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Weyl-Hamilton Equations on 3-Dimensional Normal Almost Paracontact Metric Manifold

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Abstract

Paracontact geometry is in many ways an odd-dimensional counterpart of symplectic geometry. Both paracontact and symplectic geometry are motivated by the mathematical formalism of classical, analytical and dynamical mechanics. A formulation of classical mechanics is Hamiltonian mechanics. The purpose of this paper is to study Weyl-Hamiltonian differential (move) equations using Weyl theorem for mechanical systems on 3-dimensional normal almost paracontact metric manifolds and is to get a general form for any movement of the object.

Keywords: *Symplectic geometry, paracontact manifold, Hamiltonian formalism, mechanical system, dynamic equation.*

Özet

Paracontact geometrisi birçok yönden semplektik geometrinin benzer boyutlu bir karşılığıdır. Hem paracontact hem de semplektik geometri, klasik, analitik ve dinamik mekaniğin matematiksel formülasyonu tarafından motive edilir. Klasik mekaniğin bir formülasyonu Hamilton mekaniğidir. Bu makalenin amacı, Weyl teoremini kullanarak 3 boyutlu normal hemen hemen paracontact metrik manifoldlar üzerinde Weyl teoremini kullanarak Weyl-Hamiltonian diferansiyel (hareket) denklemlerini incelemek ve nesnenin herhangi bir hareketi için genel bir form elde etmektir.

Anahtar Kelimeler: *Semplektik geometri, paracontact manifoldu, Hamilton formalizmi, mekanik sistem, dinamik denklem.*

1. Introduction

One way of solving problems in classical and analytical mechanics is through use of the Hamilton equations. The Hamiltonian formulation is an important tramplen from which to develop another useful formulation of classical mechanics known. Classical field theory utilizes traditionally the language of Hamiltonian dynamics. This theory was extended to time-dependent classical mechanics. A Hamiltonian space has been certified as an

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excellent model for some important problems in relativity, gauge theory and electromagnetism.
Symplectic and paracontact geometry are theories such that naturally emerged from the mathematical description of classical physics. They were revolutionized in the early 1980s with the discovery of new rigidity phenomena and properties satisfied by these geometric structures. They have been very useful in the development of many areas of mathematics and modern mathematical physics. Paracontact manifolds are the natural framework for geometric optics. Paracontact structures arise naturally on energy levels of autonomous Hamiltonian systems.
Tripathi et al submitted the concept of ε-almost paracontact manifolds of ε-para-Sasakian manifolds [1]. Kr. Srivastava et al introduced the concept of (ε)-almost paracontact manifolds [2]. Atceken introduced the existence of warped product semi-invariant submanifolds in almost para contact metric manifolds [3]. Shukla and Verma investigated the notion of paracomplex paracontact pseudo-Riemannian submersions from almost para-Hermitian manifolds onto almost paracontact metric manifolds [4]. Gunduzalp and Sahin first defined the concept of paracontact semi-Riemannian submersions between almost paracontact metric manifolds [5]. Erken and Murathan completed study of three-dimensional paracontact metric $(\tilde{\kappa},\tilde{\mu},\tilde{\upsilon})$-manifolds [6]. Manev and Staikova studied a classification with eleven basic classes of almost paracontact Riemannian manifolds of type (n,n) [7]. Bucki shown that for an almost r-paracontact manifold of P. Sasakian type there exists a product submanifold [8]. Acet et al gave canonical paracontact connection on a para-Sasakian manifold [9]. Ahmad et al defined a quarter symmetric semi-metric connection in an almost r-paracontact Riemannian manifold and consider invariant, non-invariant and anti-invariant hypersurfaces [10]. Nakova and Zamkovoy considered almost paracontact pseudo-Riemannian manifolds with indefinite metric g [11]. Kasap and Tekkoyun found Lagrangian and Hamiltonian formalism for mechanical systems using para/pseudo-Kähler manifolds [12].

2. Preliminaries

Definition 1. Let M be a differentiable manifold of dimension (2n+1) and suppose J is a differentiable vector bundle isomorphism $J:TM\to TM$ such that $J:TM\to TM$ is a almost complex structure for TM. An almost complex structure J on M assigns to each $p\in M$ a linear map $J_p:T_pM\to T_pM$ that is smooth in p and satisfies $J_p{}^2=Id$ for all p. The pair (M,J) is called an almost paracomplex manifold. Any paracomplex manifold M is also an almost paracomplex manifold.

A celebrated theorem of Newlander and Nirenberg [13] says that an almost paracomplex structure is a paracomplex structure if and only if its Nijenhuis tensor or torsion vanishes. The almost paracomplex structure J on M is integrable if and only if the tensor N_J vanishes identically, where N_J is defined on two vector fields X and Y by

$$N_J[X,Y]=[JX,JY]-J[X,JY]-J[JX,Y]-[X,Y]. \tag{1}$$

The tensor (2,1) is called the Nijenhuis tensor (1). We say that J is torsion free if $N_J=0$. An almost product structure is integrable if its Nijenhuis tensor vanishes. An almost complex manifold (M,J) is complex if and only if J is integrable.

Definition 2. A (2n+1)-dimensional manifold M is said to be a contact manifold if it admits a global 1-form η, such that $\eta\wedge(d\eta)^n\neq0$.

