\lambda_p,\mu_s)}\leq R$ for all $p\geq r$ and $s\geq t$ then $[S_{\lambda\mu}^{g}]_f\subseteq[S_{\lambda\mu}]_f$.*

Theorem 3.3. *Let $\lambda,\mu\in\Delta$ and $g\in\mathbf{G}_2$. $[S_f]\subseteq[S_{\lambda\mu}^{g}]_f$ if $P-\liminf_{p,s\to\infty}\frac{g(\lambda_p,\mu_s)}{ps}>0$.*

Proof. For any $\varepsilon>0$, we obtain

$$\left\{\xi\leq p,\ \eta\leq s \ : \ \left|f(\xi,\eta)-L\right|\geq\varepsilon\right\}\supseteq\left\{(\xi,\eta)\in I_{p,s} \ : \ \left|f(\xi,\eta)-L\right|\geq\varepsilon\right\}.$$

Thus it will follow that for $p,s\in\mathrm{N}$

$$\frac{1}{ps}\left|\left\{\xi\leq p,\ \eta\leq s \ : \ \left|f(\xi,\eta)-L\right|\geq\varepsilon\right\}\right|$$
$$\geq\frac{1}{ps}\left|\left\{(\xi,\eta)\in I_{p,s} \ : \ \left|f(\xi,\eta)-L\right|\geq\varepsilon\right\}\right|$$
$$\geq\frac{g(\lambda_p,\mu_s)}{ps}\cdot\frac{1}{g(\lambda_p,\mu_s)}\left|\left\{(\xi,\eta)\in I_{p,s} \ : \ \left|f(\xi,\eta)-L\right|\geq\varepsilon\right\}\right|.$$

If $f(\xi,\eta)\to L(S_f)$, then $\frac{1}{ps}\left|\left\{\xi\leq p,\ \eta\leq s \ : \ \left|f(\xi,\eta)-L\right|\geq\varepsilon\right\}\right|\to 0$ as $p,s\to\infty$, and so

$$\frac{1}{g(\lambda_p,\mu_s)}\left|\left\{(\xi,\eta)\in I_{p,s} \ : \ \left|f(\xi,\eta)-L\right|\geq\varepsilon\right\}\right|\to 0$$

as $p,s\to\infty$. Hence, $f(\xi,\eta)\to L\left[S_{\lambda\mu}^{g}\right]_f$.

In view of Theorem 3.2, the following consequence will be stated without proof.

Theorem 3.4. *Let* $\lambda,\mu\in\Delta$. *If* $g_1,g_2\in\mathbf{G}_2$ *be such that there exists* $R>0$ *and* $(r,t)\in\mathrm{N}\times\mathrm{N}$ *such that* $g_1(\lambda_p,\mu_s)/g_2(\lambda_p,\mu_s)\leq R$ *for all* $p\geq r$ *and* $s\geq t$ *then* $\left[W_{\lambda\mu}^{g_1}\right]_f^q\subseteq\left[W_{\lambda\mu}^{g_2}\right]_f^q$.

We have the following result from Theorem 3.4.

Corollary 3.2. *Let* $\lambda,\mu\in\Delta$, *and* q *be a positive real number, then* $\left[W_{\lambda\mu}^{g_1}\right]_f^q\subseteq\left[W_{\lambda\mu}^{g_2}\right]_f^q$.

Theorem 3.5. *Let* $\lambda,\mu\in\Delta$ *and* q *be a positive real number. Let* $g_1,g_2\in\mathbf{G}_2$ *be such that there exist* $R>0$ *and* $(r,t)\in\mathrm{N}\times\mathrm{N}$ *such that* $g_1(\lambda_p,\mu_s)/g_2(\lambda_p,\mu_s)\leq R$ *for all* $p\geq r$ *and* $s\geq t$. *Let* $0<q<\infty$. *If a function* $f(\xi,\eta)$ *is strongly* $\left[W^g,\lambda,\mu\right]_f^q-$*strongly double summable with respect to weight* g_1 *to* L *then it is* $\lambda\mu-$*double statistically convergent with respect to weight* g_2 *to* L *in other way* $[W_{\lambda\mu}^{g_1}]_f^q\subset\left[S_{\lambda\mu}^{g_2}\right]_f$.

