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On a Generalized Difference Sequence Spaces of Fractional Order associated with Multiplier Sequence Defined by a Modulus Function

Taja YAYING*1

Abstract

Let $\Gamma(m)$ denotes the gamma function of a real number $m \notin \{0,-1,-2,\dots\}$. Then the difference matrix Δ^{α} of a fractional order α is defined as

$$(\Delta^{\alpha} v)_k = \sum_i (-1)^i \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)} v_{k+i}.$$

Using the difference operator Δ^{α}, we introduce paranormed difference sequence spaces $N_{\theta}(\Delta^{\alpha}, f, \Lambda, p)$ and $S_{\theta}(\Delta^{\alpha}, f, \Lambda, p)$ of fractional orders involving lacunary sequence, θ; modulus function, f and multiplier sequence, $\Lambda = (\lambda_k)$. We investigate topological structures of these spaces and examine various inclusion relations.

Keywords: Difference operator Δ^{α}, Paranormed sequence space, Lacunary sequence, Modulus function, Multiplier sequence.

1. INTRODUCTION

Let w denotes the space of all real valued sequences. Also ℓ_{∞}, c and c_0 will denote the spaces of bounded, convergent and null sequences, respectively. The spaces ℓ_{∞}, c and c_0 are Banach spaces normed by $\|v\|_{\infty} = \sup_k |v_k|$.

The notion of difference sequence spaces was first introduced by Kızmaz [1]. Later on, the notion was generalized by Et and Colak [2] as given below:

Let m be a non negative integer, then

$$\Delta^m(V) = \{v = (v_k) : \Delta^m v \in V\} \text{ for } V \in \{\ell_{\infty}, c, c_0\},$$

where$(\Delta^m v)_k = (\Delta^{m-1} v)_k - (\Delta^{m-1} v)_{k+1})$, $(\Delta^0 v)_k = v_k$ and

$$(\Delta^m v)_k = \sum_{i=0}^{m} (-1)^i \binom{m}{i} v_{k+i}.$$

These spaces are Banach spaces with the norm defined by

$$\|v\|_{\Delta} = \sum_{i=0}^{m} |v_i| + \sup_k |(\Delta^m v)_k|.$$

Furthermore, generalized difference sequence spaces were studied by Et and Esi [3], Et and

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Başarır [4], Malkowsky and Parashar [5], Et and Tripathy [38], Çolak [6], and many others.

The notion of statistical convergence was independently introduced by Fast [31] and Schoenberg [32]. The concept lies on the asymptotic density of the subset E of natural numbers $\mathbb{N}$. A subset E of $\mathbb{N}$ is said to have asymptotic density $\delta(E)$, if $\delta(E) = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\chi_E(k)$ exists, where χ_E is the characteristic function of E.

A sequence $v = (v_i)$ is said to be statistically convergent to L if for every $\varepsilon > 0$,

$$\lim_{k\to\infty}\frac{1}{k}|\{k \in \mathbb{N}: |v_k - L| \geq \varepsilon\}| = 0,$$

where $|E|$ denotes the cardinality of the set E. In this case, we write $S - \lim v_k = L$ or $v_k \to L(S)$.

Let $\theta = (k_r)$ be the sequence of positive integers such that $k_0 = 0$, $0 < k_r < k_{r+1}$ and $h_r = k_r - k_{r-1} \to 0$ as $r \to \infty$. Then θ is called lacunary sequence. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ will be denoted by q_r. Freedman et. Al [44] introduced the sequence space N_θ given by

$$N_\theta = \left\{v = (v_k) \in w: h_r^{-1}\sum_{k\in I_r}|v_k - L| \to 0, \text{for some } L\right\};$$

and showed that the space N_θ is a BK space with the norm defined by

$$\|v\|_\theta = \sup_r\left(h_r^{-1}\sum_{k\in I_r}|v_k|\right).$$

The study on sequence spaces was extended by using the notion of associated multiplier sequences. Goes and Goes [40] defined the differentiated sequence space dE and the integrated sequence space $\int E$ for a given sequence space E, using the multiplier sequences (k^{-1}) and (k) respectively. Different authors took different types of multiplier sequences for their study. In this article we shall consider a general multiplier sequence $\Lambda = (\lambda_k)$ of non-zero scalars.