Given such a form η, there exists a unique vector field ξ, called the characteristic vector field, such that $\eta(\xi)=1$ and $d\eta(\xi,\ )=0$. A semi-Riemannian metric g is said to be an associated metric if there exists a tensor φ of type (1,1), such that

$$\varphi^2X=X-\eta(X)\xi,$$
$$\varphi\xi=0,\ \eta(\varphi X)=0,\ \eta(\xi)=1,$$
$$g(\varphi X,\varphi Y)=g(X,Y)-\eta(X)\eta(Y),$$
$$\eta(X)=g(X,\xi),$$
$$d\eta(\ ,\ )=g(\ ,\varphi). \tag{2}$$

Then, (φ,ξ,η,g) (more briefly, (η,g)) is called a paracontact metric structure, and (M,φ,ξ,η,g) or M a paracontact metric manifold [14].

Definition 3. Let M be an almost paracontact manifold with almost paracontact structure (φ,ξ,η,g) and consider the product manifold $M\times\mathbb{R}$, where $\mathbb{R}$ is the real line. A vector field on $M\times\mathbb{R}$ can be represented by (X,f(d/dt)), where X is tangent to M, f a smooth function on $M\times\mathbb{R}$, and t the coordinates of $\mathbb{R}$. For any two vector fields (X,f(d/dt)) and (Y,h(d/dt)) , it is easy to verify the following:

$$[(X,f(d/dt)),(Y,h(d/dt))]=([X,Y],(Xh-Yf)(d/dt)). \tag{3}$$

If the induced almost product structure J on $M\times\mathbb{R}$ defined by

$$J(X,f(d/dt))=(\varphi X+f\xi,\eta(X)(d/dt)), \tag{4}$$

is integrable, then we say that the almost paracontact structure (φ,ξ,η,g) is normal. Let M be an almost paracontact manifold and for any vector fields X,Y on M if it is additionally endowed with a pseudo-Riemann metric g of signature (n+1,n) and such that

$$g(\varphi X,\varphi Y)=-g(X,Y)+\eta(X)\eta(Y). \tag{5}$$

3. Almost Paracontact 3-Structure on A Differentiable Manifold

Theorem 1. Let (M,φ,ξ,η) bean almost paracontact manifold, and let $\mu\neq 0$ be (1,1) tensor field defined on M. If we put

$$\varphi' X=\mu^{-1}\varphi\mu X,\ \eta'(X)=\eta(\mu X) \text{ and } \mu\xi'=\xi\,, \tag{6}$$

then we observe that (φ′, ξ′, η′) is also an almost paracontact structure defined on M, thereby indicating that an almost paracontact structure on a differentiable manifold is not unique. This leads us to define an almost paracontact 3-structure on a differentiable manifold.

Proof: Suppose a differentiable manifold M admits three almost paracontact structures (φ_i,ξ_i,η_i), i=1,2, satisfying

$$\eta_i(\xi_j)=\eta_j(\xi_i)=0,$$
$$\varphi_i(\xi_j)=\varphi_j(\xi_i)=0,$$
$$\eta_i\circ\varphi_j=\eta_j\circ\varphi_i=\xi_K,$$

and

$$\varphi_i\circ\varphi_j+\eta_j\otimes\xi_i=\varphi_j\circ\varphi_i+\eta_i\otimes\xi_j=\varphi_K. \tag{7}$$

for a cyclic permutation (i,j,k) of (1,2,3), then M is said to have an almost paracontact 3-structure.

Theorem 2. If a differentiable manifold M admits two almost paracontact structures (φ_i,ξ_i,η_i), i=1,2, satisfying

$$\eta_1(\xi_2)=\eta_2(\xi_1)=0,$$

$$\varphi_1(\xi_2)=\varphi_2(\xi_1),$$

$$\eta_1\circ\varphi_2=\eta_2\circ\varphi_1,$$

and

$$\varphi_1\circ\varphi_2+\eta_2\otimes\xi_1=\varphi_2\circ\varphi_1+\eta_1\otimes\xi_2. \quad (8)$$

then it admits an almost paracontact 3-structure.

Proof: Define a triplet (φ_3,ξ_3,η_3) on M by

$$\varphi_3=\varphi_1\circ\varphi_2+\eta_2\otimes\xi_1, \xi_3=\varphi_1(\xi_2) \text{ and } \eta_3=\eta_1\circ\varphi_2. \quad (9)$$

We can be easy shown that (φ_3,ξ_3,η_3) is also an almost paracontact 3-structure on M.

Theorem 3. Suppose a differentiable manifold M admits two almost paracontact structures (φ_i,ξ_i,η_i), i=1,2, and let there be given a Riemannian metric on M associated to both the structures and if

$$\varphi_1\circ\varphi_2+\eta_2\otimes\xi_1=\varphi_2\circ\varphi_1+\eta_1\otimes\xi_2, \quad (10)$$

then

(a) $\eta_1(\xi_2)=\eta_2(\xi_1)=0,$

(b) $\varphi_1(\xi_2)=\varphi_2(\xi_1),$

(c) $\eta_1\circ\varphi_2=\eta_2\circ\varphi_1,$ (11)

Proof: Since g is associated metric for the structure we have

$$g(\xi_1,\xi_2)=\eta_1(\xi_2)=\eta_2(\xi_1). \quad (12)$$

Using the given condition, we have

$$g(\varphi_1\varphi_2X+\eta_2(X)\xi_1,Y) = g(\varphi_2\varphi_1X,Y)+\eta_1(X)\eta_2(Y),$$

$$g(\varphi_2X,\varphi_1(Y))+\eta_2(X)\eta_1(Y) = g(\varphi_1X,\varphi_2Y)+\eta_1(X)\eta_2(Y). \quad (13)$$

Put $X=\xi_1$ and $Y=\xi_2$ in the above equation and using (12) we obtained

$$g(\varphi_2\xi_1,\varphi_1(\xi_2))=g(\xi_1,\xi_1)-g(\xi_1,\xi_2)\eta_1(\xi_2)=g(\xi_1,\xi_1-\eta_1(\xi_2)\xi_2)$$

$$=g(\xi_1,\xi_1-\eta_2(\xi_1)\xi_2)=g(\xi_1,\varphi_2{}^2\xi_1)=g(\varphi_2\xi_1,\varphi_2\xi_1,). \quad (14)$$