Proof. Let function $f(\xi,\eta))\in[W_{\lambda\mu}^g]_f^q$ and let $\varepsilon>0$ be given. We observe that

$$\begin{aligned}\int\limits_{(\xi,\eta)\in I_{p,s}}\left|f(\xi,\eta)-L\right|^q d\xi d\eta &= \int\limits_{(\xi,\eta)\in I_{p,s},|f(\xi,\eta)-L|\geq\varepsilon}\left|f(\xi,\eta)-L\right|^q d\xi d\eta\\ &\quad+\int\limits_{(\xi,\eta)\in I_{p,s},|f(\xi,\eta)-L|\leq\varepsilon}\int\left|f(\xi,\eta)-L\right|^q d\xi d\eta\\ &\geq \int\limits_{(\xi,\eta)\in I_{p,s},|f(\xi,\eta)-L|\geq\varepsilon}\left|f(\xi,\eta)-L\right|^q d\xi d\eta\\ &\geq\left|\{(\xi,\eta)\in I_{p,s}\ :\ \left|f(\xi,\eta)-L\right|\geq\varepsilon\right|\cdot\varepsilon^q.\end{aligned}$$

Now, it follows that

$$\begin{aligned}&\frac{1}{g_1(\lambda_p,\mu_s)}\int\limits_{(\xi,\eta)\in I_{p,s}}\left|f(\xi,\eta)-L\right|^q\\ &\geq\frac{1}{g_1(\lambda_p,\mu_s)}\left|\{(\xi,\eta)\in I_{m,n}\ :\ \left|f(\xi,\eta)-L\right|\geq\varepsilon\right|\cdot\varepsilon^q\\ &=\frac{g_2(\lambda_p,\mu_s)}{g_1(\lambda_p,\mu_s)}\cdot\frac{1}{g_2(\lambda_p,\mu_s)}\left|\{(\xi,\eta)\in I_{p,s}\ :\ \left|f(\xi,\eta)-L\right|\geq\varepsilon\right|\cdot\varepsilon^q\\ &\geq\frac{1}{M}\cdot\frac{1}{g_2(\lambda_p,\mu_s)}\left|\{(\xi,\eta)\in I_{p,s}\ :\ \left|f(\xi,\eta)-L\right|\geq\varepsilon\right|\cdot\varepsilon^q\end{aligned}$$

for all $p\geq r$ and $s\geq t$. If $f(\xi,\eta)\to L([W_{\lambda\mu}^g]_f^p)$, then the left hand side leans towards to 0 and as a result the right hand side also leans towards to 0. Thus, $f(\xi,\eta)\to L\left[S_{\lambda\mu}^{g_2}\right]_f$.

Corollary 3.3. *If* $g\in\mathbf{G}_2$ *be such that there exist* $R>0$ *and* $(r,t)\in\mathrm{N}\times\mathrm{N}$ *such that* $\frac{ps}{g(\lambda_p,\mu_s)}\leq R$ *for all* $p\geq r$, $s\geq t$, *and* $0<q<\infty$, *then* $[W_{\lambda\mu}^g]_f^q\subseteq\left[S_{\lambda\mu}\right]_f$.

Theorem 3.6. *Let* $\lambda = (\lambda_p)$, $\tau = (\tau_p)$, *and* $\mu = (\mu_s)$, $\nu = (\nu_s)$ *be four sequences in* Δ *such that* $\lambda_p \le \tau_p$, *and* $\mu_s \le \nu_s$ *for all* $p, s \in \mathrm{N}$, *and* $g_1, g_2 \in \mathbf{G}_2$. *If*

$$P - \liminf_{p,s\to\infty} \frac{g_1(\lambda_p,\mu_s)}{g_2(\tau_p,\nu_s)} > 0 \qquad (1)$$

then $[S_{\tau\nu}^{g_2}]_f \subseteq [S_{\lambda\mu}^{g_1}]_f$.