Let $\Lambda = (\lambda_k)$ be a sequence of non-zero scalars. Then the multiplier sequence space $E(\Lambda)$, associated with the multiplier sequence Λ, of the sequence space E is defined as

$$E(\Lambda) = \{v = (v_k) \in w: (\lambda_k v_k) \in E\}.$$

The notion of a modulus function was introduced by Nakano [34]. A modulus is a function $f: [0,\infty) \to [0,\infty)$ such that

1. $f(v) = 0$ if and only if $v = 0$;
2. $f(v + u) \leq f(v) + f(u)$;
3. f is increasing;
4. f is continuous from right at 0.

Ruckle [36] and Maddox [35] used modulus function f to construct various sequence spaces. The following inequality (see [37]) will be used throughout in this article:

$$|a_k + b_k|^{p_k} \leq C(|a_k|^{p_k} + |b_k|^{p_k});$$

where $a_k, b_k \in \mathbb{C}$, $0 < p_k \leq \sup p_k = H$, $C = \max(1, 2^{H-1})$.

Proposition 1.1 [43] Let f be a modulus function and let $0 < \delta < 1$. Then for each $v \geq \delta$ we have $f(v) \leq 2f(1)\delta^{-1}v$.

2. FRACTIONAL DIFFERENCE OPERATOR AND GENERALIZED DIFFERENCE SEQUENCE SPACE OF FRACTIONAL ORDERS

Let $\Gamma(m)$ be the Gamma function of a real number m and $m \notin \{0, -1, -2, \dots\}$. Gamma function can be expressed as an improper integral

$$\Gamma(m) = \int_0^\infty e^{-t}t^{m-1}dx.$$

Recently, Baliarsingh and Dutta [10, 11] have introduced the generalized difference operator Δ^α, for a positive fraction α as follows:

$$(\Delta^{\alpha}v)_k = \sum_{i=0}^{\infty} (-1)^i \frac{\Gamma(\alpha+1)}{i!\,\Gamma(\alpha-i+1)} v_{k+i}.$$

In particular, we have

1. $(\Delta^{\frac{1}{2}}v)_k = v_k - \frac{1}{2}v_{k+1} - \frac{1}{8}v_{k+2} - \frac{1}{16}v_{k+3} - \frac{5}{128}v_{k+4} - \cdots$
2. $(\Delta^{\frac{-1}{2}}v)_k = v_k + \frac{1}{2}v_{k+1} + \frac{3}{8}v_{k+2} + \frac{5}{16}v_{k+3} + \frac{35}{128}v_{k+4} + \cdots$
3. $(\Delta^{\frac{2}{3}}v)_k = v_k - \frac{2}{3}v_{k+1} - \frac{1}{9}v_{k+2} - \frac{4}{81}v_{k+3} - \frac{7}{243}v_{k+4} - \cdots$

Baliarsingh [12] defined the spaces $V(\Gamma, \Delta^{\alpha}, u)$ for $V \in \{\ell_{\infty}, c, c_0\}$ using the fractional difference operator Δ^{α} and studied their topological properties and obtained their α, β, and γ duals.

The studies on generalized difference sequence spaces of fractional orders were extended by Baliarsingh and Dutta [10, 11], Dutta and Baliarsingh [18], Kadak and Baliarsingh [19], Baliarsingh and Kadak [13], Meng and Mei [14], Yaying and Hazarika [16], Yaying et. al [15], Nayak et. al [26], Kadak [21, 22], Furkan [28], Özger [29, 30] etc. They studied different sequence spaces of fractional orders. Kadak in [23] determined a new classes of fractional difference sequence spaces $\Delta_v^{\alpha}(V)$ as follows:

$$\Delta_u^{\alpha}(V) = \{v = (v_k) \in w : (\Delta_u^{\alpha}v)_k \in V\},$$

where $(\Delta_u^{\alpha}v)_k = \sum_{i=0}^{\infty} (-1)^i \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)} u_{k+i}v_{k+i}$ and $u = (u_k)$ is a sequence of positive real numbers. Using the fractional difference operator Δ_u^{α}, he defined strongly Cesàro summable and statistical difference sequence spaces of fractional orders involving lacunary sequence, θ and arbitrary sequence $p = (p_k)$ of positive real numbers.