This gives $\varphi_1\xi_2=\varphi_2\xi_1$. Using (14) we get

$$\varphi_1(\xi_2)=\varphi_2(\xi_1)=\varphi_2(\xi_1)-\eta_1(\varphi_1\xi_2)\xi_1=\varphi_2(\xi_1)-\eta_1(\varphi_2\xi_1)\xi_1=\varphi_1{}^2(\varphi_2(\xi_1)), \quad (15)$$

which gives $\xi_2=\varphi_1\varphi_2\xi_1=\varphi_1\varphi_1\xi_2=\xi_2-\eta_1(\xi_2)\xi_1$. Hence $\eta_1(\xi_2)\xi_1=0$, giving $\eta_1(\xi_2)=0$ and by (12) we have $\eta_1(\xi_2)=\eta_2(\xi_1)=0$ [15].

Theorem 4. For an almost paracontact 3-structures (φ_i,ξ_i,η_i), i=1,2,3, on a differentiable manifold M there exist a Riemannian metric g such that

$$g(X,\xi_i)=\eta_i(X),\ i=1,2,3,\ X\in\chi(M). \tag{16}$$

Proof: Let g_1 be the associated Riemannian metric to (φ_1,ξ_1,η_1) and define a metric g_2 by

$$g_2(X,Y)=g_1(X-\eta_2(X)\xi_2,Y-\eta_2(Y)\xi_2)+\eta_2(X)\eta_2(Y). \tag{17}$$

Now define g by

$$g(X,Y)=g_2(X-\eta_3(X)\xi_3,Y-\eta_3(Y)\xi_3)+\eta_3(X)\eta_3(Y). \tag{18}$$

Then clearly g is Riemannian metric defined on M, and we have

$$\begin{aligned} g(X,\xi_1)&= g_2(X-\eta_3(X)\xi_3,\xi_1)=g_1(X-\eta_3(X)\xi_3-\eta_2(X-\eta_3(X)\xi_3)\xi_2,\xi_1) \\ &= g_1(X,\xi_1)-\eta_3(X)g_1(\xi_3,\xi_1)-\eta_2(X)g_1(\xi_2,\xi_1)=\eta_1(X). \end{aligned} \tag{19}$$

Further we have

$$g(X,\xi_2)=g_2(X-\eta_3(X)\xi_3,\xi_2)=\eta_2(X). \tag{20}$$

and

$$g(X,\xi_3)=g_2(X-\eta_3(X)\xi_3,\xi_3)=\eta_3(X). \tag{21}$$

Lemma 1. In differentiable manifold M with almost paracontact 3-structures (φ_i,ξ_i,η_i), i=1,2,3, and associated metric g we have

$$g(\varphi_i X,\varphi_j Y)=g(\varphi_k X,Y)-\eta_i(X)\eta_j(Y) \quad \text{(proof see[16]).} \tag{22}$$

4. Gauge Theory and Weyl Geometry

A conformal manifold is a differentiable manifold equipped with an equivalence class of (pseudo) Riemann metric tensors, in which two metrics g_2 and g_1 are equivalent if and only if

$$g_2=\Psi^2 g_1, \tag{23}$$

where $\Psi>0$ is a smooth positive function. An equivalence class of such metrics is known as a conformal metric or conformal class and a manifold with a conformal structure (23) is called a conformal manifold.

Hermann Weyl (1885-1955) made many fundamental and important contributions to physics. Weyl's gauge theory sprang from an even earlier (1918) theory in which Weyl demanded that Einstein's theory of general relativity should be invariant with respect to the similar replacement

$$g_{\mu\nu}(x)\rightarrow e^{\lambda(x)}g_{\mu\nu}(x), \tag{24}$$

which we shall call a metric gauge transformation (24) and it has emerged effect of these transformations on Riemannian and non-Riemannian geometry. Weyl, using this gauge principle, was able to derive all of electrodynamics from a generalized Einstein-Maxwell Lagrangian. Today, the gauge principle is arguably the most powerful concept in all of modern physics. This gauge principle underlies all of the Yang-Mills theories and is a key component in string theory and its more recent variant, M theory.

Two Riemann metrics g_1 and g_2 on M are said to be conformally equivalent iff there exists a smooth function f:M→ℝ with

$$e^f g_1=g_2. \quad (25)$$

In this case, $g_1 \sim g_2$. Let M be an n-dimensional smooth manifold. A pair (M,G), a conformal structure on M is an equivalence class G of Riemann metrics on M, is called a conformal structure [17].

Theorem 5. **(a)** Let ∇ be a connection on M and g∈G a fixed metric. ∇ is compatible with (M,G)⇔ there exists a 1-form ω with $\nabla_X g+\omega(X)g=0$. A compatible torsion-free connection is called a Weyl connection. The triple (M,G,∇) is a Weyl structure.

(b) To each metric g∈G and 1-form ω, there corresponds a unique Weyl connection ∇ satisfying $\nabla_X g+\omega(X)g=0$. Define a function F:{ 1-forms on M }×G→{Weyl connections} by F(g,ω)=∇, where ∇ is the connection guaranteed by Theorem 5. We say that ∇ corresponds to (g,ω) (poof see [16]).

Proposition 1.

(a) F is surjective.

Proof: F is surjective by Theorem 5.