Proof. Suppose that $\lambda_p \le \tau_p$, and $\mu_s \le \nu_s$ for all $p, s \in \mathrm{N}$ and let (1) be satisfied. Then $I_{p,s} \subset I_{p,s}^{*}$, and so that $\varepsilon > 0$ we shall write

$$\{(\xi,\eta) \in I_{p,s}^{*} \ :|\ f(\xi,\eta) - L| \ge \varepsilon\} \supset \{(\xi,\eta) \in I_{p,s} \ :|\ f(\xi,\eta) - L| \ge \varepsilon\}$$

and so

$$\frac{1}{g_2(\tau_p,\nu_s)} \left|\{(\xi,\eta) \in I_{p,s}^{*} \ :|\ f(\xi,\eta) - L| \ge \varepsilon\}\right|$$
$$\ge \frac{g_1(\lambda_p,\mu_s)}{g_2(\tau_p,\nu_s)} \frac{1}{g_1(\lambda_p,\mu_s)} \left|\{\xi,\eta) \in I_{p,s} \ :|\ f(\xi,\eta) - L| \ge \varepsilon\}\right|$$

for all $p, s \in \mathrm{N}$, where $I_{p,s}^{*} = [p - \tau_p + 1, p] \times [s - \nu_s + 1, s]$. Now, taking the Pringsheim limit as $p, s \to \infty$ in the last inequality and considering (1), we obtain $[S_{\tau\nu}^{g_2}]_f \subseteq [S_{\lambda\mu}^{g_1}]_f$.

From Theorem 3.6 we are granted the following consequence.

Corollary 3.4. *Let* $\lambda = (\lambda_p)$, $\tau = (\tau_p)$, $\mu = (\mu_s)$, *and* $\nu = (\nu_s)$ *be four sequences in* Δ *such that* $\lambda_p \le \tau_p$ *and* $\mu_s \le \nu_s$ *for all* $p, s \in \mathrm{N}$. *If (1) holds, at that point*

(i) $[S_{\tau\nu}^{g_1}]_f \subseteq [S_{\lambda\mu}^{g_1}]_f$,
(ii) $[S_{\tau\nu}]_f \subseteq [S_{\lambda\mu}^{g_1}]_f$,
(iii) $[S_{\tau\nu}]_f \subseteq [S_{\lambda\mu}]_f$.

Theorem 3.7. *Let* $\lambda = (\lambda_p)$, $\tau = (\tau_p)$, $\mu = (\mu_s)$, *and* $\nu = (\nu_s)$ *be four sequences in* Δ *such that* $\lambda_p \le \tau_p$ *and* $\mu_s \le \nu_s$ *for every* $p, s \in \mathrm{N}$, *and* $g_1, g_2 \in \mathbf{G}_2$. *If (1) holds, then* $[W_{\tau\nu}^{g_2}]_f^p \subseteq [W_{\lambda\mu}^{g_1}]_f^p$.

Proof. The proof is clear, therefore omitted.

Theorem 3.8. *Let* $\lambda = (\lambda_p)$, $\tau = (\tau_p)$, *and* $\mu = (\mu_s)$, $\nu = (\nu_s)$ *be four sequences in* Δ *such that* $\lambda_p \le \tau_p$, *and* $\mu_s \le \nu_s$ *for all* $p, s \in \mathrm{N}$, $g_1, g_2 \in \mathbf{G}_2$, *and (1) holds. If a real-valued bivariate measurable function* $f(\xi,\eta)$ *is* $\left[W_{\lambda\mu}^{g}\right]_f^q -$ *strongly double summable* *with respect to weight* g_2 *to* L*, then it is* $\lambda\mu-$*double statistically convergent with respect to weight* g_1 *to* L*.*