Theorem 2.1. [12]

1. For a proper fraction α, $\Delta^{\alpha}: w \to w$ is a linear operator.
2. For proper fractions $\alpha, \beta > 0$, $\Delta^{\alpha}((\Delta^{\beta}v)_k) = (\Delta^{\alpha+\beta}v)_k$ and $\Delta^{\alpha}((\Delta^{-\alpha}v)_k) = v_k$.

The main objective of this article is to introduce generalized paranormed difference sequence spaces $N_{\theta}(\Delta^{\alpha}, f, \Lambda, p)$ and $S_{\theta}(\Delta^{\alpha}, f, \Lambda, p)$ of fractional orders involving lacunary sequence, θ; modulus function, f and multiplier sequence, Λ and to investigate topological structures of these spaces and examine various inclusion relations.

3. MAIN RESULTS

Throughout the paper, $p = (p_k)$ is a sequence of positive scalars. By using the fractional difference operator Δ^{α}, we introduce some new generalized difference sequence spaces $N_{\theta}^{0}(\Delta^{\alpha}, f, \Lambda, p)$, $N_{\theta}(\Delta^{\alpha}, f, \Lambda, p)$ and $N_{\theta}^{\infty}(\Delta^{\alpha}, f, \Lambda, p)$ involving lacunary sequence, θ; modulus function, f and multiplier sequence, Λ as follows:

$$N_{\theta}(\Delta^{\alpha}, f, \Lambda, p) = \Big\{v = (v_k) \in w : \lim_{r\to\infty} h_r^{-1} \sum_{k\in I_r} f(|\lambda_k(\Delta^{\alpha}v)_k - L|)^{p_k} = 0, \text{for some } L\Big\};$$

$$N_{\theta}^{0}(\Delta^{\alpha}, f, \Lambda, p) = \Big\{v = (v_k) \in w : \lim_{r\to\infty} h_r^{-1} \sum_{k\in I_r} f(|\lambda_k(\Delta^{\alpha}v)_k|)^{p_k} = 0\Big\};$$

$$N_{\theta}^{\infty}(\Delta^{\alpha}, f, \Lambda, p) = \Big\{v = (v_k) \in w : \lim_{r\to\infty} h_r^{-1} \sum_{k\in I_r} f(|\lambda_k(\Delta^{\alpha}v)_k|)^{p_k} < \infty\Big\};$$

Note that:

1. When $f(v) = v$, $\alpha = 0$, $\lambda_k = 1$, for all k and $p_k = 1$ for all k, then the above sequence spaces reduces to ordinary lacunary convergent sequence spaces as studied by Freedman et. Al [44].
2. When $\alpha = m \in \mathbb{N}$, $\lambda_k = 1$, for all k and $p_k = 1$ for all k, then the above sequence spaces reduces to $V(\Delta^m, f)$ where $V \in \{N_{\theta}^{0}, N_{\theta}, N_{\theta}^{\infty}\}$ as studied by Çolak [6].
3. When $\alpha = m \in \mathbb{N}$, and $\lambda_k = 1$, for all k, then the above sequence spaces reduces to the sequence spaces studied by Tripathy and Et [38].

4. When $\lambda_k = 1$ for all $k \in \mathbb{N}$, then the above class of sequence spaces reduce to $N_\theta^0(f,p)$, $N_\theta(f,p)$, $N_\theta^\infty(f,p)$ as studied by Yaying [27].

Theorem 3.1. The sequence spaces $N_\theta(\Delta^\alpha, f, \Lambda, p)$, $N_\theta^0(\Delta^\alpha, f, \Lambda, p)$ and $N_\theta^\infty(\Delta^\alpha, f, \Lambda, p)$ are linear spaces.

Proof. We shall prove for $N_\theta^0(\Delta^\alpha, f, \Lambda, p)$. Others can be proved in a similar fashion. Let $v, u \in N_\theta^0(\Delta^\alpha, f, \Lambda, p)$ and α' and β' be two scalars. Then there exist $M_{\alpha'} > 0$ and $K_{\beta'} > 0$ such that $|\alpha'| \le M_{\alpha'}$ and $|\beta'| \le K_{\beta'}$. Since f is subadditive and Δ^α is linear, we have

$$h_r^{-1} \sum_{k\in I_r} f(|\lambda_k(\Delta^\alpha(\alpha' v + \beta' u))_k|)^{p_k}$$

$$\le h_r^{-1} \sum_{k\in I_r} [f(|\alpha'|\lambda_k|(\Delta^\alpha v)_k|) + f(|\beta'||\lambda_k(\Delta^\alpha u)_k|)]^{p_k}$$

$$\le C(M_{\alpha'})^H h_r^{-1} \sum_{k\in I_r} f(|\lambda_k(\Delta^\alpha v)_k|)^{p_k} + C\left(K_{\beta'}\right)^H h_r^{-1} \sum_{k\in I_r} f(|\lambda_k(\Delta^\alpha u)_k|)^{p_k} \to 0$$

as $r \to \infty$. This proves the linearity of $N_\theta^0(\Delta^\alpha, f, \Lambda, p)$.