(b) $F(g,\omega)=F(e^f g,\eta)$ iff η=ω-df. So

$$F(e^f g)=F(g)\text{-}df. \quad (26)$$

Where G is a conformal structure. Note that a Riemann metric g and a one-form ω determine a Weyl structure, namely $F:G\to\Lambda^1 M$ where G is the equivalence class of g and $F(e^f g)=\omega-df$.

Proof: Suppose $F(g,\omega)=F(e^f g,\eta)=\nabla$. We have

$$\nabla_X(e^f g)+\eta(X)e^f g=X(e^f)g+e^f\nabla_X g+\eta(X)e^f g=df(X)e^f g+e^f\nabla_X g+\eta(X)e^f g=0. \quad (27)$$

Therefore $\nabla_X(e^f g)=-(df(X)+\eta(X))$. On the other hand $\nabla_X g+\omega(X)g=0$. Therefore ω=η+df. Set ∇=F(g,ω). To show $\nabla=F(e^f g,\eta)$ and $\nabla_X(e^f g)+\eta(X)e^f g=0$. To calculate

$$\nabla_X(e^f g)+\eta(X)e^f g=e^f df(X)g+e^f\nabla_X g+(\omega(X)-df(X))e^f g=e^f(\nabla_X g+\omega(X)g)=0 \text{ [17]}. \quad (28)$$

Definition 4. Kähler geometry can be thought of as a compatible intersection of complex and symplectic geometries. Indeed, the triple (M^{2n},J,ω), with 2n the real dimension of M, is a Kähler manifold if

(i) (M^{2n},J) is a complex manifold, i.e. the automorphism J:TM→TM, $J^2=-I$, is an integrable complex structure

(ii) (M^{2n},ω), is a symplectic manifold, i.e. the 2-form ω is closed and nondegenerate

(iii) J and ω are compatible in the sense that the bilinear form ω(·,J·) is a Riemannian metric, i.e. symmetric and positive definite [18].

Definition 5. Consider a triple (M,g,∇) where g is a pseudo Riemannian metric on a smooth n dimensional manifold M and where ∇ is a torsion free connection on the tangent bundle TM of M. We suppose n≥2 henceforth. We say that (M,g,∇) is a Weyl manifold if the following identity is satisfied: $\nabla g=-2\varphi\otimes g$ for some $\varphi\in\mathbb{C}^\infty(T^*M)$. This notion is conformally invariant. If (M,g,∇) is a Weyl manifold, then $(M,e^{2f}g,\nabla)$ is again a Weyl manifold where φ:=φ-df. The simultaneous transformation of the pair (g,φ) is called a gauge transformation, properties of the Weyl geometry that are invariant under gauge transformations are called gauge invariants [19].

Let ∇ be a torsion free connection on the tangent bundle of M and m≥6. If (M,g,∇,J) is a Kähler-Weyl structure, then the associated Weyl structure is trivial, i.e. there is a conformally equivalent metric

$$g_1=e^{2f}g, \tag{29}$$

so that (M,g_1,J) is Kähler and so that $\nabla=\nabla^{g_1}$ [20].

Let (M,g) is conformally flat if for each point x in M, there exists a neighborhood U of x and a smooth function f defined on U such that $(U,e^{2f}g)$ is flat. The function f need not be defined on all of M.

Let m≥6. If (M,g,J,∇) is a (para)-Kähler-Weyl structure, then the associated Weyl structure is trivial, i.e. there is a conformally equivalent metric

$$g_1=e^{2f}g, \tag{30}$$

so that (M,g_1,J) is (para)-Kähler and so that $\nabla=\nabla^{g_1}$ [21].

Weyl transformation is a local rescaling of the metric tensor: $g_{ab}(x)\rightarrow e^{-2\omega(x)}g_{ab}(x)$ which produces another metric in the same conformal class. A theory or an expression invariant under this transformation is called conformally invariant, or is said to possess Weyl symmetry. The Weyl symmetry is an important symmetry in conformal field theory.

Also, in three dimensions, the vector from the origin to the point with Cartesian coordinates (x,y,z) can be written as:

$$r=x\vec{i}+y\vec{j}+z\vec{k}=x(\partial/(\partial x))+y(\partial/(\partial y))+z(\partial/(\partial z)).$$

Example 1. Let $\mathbb{R}^3$ be the Cartesian space and (x,y,z) be the Cartesian coordinates in it define the standard almost paracontact structure (φ,ξ,η) on $\mathbb{R}^3$ by

$$\varphi(\partial_1)=\partial_2\text{-}2x\partial_3,$$

$$\varphi(\partial_2)=\partial_1,$$

$$\varphi(\partial_3)=0,$$

$$\xi=\partial_3,\quad \eta=2xdy+dz. \tag{31}$$

We research "almost paracontact 3-structure on a differentiable manifold" conditions and features of (31) for $\partial_1=\partial/\partial x$, $\partial_2=\partial/\partial y$, $\partial_3=\partial/\partial z$;

1. $\varphi\xi=\varphi(\partial/\partial z)=0$,
2. $\eta(\varphi X)=[2x(\partial/\partial y)+\partial/\partial z](\varphi X)=0$,
3. $\eta(\xi)=[2x(\partial/\partial y)+\partial/\partial z](\partial/\partial z)=1$ [22]. (32)

Proposition 2. φ* homomorphic structure is the dual of φ homomorphic structure. We, using (29), transferred to Weyl structure of the system (31).

1. $\varphi^{*}dx)=e^{2f}dy\text{-}2xe^{2f}dz$,
2. $\varphi^{*}(dy)=e^{-2f}dx$,
3. $\varphi^{*}(dz)=0$. (33)

Proof: 1. $\varphi^{*2}(dx)=e^{2f}\varphi^{*}(dy)-2xe^{2f}\varphi^{*}(dz)=dx$,

2. $\varphi^{*2}(dy)=e^{-2f}(e^{2f}dy-2xe^{2f}dz)=dy-2xdz$,

3. $\varphi^{*2}(dz)=0$. (34)

As seen above (33) provided for the condition (2) $\varphi^{*2}X=X-\eta(X)\xi$.