Proof. Let a function $f(\xi,\eta)$ is defined to be $[W_{\tau v}^{g_2}]_f^q$ $-$strongly double summable with respect to weight g_2 to L and $\varepsilon>0$. Then we are granted the following

$$\begin{aligned}\int_{(\xi,\eta)\in I_{p,s}^*} |f(\xi,\eta)-L|^q\,d\xi d\eta &= \int_{(\xi,\eta)\in I_{p,s}^*,|f(\xi,\eta)-L|\geq\varepsilon} |f(\xi,\eta)-L|^q\,d\xi d\eta \\ &\quad+\int_{(\xi,\eta)\in I_{p,s}^*,|f(\xi,\eta)-L|\leq\varepsilon} \int |f(\xi,\eta)-L|^q\,d\xi d\eta \\ &\geq \int_{(\xi,\eta)\in I_{p,s},|f(\xi,\eta)-L|\geq\varepsilon} |f(\xi,\eta)-L|^q\,d\xi d\eta \\ &\geq \left|\{(\xi,\eta)\in I_{p,s}\ :\ |f(\xi,\eta)-L|\geq\varepsilon\right|\cdot\varepsilon^q.\end{aligned}$$

and so that

$$\begin{aligned}&\frac{1}{g_2(\tau_p,v_s)}\int_{(\xi,\eta)\in I_{p,s}^*} |f(\xi,\eta)-L|^q\,d\xi d\eta \\ &\geq \frac{g_1(\lambda_p,\mu_s)}{g_2(\tau_p,v_s)}\frac{1}{g_1(\lambda_p,\mu_s)}|\{(\xi,\eta)\in I_{p,s}\ :|\ f(\xi,\eta)-L|\geq\varepsilon\}|\cdot\varepsilon^q.\end{aligned}$$

Since (1) holds, it follows that if $f(\xi,\eta)$ is $[W_{\tau v}^{g_2}]_f^q$ $-$strongly double summable with respect to weight g_2 to L, then it is $\lambda\mu-$double statistically convergent with respect to weight g_1 to L.

4. Conclusion and Comment

Our results provide new suitable tools to deal with many situations of uncertainty for bivariate measurable functions. Additionally, Definition 2.4 gives a new idea to study double statistically convergent functions with respect to weight g. These results can be utilized to study other summability methods.

Author Statement

Rabia Savaş: Methodology, Conceptualization, Review and Editing.

Acknowledgment

As the author of this study, I declare that I do not have any support and thank you statement.

Conflict of Interest

As the author of this study, I declare that I do not have any conflict of interest statement.

Ethics Committee Approval and Informed Consent

As the author of this study, I declare that I do not have any ethics committee approval and/or informed consent statement.

References

[1] M. Balcerzak, P. Das, and M. Filipczak, J. Swaczyna, “Generalized kinds of density and the associated ideals”, *Acta Math. Hungar.*, 147(1), 97-115, 2015.

[2] D. Borwein, “Linear functionals with strong Cesáro Summability”, *J. Lond. Math. Soc.*, 40, 628-634, 1965.

[3] R. C. Buck, “Generalized asymptotic density”, *Amer. J. Math.*, 75 (1953), 335-346.

[4] J. S. Connor, “The statistical and strongly $p-$Cesáro convergence of sequences”, *Analysis (Munich)*, 8(1-2) , 47-63, 1988.

[5] J. Connor, and M. Ganichev, V. Kadets, “A characterization of Banach Spaces with separable duals via week statistical convergence”, *J. Math. Anal. Appl.*, 244, 251-261, 2000.

[6] R. Çolak, “Statistical convergence of order α ”, Modern methods in Analysis and its Applications, New Delhi, India, Anamaya Pub., 121-129, 2010.

[7] R. Çolak, and C. A. Bektaş, “ λ -statistical convergence of order α ”, *A Acta Math. Sci. Ser. A Chin. Ed.*, 31B(3), 953-959, 2011.

[8] O. Duman, M. K. Khan, and C. Orhan, “A-statistical convergence of approximating operators”, *Math. Inequal. Appl.*, 6, 689-699, 2003.