Theorem 3.2. $N_\theta^0(\Delta^\alpha, f, \Lambda, p)$ is a paranormed sequence space paranormed by

$$g(v) = \sup_r \left(h_r^{-1} \sum_{k\in I_r} f(|\lambda_k(\Delta^\alpha v)_k|)^{p_k} \right)^{1/M};$$

where $M = \max(1, \sup_k p_k)$.

Proof. Clearly $g(\theta) = 0$ and $g(v) = g(-v)$ for all $v \in N_\theta^0(\Delta^\alpha, f, \Lambda, p)$. Using the linearity of Δ^α, definition of f and Minkowski's inequality, it is not difficult to show that $g(v+u) \le g(v) + g(u)$, for any two sequences $v, u \in N_\theta^0(\Delta^\alpha, f, \Lambda, p)$.

It remains to show the continuity of the scalar multiplication. Let β be any scalar. By definition of modulus f, we have

$$g(\beta v) = \sup_r \left(h_r^{-1} \sum_{k\in I_r} f(|\lambda_k(\Delta^\alpha \beta v)_k|)^{p_k} \right)^{1/M} \le N_\beta^{H/M} g(v),$$

where N_β is a positive number such that $|\beta| \le N_\beta$ and $H = \sup p_k$.

Now, since f is modulus, we have $x \to 0$ implies $g(\beta v) \to 0$. Similarly, $v \to 0$ and $\beta \to 0$ implies $g(\beta v) \to 0$. Finally, keeping v fixed and letting $\beta \to 0$ implies $g(\beta v) \to 0$. This completes the proof.

Theorem 3.3. Let f be a modulus function, then $N_\theta^0(\Delta^\alpha, f, \Lambda, p) \subset N_\theta(\Delta^\alpha, f, \Lambda, p) \subset N_\theta^\infty(\Delta^\alpha, f, \Lambda, p)$.

Proof. The first inclusion is obvious. We provide the proof of the second inclusion.

Let $v \in N_\theta(\Delta^\alpha, f, \Lambda, p)$. By definition of f, we have,

$$h_r^{-1} \sum_{k\in I_r} f(|\lambda_k(\Delta^\alpha v)_k|)^{p_k}$$

$$= h_r^{-1} \sum_{k\in I_r} f(|\lambda_k(\Delta^\alpha v)_k - L + L|)^{p_k}$$

$$\le C h_r^{-1} \sum_{k\in I_r} f(|\lambda_k(\Delta^\alpha v)_k - L|)^{p_k} + C h_r^{-1} \sum_{k\in I_r} f(|L|)^{p_k}.$$

Now, there exist a positive integer K_L such that $|L| \le K_L$. Hence, we have,

$$h_r^{-1} \sum_{k\in I_r} f(|\lambda_k(\Delta^\alpha v)_k|)^{p_k}$$

$$\le C h_r^{-1} \sum_{k\in I_r} f(|\lambda_k(\Delta^\alpha v)_k - L|)^{p_k} + C(K_L f(1))^H.$$

This proves the result.

Theorem 3.4. If f, f_1, f_2 be modulus functions and $V \in \{N_\theta, N_\theta^0, N_\theta^\infty\}$, then

1. $V(\Delta^\alpha, f, \Lambda, p) \subset V(\Delta^\alpha, f \circ f_1, \Lambda, p)$.
2. $V(\Delta^\alpha, f_1, \Lambda, p) \cap V(\Delta^\alpha, f_2, \Lambda, p) \subset V(\Delta^\alpha, f_1 + f_2, \Lambda, p)$.