5. Hamilton Dynamics Equations

The vector field X on $T^{*}M$ given by $i_X\omega=dH$ is called the geodesic flow of the metric g. Suppose that ξ is a vector field: that is, a vector-valued function with Cartesian coordinates $(\xi_1,...,\xi_n)$; and x(t) a parametric curve with Cartesian coordinates $(x_1(t),...,x_n(t))$. Then x(t) is an integral curve of ξ if it is a solution of the following autonomous system of ordinary differential equations:

$dx_1/dt=\xi_1(x_1,...,x_n),...,dx_n/dt=\xi_n(x_1,...,x_n)$. Such a system may be written as a single vector equation:

$$\xi(x(t))=x'(t)=(\partial/\partial t)(x(t)). \quad (35)$$

Let M is the configuration manifold and its cotangent manifold $T^{*}M$. By a symplectic form we mean a 2-form Φ on $T^{*}M$ such that:

(i) Φ is closed , that is, $d\Phi=0$; (ii) for each $z\in T^{*}M$, $\Phi:T^{*}M\times T^{*}M\to\mathbb{R}$ is weakly nondegenerate. If Φ_z in (ii) is nondegenerate, we speak of a strong symplectic form. If (ii) is dropped we refer to Φ as a presymplectic form. Let $(T^{*}M,\Phi)$ be a symplectic manifold. A vector field $X_H:T^{*}M\to T^{*}M$ is called Hamiltonian if there is a C^1 function $H:T^{*}M\to\mathbb{R}$ such that dynamical equation is determined by

$$i_{X_H}\Phi=dH \quad [23]. \quad (36)$$

We say that X_H is locally Hamiltonian vector field if $i_{X_H}\Phi$ is closed and where Φ shows the canonical symplectic form so that $\Phi=-d\Omega$, $\Omega=J^{*}(\omega)$, J^{*} a dual of J, ω a 1-form on $T^{*}M$. The trio $(T^{*}M,\Phi,X_H)$ is named Hamiltonian system which it is defined on the cotangent bundle $T^{*}M$. Recall from elementary physics that momentum of a particle, p_i, is defined in terms of its velocity $\dot{q}_i$ by $p_i=m_i\dot{q}_i$. In fact, the more general definition of conjugate momentum, valid for any set of coordinates, is given in terms of the Lagrangian: $p_i=\partial L/\partial\dot{q}_i$, $\dot{p}_i=\partial L/\partial q_i$. Note that these two definitions are equivalent for Cartesian variables. In terms of Cartesian momenta, the kinetic energy is given by $T=\sum_{i=1}^{n}(p_i^2)/(2m_i)$. Then, the Hamiltonian, which is defined to be the sum, H=T+V, expressed as a function of positions and momenta, will be given by $H(q_i,p_i)=V+T=\sum_{i=1}^{n}(p_i^2)/(2m_i)+(q_1,...,q_n)$ where $p=p_1,...,p_n$. The function H is equal to the total energy of the system. In terms of the Hamiltonian, the equations of motion of a system are given by Hamilton's equations:

$$\dot{q}_i=\partial H/\partial p_i,\ \dot{p}_i=-\partial H/\partial q_i \quad [24]. \quad (37)$$

6. An Example for Contact Manifold

Differential geometry provides and mathematical physics a good workspace for studying Hamiltonians of classical mechanics and field theory. The dynamic equations for moving bodies are obtained for Hamiltonian mechanics. Contact geometry has a practical usage in physics, geometrical optics, classical mechanics, analytical mechanics, mechanical systems, thermodynamics, geometric quantization, applied mathematics and differential geometry. The some examples of the Hamiltonian are applied to model the problems include harmonic oscillator, charge Q in electromagnetic fields, Kepler problem of the earth in orbit around the sun, rotating/spherical/plane pendulum, molecular and fluid dynamics, LC networks, Atwood's machine, symmetric top etc. In this section, an oscillator is given as an example for contact Hamiltonian system.

Lemma 2. Assume that H(t,q,p,S) is a C^1 contact Hamiltonian function and (q,p,S) are the coordinates of a point of a contact manifold with the one form η=dS-p(dq). Let λ be the number by which we must multiply to η obtain the given point of the symplectified space. In these coordinates, we have ω=λdS-λp(dq).

A Hamiltonian system with Hamiltonian function on a contact manifold:

$$\partial q/\partial t=\partial H(t,q,p)/\partial p,$$

$$\partial p/\partial t=-\partial H(t,q,p)/\partial q, \quad (38)$$

describes reversible systems such as in mechanics and electromagnetism, where dissipation effects are neglected. Hamiltonian system (38) extends to the contact Hamiltonian system on contact manifold with a contact Hamiltonian function $H:\mathbb{R}\times T^{*}\mathbb{R}^{n}\times\mathbb{R}\to\mathbb{R}$ defined by

$$\partial q/\partial t=\partial H(t,q,p,S)/\partial p,$$

$$\partial p/\partial t=-\partial H(t,q,p,S)/\partial q-p.(\partial H(t,q,p,S)/\partial S),$$

$$\partial S/\partial t=p.(\partial H(t,q,p,S)/\partial p)-H(t,q,p,S). \quad (39)$$

In the coordinates (q,p,S) of phase space, the contact form is $\omega = dS - \sum_{i=1}^{n} p_i.(dq_i)$ (proof see [25]).

