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A Solution Form of A Higher Order Difference Equation

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Abstract

The main aim of this paper is to investigate the solutions of the difference equation

$$x_{n+1}=\frac{(-1)^n a x_{n-2k}}{a+(-1)^n \prod_{i=0}^{2k} x_{n-i}}\ ,\ n=0,1,...$$

where k is a positive integer and initial conditions are non zero real numbers with $\prod_{i=0}^{2k} x_{n-i}\neq\mp a$.

Keywords: *Difference Equation, Solutions, Boundedness, Periodicity.*
2010 Mathematics Subject Classification: *39A10*

1. Introduction

Recently, investigating the qualitative behavior of nonlinear difference equations is a topic of a great interest. The theory of difference equations play inportant role in applicable analysis. Applications of difference equations have appeared in many areas such as ecology, population dynamics, genetics in biologiy, physics and engineering.
Many researchers have investigated the behavior of the solution of rational difference equations. For example see Refs. [1-12].
Ergin and Karatas[11] obtained the formulas of the solution of the difference equation

$$x_{n+1}=\frac{a x_{n-k}}{a-\prod_{i=0}^{k} x_{n-i}}.$$

Simsek and Abdullayev [4] studied a solution of the difference equation

$$x_{n+1}=\frac{x_{n-(k+1)}}{1+x_n x_{n-1}...x_{n-k}}.$$

Karatas [9] studied the global behavior of the nonnegative equilibrium points of the difference equation

$$x_{n+1}=\frac{A x_{n-m}}{B+C\prod_{i=0}^{2k+1} x_{n-i}}.$$

Abo-Zeid [8] investigated the global behavior of all solutions of the difference equation

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$$x_{n+1}=\frac{Ax_{n-k}}{B+C\prod\limits_{i=0}^{k}x_{n-i}}.$$

El-Sayed et al. [5] obtained the formulas of the recursive sequences

$$x_n=\frac{x_nx_{n-5}}{x_{n-4}(\pm1\pm x_nx_{n-5})}.$$

Our aim in this paper is to obtain the solutions of the difference equation

$$x_{n+1}=\frac{(-1)^n ax_{n-2k}}{a+(-1)^n\prod\limits_{i=0}^{2k}x_{n-i}},\ \ n=0,1,... \tag{1.1}$$

where k is a positive integer and initial conditions are non zero real numbers with $\prod\limits_{i=0}^{2k}x_{n-i}\neq\mp a$.

Definition 1.1. *Let I be some interval of real numbers and let $f:I^{k+1}\to I$ be a continuously differentiable function. Then for every set of initial conditions $x_{-k},x_{-(k+1)},...,x_0\in I$, the difference equation*

$$x_{n+1}=f\left(x_n,x_{n-1},...,x_{n-k}\right),\ n=0,1,... \tag{1.2}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

Definition 1.2. *A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if*

$x_{n+p}=x_n$ for all $n\geq -k$.

2. Main Results

Before we obtain main results we will give a few lemmas for future use.

Lemma 2.1. *Let $\{x_n\}_{n=-2k}^{\infty}$ be a solution of $Eq.(1.1)$.Assume that $\prod\limits_{i=0}^{2k}x_{-i}\neq\pm a$.Then we take following equalities*

$$x_1=\frac{ax_{-2k}}{a+\prod\limits_{i=0}^{2k}x_{-i}},x_2=\frac{1}{a}\left[-x_{-(2k-1)}\left(a+\prod_{i=0}^{2k}x_{-i}\right)\right],$$

$$x_3=\frac{ax_{-(2k-2)}}{a-\prod\limits_{i=0}^{2k}x_{-i}},x_4=\frac{1}{a}\left[-x_{-(2k-3)}\left(a-\prod_{i=0}^{2k}x_{-i}\right)\right],...,$$

$$x_{2k-2}=\frac{1}{a}\left[-x_{-3}\left(a-\prod_{i=0}^{2k}x_{-i}\right)\right],x_{2k-1}=\frac{ax_{-2}}{a+\prod\limits_{i=0}^{2k}x_{-i}},$$

$$x_{2k}=\frac{1}{a}\left[-x_{-1}\left(a+\prod_{i=0}^{2k}x_{-i}\right)\right],x_{2k+1}=\frac{ax_0}{a-\prod\limits_{i=0}^{2k}x_{-i}}.$$