Proof. We shall prove for $N_\theta^0(\Delta^\alpha, f, \Lambda, p)$. Let $\varepsilon > 0$ and choose $0 < \delta < 1$ such that $f(t) < \varepsilon$ for $0 \le t \le \delta$. We write $u_k = f_1(|\lambda_k(\Delta^\alpha v)_k|)$ and consider

$$\sum_{k \in I_r} f(u_k)^{p_k} = \sum_1 f(u_k)^{p_k} + \sum_2 f(u_k)^{p_k}$$

where the first summation runs over $u_k \le \delta$ and the second summation runs over $u_k > \delta$. Since f is continuous, we have

$$\textstyle\sum_1 f(u_k)^{p_k} < h_r \varepsilon^H. \tag{3.1}$$

Also,

$$u_k < \frac{u_k}{\delta} \le 1 + \frac{u_k}{\delta}.$$

Hence, by using Proposition 1.1, we can write

$$h_r^{-1} \textstyle\sum_2 f(u_k)^{p_k} \le \max\{1, (2f(1)\delta^{-1})^H\}\, h_r^{-1} \sum_{k \in I_r} u_k. \tag{3.2}$$

Using equations (3.1) and (3.2), we get $N_\theta^0(\Delta^\alpha, f, \Lambda, p) \subset N_\theta^0(\Delta^\alpha, f \circ f_1, \Lambda, p)$.

The proof of (ii) follows from the inequality

$$(f_1 + f_2)(|\lambda_k(\Delta^\alpha v)_k|)^{p_k}$$

$$\le Cf_1(|\lambda_k(\Delta^\alpha v)_k|)^{p_k} + Cf_2(|\lambda_k(\Delta^\alpha v)_k|)^{p_k}.$$

The following result is an immediate consequence of Theorem 3.4 (i).

Corollary 3.5 Let f be a modulus function. Then $V(\Delta^\alpha, \Lambda, p) \subset V(\Delta^\alpha, f, \Lambda, p)$ where $V \in \{N_\theta, N_\theta^0, N_\theta^\infty\}$.

Theorem 3.6 *Let* $0 < p_k < q_k$ *and* $\left(\frac{q_k}{p_k}\right)$ *be bounded then* $V(\Delta^\alpha, f, \Lambda, q) \subset V(\Delta^\alpha, f, \Lambda, p)$.

Proof. The proof of the theorem is easy and hence omitted.

4. LACUNARY STATISTICAL CONVERGENCE OF FRACTIONAL ORDER DEFINED BY MODULUS FUNCTION

In this section we introduce the set of generalized lacunary statistical convergence of fractional orders associated with a multiplier sequence defined by a modulus function as follows:

$$S_\theta(\Delta^\alpha, f, \Lambda, p) = \Big\{v = (v_k) \in w : \lim_{r\to\infty} h_r^{-1}|\{k \in I_r : f(|\lambda_k(\Delta^\alpha v)_k - L|)^{p_k} \ge \varepsilon\}| = 0, \text{for some } L\Big\}.$$

When $p_k = 1$ for all $k \in \mathbb{N}$, we shall denote $S_\theta(\Delta^\alpha, f, \Lambda, p)$ by $S_\theta(\Delta^\alpha, f, \Lambda)$.

Note that:

1. When $f(v) = v, \alpha = 0, \lambda_k = 1$ for all $k \in \mathbb{N}$ and $p_k = 1$ for all $k \in \mathbb{N}$, then the above sequence spaces reduce to ordinary lacunary statistical convergent sequence spaces as studied by Fridy and Orhan [42].
2. When $\alpha = m \in \mathbb{N}$, $\lambda_k = 1$, for all $k \in \mathbb{N}$, then the above sequence spaces reduce to the sequence spaces as studied by Tripathy and Et [38].

Theorem 4.1. Let θ be a lacunary sequence. Then $S(\Delta^\alpha, f, \Lambda) \subset S_\theta(\Delta^\alpha, f, \Lambda)$, if $\liminf q_r > 1$.

Proof. Let $\liminf q_r > 1$, then there exist a $\delta > 0$ such that $1 + \delta \le q_r$, for sufficiently large r. Since $h_r = k_r - k_{r-1}$, which implies that $\frac{h_r}{k_r} \ge \frac{\delta}{1+\delta}$.