[9] P. Erdös, G. Tenenbaum, “Sur les densities de certaines suites d'entiers”, *J. Lond. Math. Soc.*, 59(3), 417-438, 1989.

[10] M. Et, S. A. Mohiuddine, and A. Alotaibi, “On λ -statistical convergence and strongly λ -summable functions of order α ”, *J. Inequal. Appl.*, 2-8, 2013.

[11] H. Fast, “Sur la convergence statistique”. *Colloq. Math.*, 2, 241-244, 1951.

[12] R. Freedman, and J. J. Sember, “Densities and summability”. *Pacific J. Math.*, 95(2), 293-305, 1981.

[13] J. A. Fridy, “On statistical convergence”, *Analysis (Munich)*, 5, 301-313, 1985.

[14] J.A. Fridy, and M. K. Khan, “Tauberian theorems via statistical convergence”, *J. Math. Anal. Appl.*, 228, 73-95, 1988.

[15] A. D. Gadjiev, and C. Orhan, “Some approximation theorems via statistical convergence”, *Rocky Mountain J. Math.*, 32(1), 508-520, 2002.

[16] L. Liendler, “ber die verallgemeinerte de la Vallee-Poussinsche Summierbarkeit allgemeiner Orthogonalreihen”, (German), *Acta Math. Hungar.*, 16, 375-387, 1965.

[17] H. I. Miller, “A measure theoretical subsequence characterization of statistical convergence”, *Trans. Amer. Math. Soc.*, 347, 1811-1819, 1995.

[18] F. Moricz, “Statistical convergence of multiple sequences”, *Arch. Math.*, 81, 82-89, 2003.

[19] M. Mursaleen, “ λ -statistical convergence”, *Math. Slovaca*, 50(1), 111-115, 2000.

[20] M. Mursaleen, C. Cakan, and S. A. Mohiuddine, E. Savas, “Generalized statistical convergence and statistical core of double sequences”, *Acta Math. Sin. (Engl. Ser.)*, 26(11), 2131-2144, 2010.

[21] M. Mursaleen, and O. H. Edely, “Statistical convergence of double sequences”, *J. Math. Anal. Appl.*, 288, 223-231, 2003.

[22] M. Mursaleen, and C. Belen, “On statistical Lacunary summability of double sequences”, *Filomat*, 28(2), 231-239, 2014.

[23] M. Mursaleen, and S. A. Mohiuddine, “Statistical convergence of double sequences in intuitionistic fuzzy normed spaces”, *Chaos Solitons Fractals*, 41, 2414-2421, 2009.

[24] F. Nuray, “ λ -strongly summable and λ -statistically convergent functions”, *Iran. J. Sci. Technol. Trans. A Sci.*, A4(34), 335-339, 2010.

[25] S. Pehlivan, and M. A. Mamedov, “Statistical cluster points and turnpike”, *Optimization*, 48, 93-106, 2000.

[26] A. Pringsheim, “Zur theorie der zweifach unendlichen zahlen folgen”. *Math. Ann.*, 53, 289-321, 1900.

[27] R. F. Patterson, R. Savaş, and E. Savaş, “Multidimensional linear functional connected with double strongly cesaro summability”. *Indian J. Pure, App. Math.*, 51(1), 143-153,2020.

[28] R. Savaş, “ $\lambda-$ double statistical convergence of function”, *Filomat*, 33(2), 519-524, 2019.

[29] R. Savaş, and R. F. Patterson, “I-lacunary strongly summability for multidimensional measurable”, preprint.

[30] I. J. Schoenberg, “The integrability of certain functions and related summability methods”, *Amer. Math. Monthly*, 66, 361-375, 1959.

[31] H. Steinhaus, “Sur la convergence ordinaire et la convergence asymptotique“, Colloq. Math, 2, 73-74, 1951.

[32] A. Zygmund, Trigonometric Series. Cambridge, Cambridge University Press, 1979.