Proof. It is obvious when applied basic itiration method for $n=0,1,...,2k$ in $Eq.(1.1)$. □

Lemma 2.2. *Let $\{x_n\}_{n=-2k}^{\infty}$ be a solution of $Eq.(1.1)$.Assume that $\prod\limits_{i=0}^{2k}x_{-i}\neq\pm a$.Then we take following equalities*

$$x_{2k+5}=-x_{-(2k-3)},x_{2k+6}=\frac{x_{-(2k-4)}\left(-a+\prod\limits_{i=0}^{2k}x_{-i}\right)}{a+\prod\limits_{i=0}^{2k}x_{-i}},$$

$$x_{2k+7}=-x_{-(2k-5)},x_{2k+8}=\frac{x_{-(2k-6)}\left(a+\prod\limits_{i=0}^{2k}x_{-i}\right)}{-a+\prod\limits_{i=0}^{2k}x_{-i}},$$

$$x_{2k+9}=-x_{-(2k-7)},...,x_{4k-1}=x_{-3},x_{4k}=\frac{x_{-2}\left(-a+\prod\limits_{i=0}^{2k}x_{-i}\right)}{a+\prod\limits_{i=0}^{2k}x_{-i}},$$

$$x_{4k+1}=-x_{-1},x_{4k+2}=\frac{x_0\left(a+\prod\limits_{i=0}^{2k}x_{-i}\right)}{-a+\prod\limits_{i=0}^{2k}x_{-i}}.$$

Proof. It is obvious when applied basic itiration method for $n=2k+4,2k+5,...,4k+1$ in $Eq.(1.1)$. □

Lemma 2.3. *Let* $\{x_n\}_{n=-2k}^{\infty}$ *be a solution of* $Eq.(1.1)$*.Assume that* $\prod_{i=0}^{2k} x_{-i} \neq \pm a$*. Then we take following equalities*

$$x_{4k+6} = \frac{1}{a}\left[x_{-(2k-3)}\left(a+\prod_{i=0}^{2k} x_{-i}\right)\right], x_{4k+7} = \frac{-ax_{-(2k-4)}}{a+\prod_{i=0}^{2k} x_{-i}},$$

$$x_{4k+8} = \frac{1}{a}\left[-x_{-(2k-5)}\left(-a+\prod_{i=0}^{2k} x_{-i}\right)\right], x_{4k+9} = \frac{ax_{-(2k-6)}}{-a+\prod_{i=0}^{2k} x_{-i}},$$

$$x_{4k+10} = \frac{1}{a}\left[x_{-(2k-7)}\left(a+\prod_{i=0}^{2k} x_{-i}\right)\right], \ldots, x_{6k} = \frac{1}{a}\left[x_{-3}\left(a+\prod_{i=0}^{2k} x_{-i}\right)\right],$$

$$x_{6k+1} = \frac{-ax_{-2}}{a+\prod_{i=0}^{2k} x_{-i}}, x_{6k+2} = \frac{1}{a}\left[-x_{-1}\left(-a+\prod_{i=0}^{2k} x_{-i}\right)\right], x_{6k+3} = \frac{ax_0}{-a+\prod_{i=0}^{2k} x_{-i}}.$$

Proof. It is obvious when applied basic itiration method for $n = 4k+5, 4k+6, ..., 6k+2$ in $Eq.(1.1)$. □

The following result is obtained directly from Lemma 2.1, Lemma 2.2 and Lemma 2.3.