Let $v \in S(\Delta^\alpha, f, \Lambda)$. Then for $\varepsilon > 0$

$$\frac{1}{k_r}|\{k \le k_r : f(|\lambda_k(\Delta^\alpha v)_k - L|) \ge \varepsilon\}|$$

$$\ge \frac{1}{k_r}|\{k \in I_r : f(|\lambda_k(\Delta^\alpha v)_k - L|) \ge \varepsilon\}|$$

$$\ge \frac{\delta}{1+\delta} h_r^{-1}|\{k \in I_r : f(|\lambda_k(\Delta^\alpha v)_k - L|) \ge \varepsilon\}|.$$

This proves the result.

Theorem 4.2. Let θ be a lacunary sequence. Then $S_\theta(\Delta^\alpha, f, \Lambda) \subset S(\Delta^\alpha, f, \Lambda)$, if $limsup\, q_r < \infty$.

Proof. Let limsup $q_r < \infty$, then there is a $K > 0$ such that $q_r < K$, for all r. Let $v \in S_\theta(\Delta^\alpha, f, \Lambda)$ and let $\tau_r = |\{k \in I_r : f(|\lambda_k(\Delta^\alpha v)_k - L|) \geq \varepsilon\}|$.

Now by definition, for $\varepsilon > 0$ there is an integer r_0 such that

$$h_r^{-1}\tau_r < \varepsilon \text{ for all r} > \text{r}_0. \qquad (4.1)$$

Now let $\gamma = \max\{\tau_r : 1 \leq r \leq r_0\}$ and let n be any integer satisfying $k_{r-1} < n \leq k_r$, then we can write

$$\frac{1}{n}|\{k \leq n : f(|\lambda_k(\Delta^\alpha v)_k - L|) \geq \varepsilon\}|$$

$$\leq \frac{1}{k_{r-1}}|\{k \leq k_r : f(|\lambda_k(\Delta^\alpha v)_k - L|) \geq \varepsilon\}|$$

$$= \frac{1}{k_{r-1}}\{\tau_1 + \tau_2 + \cdots + \tau_{r_0} + \tau_{r_0+1} + \cdots + \tau_r\}$$

$$\leq \frac{\gamma}{k_{r-1}} r_0 + \frac{1}{k_{r-1}}\left\{h_{r_0+1}\frac{\tau_{r_0+1}}{h_{r_0+1}} + \cdots + h_r\frac{\tau_r}{h_r}\right\}$$

$$\leq \frac{\gamma}{k_{r-1}} r_0 + \frac{1}{k_{r-1}}\left(\sup_{r<r_0}\frac{\tau_r}{h_r}\right)\left(h_{r_0+1} + \cdots + h_r\right)$$

$$\leq \frac{\gamma}{k_{r-1}} r_0 + \varepsilon\frac{k_r - k_{r_0}}{k_{r-1}} \quad \text{(using equation (4.1)}$$

$$\leq \frac{\gamma}{k_{r-1}} r_0 + \varepsilon q_r$$

$$\leq \frac{\gamma}{k_{r-1}} r_0 + \varepsilon K.$$

This proves the result.

Following result is the direct consequence of theorems (4.1) and (4.2).

Corollary 4.3. Let θ be a lacunary sequence. Then $S(\Delta^\alpha, f, \Lambda) = S_\theta(\Delta^\alpha, f, \Lambda)$, if $1 < liminf\, q_r \leq limsup\, q_r < \infty$.

Theorem 4.4. Let f be a modulus function and $H = sup_k p_k$. *Then* $N_\theta(\Delta^\alpha, f, \Lambda, p) \subset S_\theta(\Delta^\alpha, \Lambda)$.

Proof. Let $v \in N_\theta(\Delta^\alpha, f, \Lambda, p)$ and $\varepsilon > 0$ be given. Then,

$$h_r^{-1}\sum_{k\in I_r} f(|\lambda_k(\Delta^\alpha v)_k - L|)^{p_k}$$

$$= h_r^{-1}\sum_{\substack{k\in I_r \\ |\lambda_k(\Delta^\alpha v)_k - L|\geq\varepsilon}} f(|\lambda_k(\Delta^\alpha v)_k - L|)^{p_k} + h_r^{-1}\sum_{\substack{k\in I_r \\ |\lambda_k(\Delta^\alpha v)_k - L|<\varepsilon}} f(|\lambda_k(\Delta^\alpha v)_k - L|)^{p_k}$$