Time-dependent Harmonic oscillators:

Cha at al introduced the damped harmonic oscillator with a time-dependent damping constant and a time dependent angular frequency, which is the generalization of Caldirola-Kanai Hamiltonian. Then, the equation of motion reads $\ddot{x}$ +γ(t) $\dot{x}$ $+\omega_0^2(t)x=0$,where γ(t) and $\omega_0(t)$ imply a time-dependent damping constant and a time-dependent angular frequency respectively, and the mass of a particle is determined in unity. The corresponding Lagrangian then reads $L=(1/2)g(t)(\dot{x}^2-\omega_0^2t^2)$, where $g(t)=\exp[\int^t\gamma(t')dt']$. The canonical momentum is given by $p=g(t).\dot{x}$. The corresponding Hamiltonian is obtained via a Legendre transformation: $H=p^2/2g(t)+(1/2).g(t)\omega_0^2x^2$ [26].

Consider the one-dimensional damped oscillator with changing-sign damping coefficient γ(t), mass g(t) and time-dependent frequency ω(t), whose contact Hamiltonian is

$$H(t,q,p,S)=p^2/2g(t)+(1/2).g(t)\omega^2(t)q^2+\gamma(t)S. \quad (40)$$

The contact Hamiltonian system of motions reads

$$\partial q/\partial t=p/g(t),$$

$$\partial p/\partial t=-g(t)\omega^2(t)q-p\gamma(t),$$

$$\partial S/\partial t=p^2/2g(t)-(1/2).g(t)\omega^2(t)q^2-\gamma(t)S. \quad (41)$$

7. Hamiltonian Mechanical Systems

We, using (36), present Hamilton equations and Hamiltonian mechanical systems for quantum and classical mechanics constructed on 3-dimensional normal almost-paracontact metric manifolds (φ^*,ξ,η,g).

Proposition 3. Let (M,g,φ^*) be the for (33) on 3-dimensional normal almost paracontact metric manifolds. Suppose that the structures, a Liouville form and a 1-form on 3-dimensional normal almost-paracontact metric manifolds are shown by $\phi^*,\Phi=-d\Omega$, $\Omega=\phi^*(\omega)$ and $\omega=(1/2)[dx+dz]$ is a 1-form. If (36) is used, the Hamilton equations below are obtained.

dif1. $dx/dt=-(C/(2\cdot A\cdot B))\cdot(\partial H/\partial x)+(1/A)(\partial H/\partial y)+(1/(2\cdot B))\cdot(\partial H/\partial z)$,

dif2. $dy/dt=(1/A)(\partial H/\partial x)-(B/(2\cdot C.A))\cdot(\partial H/\partial y)+(1/(2\cdot C))\cdot(\partial H/\partial z)$,

dif3. $dz/dt=((-1)/(2\cdot B))\cdot(\partial H/\partial x)-(1/(2\cdot C))\cdot(\partial H/\partial y)+(A/(2\cdot B\cdot C))\cdot(\partial H/\partial z))$.

(42)

The equations introduced in are named Weyl-Hamilton equations on 3-dimensional normal almost-paracontact metric manifolds (ϕ^*,ξ,η,g) and then the triple (M,Φ,ω) is said to be a Hamiltonian mechanical system on (M,g,ϕ^*).

Proof: The steps in the Hamilton dynamic equation section will be followed for proof. We obtain the Liouville form as follows:

$$\Omega = \varphi^*(\omega) = (1/2)[\varphi^*(dx)+\varphi^*(dz)]= (1/2)[e^{2f}dy-2xe^{2f}dz]. \quad (43)$$

It is well known that if Φ is a closed on 3-dimensional normal almost-paracontact metric manifolds (M,g,φ^*), then Φ is also a symplectic structure on (M,g,φ^*). Therefore the 2-form Φ indicates the canonical symplectic form and derived from the 1-form Ω to find to mechanical equations. Then the 2-form is calculated as:

$$\Phi=[-(\partial f/\partial x)e^{2f}dx\wedge dy+(e^{2f}+2x(\partial f/\partial x)e^{2f})dx\wedge dz$$
$$+2x(\partial f/\partial y)e^{2f}dy\wedge dz-(\partial f/\partial z)e^{2f}dz\wedge dy]. \quad (44)$$

Take a vector field X_H so that called to be Hamiltonian vector field associated with Hamiltonian energy H and determined by

$$X_H=X(\partial/\partial x)+Y(\partial/\partial y)+Z(\partial/\partial z). \quad (45)$$

So, we have

$$i_{X_H}\varphi = \varphi(X_H)$$
$$= X\Big\{ [-(\partial f/\partial x)e^{2f}](dx(\partial/\partial x)dy-dy(\partial/\partial x)dx)$$
$$+[e^{2f}+2x(\partial f/\partial x)e^{2f}](dx(\partial/\partial x)dz-dz(\partial/\partial x)dx)+[2x(\partial f/\partial y)e^{2f}$$
$$+(\partial f/\partial z)e^{2f}](dy(\partial/\partial x)dz-dz(\partial/\partial x)dy)] \Big\}$$

$$+Y\Big\{ [-(\partial f/\partial x)e^{2f}](dy(\partial/\partial y)dx-dx(\partial/\partial y)dy)$$

$$+(e^{2f}+2x(\partial f/\partial x)e^{2f})(dx(\partial/\partial y)dz-dz(\partial(\partial y)dx)$$

$$+[2x(\partial f/\partial y)e^{2f}+(\partial f/\partial z)e^{2f}](dy(\partial/\partial y)dz-dz(\partial/\partial y)dy)]\Big\}$$

$$+Z\Big\{ [-(\partial f/\partial x)e^{2f}](dy(\partial\partial z)dx-dx(\partial/\partial z)dy)$$

$$+[e^{2f}+2x(\partial f/\partial x)e^{2f}](dx(\partial/\partial z)dz-dz(\partial/\partial z)dx)$$

$$+[2x((\partial f/\partial y)e^{2f}+(\partial f/\partial z)e^{2f}](dy(\partial/\partial z)dz-dz(\partial/\partial z)dy)]\Big\}. \tag{46}$$