Corollary 2.4. *Let* $\{x_n\}_{n=-2k}^{\infty}$ *be a solution of* $Eq.(1.1)$*.Assume that* $\prod_{i=0}^{2k} x_{-i} \neq \pm a$*. Then we take following equalities*

$$\prod_{i=4k+2}^{6k-1} x_{i+4} = \prod_{i=2k+1}^{4k-2} x_{i+4} = \prod_{i=0}^{2k-3} x_{i+4} = \prod_{i=0}^{2k-3} x_{-i}.$$

Theorem 2.5. *Let* $\{x_n\}_{n=-2k}^{\infty}$ *be a solution of* $Eq.(1.1)$*.Assume that* $\prod_{i=0}^{2k} x_{-i} \neq \pm a$*,* k *is a odd positive integer and* $1 \leq m \leq 2k+1$*. Then for* $n = 0, 1, \ldots$ *all solutions of* $Eq.(1.1)$ *are of the form*

for $m \equiv 1(mod4)$,

$$x_{(2k+1)n+m} = \begin{cases} \frac{ax_{-[2k-(m-1)]}}{a+\prod_{i=0}^{2k} x_{-i}}, n \equiv 0(mod4) \\ \frac{x_{-[2k-(m-1)]}\left(-a+\prod_{i=0}^{2k} x_{-i}\right)}{a+\prod_{i=0}^{2k} x_{-i}}, n \equiv 1(mod4) \\ -\frac{ax_{-[2k-(m-1)]}}{a+\prod_{i=0}^{2k} x_{-i}}, n \equiv 2(mod4) \\ x_{-[2k-(m-1)]}, n \equiv 3(mod4) \end{cases},$$

for $m \equiv 2(mod4)$,

$$x_{(2k+1)n+m} = \begin{cases} \frac{1}{a}\left[-x_{-[2k-(m-1)]}\left(a+\prod_{i=0}^{2k} x_{-i}\right)\right], n \equiv 0(mod4) \\ -x_{-[2k-(m-1)]}, n \equiv 1(mod4) \\ \frac{1}{a}\left[-x_{-[2k-(m-1)]}\left(-a+\prod_{i=0}^{2k} x_{-i}\right)\right], n \equiv 2(mod4) \\ x_{-[2k-(m-1)]}, n \equiv 3(mod4) \end{cases},$$

for $m \equiv 3(mod4)$,

$$x_{(2k+1)n+m} = \begin{cases} \frac{-ax_{-[2k-(m-1)]}}{-a+\prod_{i=0}^{2k} x_{-i}}, n \equiv 0(mod4) \\ \frac{x_{-[2k-(m-1)]}\left(a+\prod_{i=0}^{2k} x_{-i}\right)}{-a+\prod_{i=0}^{2k} x_{-i}}, n \equiv 1(mod4) \\ \frac{ax_{-[2k-(m-1)]}}{-a+\prod_{i=0}^{2k} x_{-i}}, n \equiv 2(mod4) \\ x_{-[2k-(m-1)]}, n \equiv 3(mod4) \end{cases},$$

for $m \equiv 0(mod4)$,

$$x_{(2k+1)n+m} = \begin{cases} \frac{1}{a}\left[x_{-[2k-(m-1)]}\left(-a+\prod_{i=0}^{2k} x_{-i}\right)\right], n \equiv 0(mod4) \\ -x_{-[2k-(m-1)]}, n \equiv 1(mod4) \\ \frac{1}{a}\left[x_{-[2k-(m-1)]}\left(a+\prod_{i=0}^{2k} x_{-i}\right)\right], n \equiv 2(mod4) \\ x_{-[2k-(m-1)]}, n \equiv 3(mod4) \end{cases}.$$

Proof. Firstly we take from Lemma 2.1 for $n = 0$ and $m = 1, x_1 = \frac{ax_{-2k}}{a+\prod_{i=0}^{2k} x_{-i}}$,

for $n = 0$ and $m = 2, x_2 = \frac{1}{a}\left[-x_{-(2k-1)}\left(a+\prod_{i=0}^{2k} x_{-i}\right)\right]$,

for $n = 0$ and $m = 3, x_3 = \frac{ax_{-(2k-2)}}{a-\prod_{i=0}^{2k} x_{-i}}$,

for $n = 0$ and $m = 4, x_4 = \frac{1}{a}\left[-x_{-(2k-3)}\left(a-\prod_{i=0}^{2k} x_{-i}\right)\right]$.