$$\geq h_r^{-1}\sum_{\substack{k\in I_r \\ |\lambda_k(\Delta^\alpha v)_k - L|\geq\varepsilon}} f(|\lambda_k(\Delta^\alpha v)_k - L|)^{p_k}$$

$$\geq h_r^{-1}\sum_{k\in I_r} f(\varepsilon)^{p_k}$$

$$\geq h_r^{-1}\sum_{k\in I_r} \min\left(f(\varepsilon)^{\inf p_k}, f(\varepsilon)^H\right)$$

$$\geq h_r^{-1}|\{k \in I_r : |\lambda_k(\Delta^\alpha v)_k - L| \geq \varepsilon\}| \min\left(f(\varepsilon)^{\inf p_k}, f(\varepsilon)^H\right).$$

Taking the limit as $r \to \infty$,

$$\lim_{r\to\infty} h_r^{-1}|\{k \in I_r : |\lambda_k(\Delta^\alpha v)_k - L| \geq \varepsilon\}| \leq \frac{1}{\min(f(\varepsilon)^{\inf p_k}, f(\varepsilon)^H)}\lim_{r\to\infty} h_r^{-1}\sum_{k\in I_r} f(|\lambda_k(\Delta^\alpha v)_k - L|)^{p_k} = 0.$$

This proves the result.

Theorem 4.5. Let f be bounded and $0 < h = inf p_k \leq p_k \leq sup p_k = H < \infty$. Then $S_\theta(\Delta^\alpha, \Lambda) \subset N_\theta(\Delta^\alpha, f, \Lambda, p)$.

Proof. Since f is bounded, there exist some K such that $f(v) < K$ for all $v \geq 0$. Now,

$$h_r^{-1}\sum_{k\in I_r} f(|\lambda_k(\Delta^\alpha v)_k - L|)^{p_k}$$

$$= h_r^{-1} \sum_{\substack{k\in I_r \\ |\lambda_k(\Delta^\alpha v)_k - L|\geq\varepsilon}} f(|\lambda_k(\Delta^\alpha v)_k - L|)^{p_k} + h_r^{-1} \sum_{\substack{k\in I_r \\ |\lambda_k(\Delta^\alpha v)_k - L|<\varepsilon}} f(|\lambda_k(\Delta^\alpha v)_k - L|)^{p_k}$$

$$\leq h_r^{-1} \sum_{k\in I_r} \max(K_h, K_H) + h_r^{-1} \sum_{k\in I_r} f(\varepsilon)^{p_k}$$

$$\leq \max(K_h, K_H) h_r^{-1} |\{k \in I_r : |\lambda_k(\Delta^\alpha v)_k - L| \geq \varepsilon\}| + \max(f(\varepsilon)^h, f(\varepsilon)^H)$$

Hence $v \in N_\theta(\Delta^\alpha, f, \Lambda, p)$.

The following result is an immediate consequence of the Theorem 4.4 and Theorem 4.5.

Corollary 4.6. *Let* f *be bounded and* $0 < h = inf p_k \leq p_k \leq sup p_k = H < \infty$. *Then* $S_\theta(\Delta^\alpha, \Lambda) = N_\theta(\Delta^\alpha, f, \Lambda, p)$.

5. CONCLUSION

Fractional order difference sequence space has been an active field of research during the recent times. Many authors have introduced different classes of difference sequence spaces of fractional orders, obtained their α, β and γ duals and matrix transformations. Recently, Kadak [24] introduced the notions of statistically Ω-convergence and Ω-statistically convergence by the weighted method with respect to the fractional difference operator $\Delta_h^{\alpha,\beta,\gamma}$. In his another work, Kadak [25] introduced the concepts of statistically weighted $\psi_\Delta^{p,q}$-summability, weighted $\psi_\Delta^{p,q}$-statistical converegence and weighted strongly $\psi_\Delta^{p,q}$-summability with respect to a more generalized difference operator $\Delta_{h,p,q}^{\alpha,\beta,\gamma}$ including (p,q)-analogue of gamma function and obtained Korovkin type approximation theorems for function of two variables. In this article we tend to generalize the findings of the previous authors using modulus function and a multiplier sequence. We expect that the introduced notions and the results might be a reference for further studies in this field. For further studies one can investigate and generalize this results using sequence of modulus functions, Orlicz function, etc. One can obtain similar results by employing more generalized fractional difference operator as defined by Kadak in [24, 25].

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