Kronecker delta and external product characteristics are used here to obtain the following equation.

$$i_{X_H}\varphi=\varphi(X_H)$$

$$=X[-(\partial f/\partial x)e^{2f}]dy+X[e^{2f}+2x(\partial f/\partial x)e^{2f}]dz$$

$$+Y[-(\partial f/\partial x)e^{2f}]dx+Y[2x(\partial f/\partial y)e^{2f}+(\partial f/\partial z)e^{2f}]dz$$

$$-Z[e^{2f}+x2(\partial f/\partial x)e^{2f}]dx-Z[2x(\partial f/\partial y)e^{2f}+(\partial f/\partial z)e^{2f}]dy. \tag{47}$$

Also, the differential of Hamiltonian energy H is obtained by

$$dH=(\partial H/\partial x)dx+(\partial H/\partial y)dy+(\partial H/\partial z)dz. \tag{48}$$

From $i_{X_H}\Phi=dH$, the Hamiltonian vector field is found

$$Y[-(\partial f/\partial x)e^{2f}]dx-Z[e^{2f}+x2(\partial f/\partial x)e^{2f}]dx = (\partial H/\partial x)dx,$$

$$X[-(\partial f/\partial x)e^{2f}]dy-Z[2x(\partial f/\partial y)e^{2f}+(\partial f/\partial z)e^{2f}]dy = (\partial H/\partial y)dy,$$

$$X[e^{2f}+x2(\partial f/\partial x)e^{2f}]dz+Y[2x(\partial f/\partial y)e^{2f}+(\partial f/\partial z)e^{2f}]dz = (\partial H/\partial z)dz, \tag{49}$$

and for $A=[-(\partial f/\partial x)e^{2f}]$, $B=[e^{2f}+x2(\partial f/\partial x)e^{2f}]$, $C=[2x(\partial f/\partial y)e^{2f}+(\partial f/\partial z)e^{2f}]$ as follows:

$$X = (1/A)(\partial H/\partial y)-(A/(2\cdot B\cdot))[C\cdot(\partial H/\partial x)-A\cdot(\partial H/\partial z)+B\cdot(\partial H/\partial y)],$$

$$Y = (1/A)(\partial H/\partial x)-(1/(2\cdot A\cdot C))[C\cdot(\partial H/\partial x)-A\cdot(\partial H/\partial z)+B\cdot(\partial H/\partial y)],$$

$$Z = ((-1)/(2\cdot B\cdot C))[C\cdot(\partial H/\partial x)-A\cdot(\partial H/\partial z)+B\cdot(\partial H/\partial y)]. \tag{50}$$

So, we obtain

$$X_H = [(1/A)(\partial H/\partial y)-(A/(2\cdot B))[C\cdot(\partial H/\partial x)-A\cdot(\partial H/\partial z)+B\cdot(\partial H/\partial y)]](\partial/\partial x)$$

$$+[(1/A)(\partial H/\partial x)-(1/(2\cdot A\cdot C))[C\cdot(\partial H/\partial x)-A\cdot(\partial H/\partial z)+B\cdot(\partial H/\partial y)]](\partial/\partial y)$$

$$+[((-1)/(2\cdot B\cdot C))[C\cdot(\partial H/\partial x)-A\cdot(\partial H/\partial z)+B\cdot(\partial H/\partial y)]](\partial/\partial z). \tag{51}$$

Consider the curve and its velocity vector is

$\alpha : I\subset\mathbb{R}\to M,\ \alpha(t)=(x(t),y(t),z(t)),$

$$\dot{\alpha}(t) = (dx/dt)(\partial/\partial x)+(dy/dt)(\partial/\partial y)+(dz/dt)(\partial/\partial z). \tag{52}$$

An integral curve (35) of the Hamiltonian vector field X_H, i.e., $X_H(\alpha(t))=(\partial/\partial t)(\alpha(t))=\dot{\alpha}(t)$, t∈I, t shows the time. Then, we can be find the following equations;

dif1. $dx/dt=-(C/(2\cdot A\cdot B))\cdot(\partial H/\partial x)+(1/A)(\partial H/\partial y)+(1/(2\cdot B))\cdot(\partial H/\partial z)$,

dif2. $dy/dt=(1/A)(\partial H/\partial x)-(B/(2\cdot C.A))\cdot(\partial H/\partial y)+(1/(2\cdot C))\cdot(\partial H/\partial z)$,

dif3. $dz/dt=((-1)/(2\cdot B))\cdot(\partial H/\partial x)-(1/(2\cdot C))\cdot(\partial H/\partial y)+(A/(2\cdot B\cdot C))\cdot(\partial H/\partial z))$.

(53)

Hence, the equations introduced in are named Hamilton equations on 3-dimensional normal almost-paracontact metric manifolds (φ^*,ξ,η,g) and then the triple (M,Φ,ω) is said to be a Hamiltonian mechanical system on (M,g,φ^*).

8. Conclusion

By this study the above mentioned forms:

(a) Weyl's structure (29) on 3-dimensional normal almost-paracontact metric manifolds (ϕ^*,ξ,η,g) were transferred (33) the mechanical system for dynamical systems.

(b) Hamiltonian motion equations on 3-dimensional normal almost-paracontact metric manifolds were found using the dynamic equation (36) introduced by Klein in 1962.

(c) So, the Hamilton mechanical equations (42) with Weyl theorem (29) derived on a generalized on 3-dimensional normal almost-paracontact metric manifolds.

9. Discussion

A classical field theory explain the study of how one or more physical fields interact with matter which is used in quantum and classical mechanics. Our universe is three-dimensional such that Einstein added time as the fourth dimension in 1905.

Time-dependent moving Hamiltonian equations gives a model for both the gravitational and electromagnetic field in a very natural blending of the geometrical structures of the space with the characteristic properties of these physical fields.

The obtained time-dependent equations system (42) on 3-dimensional normal almost-paracontact metric manifolds are very important to explain and solve the rotational spatial mechanical-physical problems.

Hamilton's equation system (42) may be suggested to deal with problems in electrical, magnetically and gravitational fields force of defined space moving objects [27-29].

In addition, using these equations, the route and needs of moving of the object/system on 3-dimensional normal almost-paracontact metric manifolds can be determined.

10. Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

11. References

1. Tripathi M.M., Kilic E., Perktas S.Y., Keles S. Indefinite almost paracontact metric manifolds, International Journal of Mathematics and Mathematical Sciences, 2010; 1-19.
2. Srivastava, D. Narain, Srivastava K. Properties of ε-S paracontact panifold, VSRD-TNTJ, 2011; 2(11): 559-569.
3. Atceken M. Warped product semi-invariant submanifolds in almost paracontact metric manifolds, Mathematica Moravica, 2010; (14)1: 15-21.
4. Shukla S.S., Verma U.S. Paracomplex paracontact pseudo-Riemannian submersions, Hindawi Publishing Corporation Geometry, 2014; 1-12.
5. Gunduzalp Y., Sahin B. Paracontact semi-Riemannian Ssubmersions, TJM, 2013; 37: 114-128.
6. Erken K., Murathan C. A complete study of three-dimensional paracontact (κ,μ,ν) Spaces, arxiv.org/abs/1305.1511v3, 1-26, 2013.
7. Manev M., Staikova M. On almost paracontact Riemannian manifolds of type (n,n), Journal of Geometry, 2001.
8. Bucki A. Product submanifold almost r-paracontact Riemannian manifold of p-Sasakian type, Soochov Journal of Mathematics, 1998; 24(4): 255-259.
9. Acet B.E., Kilic E., Perktas S.Y. Some curvature conditions on a para-Sasakian manifold with canonical paracontact ponnection, IJMMS, 2012; 1-24.
10. Ahmad M., Haseeb A., Jun J-B., Rahman S. On Almost r-paracontact Riemannian manifold with a certain connection, Commun. Korean Math. Soc., 2010; 25(2): 235-243.
11. Nakova G., Zamkovoy S. Almost paracontact manifold, arXiv:0806.3859v2, 2009; 1-17.
12. Kasap Z., Tekkoyun M. Mechanical systems on almost para/pseudo-Kähler-Weyl manifolds, IJGMMP, 2013; 10(5): 1-8.
13. Newlander A., Nirenberg L. Complex analytic coordinates in almost complex manifolds, Ann. of Math., 1957; 65: 391-404.
14. Calvaruso G., Perrone D.Geometry of H-paracontact metric manifolds, arXiv:1307.7662v1, 2013.
15. Deshmukh S., Khan G. Almost paracontact 3-structures on a differentiable manifold, Ind. J. Pure Appl. Math., 2012; 101: 442-448.
16. Folland G.B. Weyl Manifolds, J. Differential Geometry, 1970; 4:145-153.
17. Abreu M., Kähler geometry of toric manifolds in symplectic coordinates, arXiv:math/0004122v1, 2000; 1-24.
18. Gilkey P., Nikcevic S. , U. Simon, Geometric realizations, curvature decompositions, and Weyl manifolds. JGP, 2011; 61: 270-275.
19. Pedersen H., Poon Y.S., Swann A., The Einstein-Weyl equations in complex and quaternionic geometry, DGIA, 1993; 3(4): 309-321.
20. Kadosh L. Topics in Weyl Geometry, Ph.D. Dissertationial, University of California, 1996.
21. Pedersen H., PoonY . S., SwannA. The Einstein-Weyl equations in complex and quaternionic Ggeometry, DGIA, 1993; 3(4), 309-321.
22. Welyczko J., Slant curves in 3-dimensional normal almost paracontact metric manifolds, arXiv:1212.5839v1, 2012.
23. Klein J. , Escapes Variationnels et Mécanique, Ann. Inst. Fourier, Grenoble, 1962; 12.
24. LeonM. De, RodriguesP.R. Methods of Differential Geometry in Analytical Mechanics, Elsevier Sc. Pub. Com. Inc., Amsterdam, 1989; 263-299.
25. Liu, Q. Contact Hamiltonian dynamics: Variational principles, invariants, completeness and periodic behavior, Annals of Physics, 2018; 95: 26-44.
26. Cha J., Lee Y., Chung W.S., Kim D., Kim D.H., Lee Y.K., Park S., Kim K.S., Chung W.S. On the exact solutions of the damped harmonic oscillator with a timedependent damping constant and a time-dependent angular frequency, Journal of the Korean Physical Society, 2015; 67: 404-408.

27. Thidé B. Electromagnetic Field Theory, Uppsala, Sweden, 2012.
28. Martín R.G. Electromagnetic Field Theory for Physicists and Engineers: Fundamentals and Applications, Asignatura: Electrodinámica, Físicas, Granada, 2007.
29. Weyl H. Space-Time-Matter, Dover Publ. 1922, Translated From the 4th German Edition by H. Brose, Dover, 1952.