Now our assumption holds that $n=1$, $m=1,2,3,4$. Let's take $\prod_{i=0}^{2k} x_{-i}=p$ for easy of writing.

We have from $Eq.(1.1)$ for $n=2k+1$,

$x_{2k+2}=\frac{-ax_1}{a-\prod_{i=0}^{2k} x_{2k+1-i}}$.

From Lemma 2.1

$$x_{2k+2}=\frac{-a\frac{ax_{-2k}}{a+p}}{a-\frac{ax_0}{a-p}\frac{-x_{-1}(a+p)}{a}\frac{ax_{-2}}{a+p}\frac{-x_{-3}(a-p)}{a}\ldots\frac{-x_{-(2k-1)(a+p)}}{a}\frac{ax_{-2k}}{a+p}}$$

$$=\frac{\frac{-a^2x_{-2k}}{a+p}}{a-\frac{-ap}{a-p}}=\frac{x_{-2k(-a+p)}}{a+p}.$$

That is, for $n=1$ and $m=1$

$$x_{2k+2}=\frac{x_{-2k}\left(-a+\prod_{i=0}^{2k} x_{-i}\right)}{a+\prod_{i=0}^{2k} x_{-i}}. \tag{2.1}$$

For $n=2k+2$, we have from $Eq.(1.1)$

$x_{2k+3}=\frac{ax_2}{a+\prod_{i=0}^{2k} x_{2k+2-i}}$.

Then from Lemma 2.1 and Eq.(2.1)

$$x_{2k+3}=\frac{a\frac{1}{a}\left[-x_{-(2k-1)}(a+p)\right]}{a+\frac{x_{-2k(-a+p)}}{a+p}\frac{ax_0}{a-p}\frac{-x_{-1}(a+p)}{a}\frac{ax_{-2}}{a+p}\frac{-x_{-3}(a-p)}{a}\ldots\frac{-x_{-(2k-1)(a+p)}}{a}}$$

$$=\frac{-x_{-(2k-1)}(a+p)}{a+p}.$$

That is, for $n=1$ and $m=2$

$$x_{2k+3}=-x_{-(2k-1)} \tag{2.2}$$

For $n=2k+3$, we have from $E.q.(1.1)$

$x_{2k+4}=\frac{-ax_3}{a-\prod_{i=0}^{2k} x_{2k+3-i}}$.

Then from Lemma 2.1 and Eqs. (2.1), (2.2)

$$x_{2k+4}=\frac{-a\frac{ax_{-(2k-2)}}{a-p}}{a-\left[-x_{-(2k-1)}\right]\frac{x_{-2k}(-a+p)}{a+p}\frac{ax_0}{a-p}\frac{1}{a}\left[-x_{-1}(a+p)\right]\frac{ax_{-2}}{a+p}\frac{1}{a}\left[-x_{-3}(a-p)\right]\ldots\frac{ax_{-(2k-2)}}{a-p}}$$

$$=\frac{-\frac{a^2x_{-(2k-2)}}{a-p}}{a-\frac{ap}{a+p}}=\frac{-x_{-(2k-2)}(a+p)}{a-p}.$$

That is, for $n=1$ and $m=3$

$$x_{2k+4}=\frac{x_{-(2k-2)}\left(a+\prod_{i=0}^{2k} x_{-i}\right)}{-a+\prod_{i=0}^{2k} x_{-i}}. \tag{2.3}$$

Lastly for $n=1$ and $m=4$ from Lemma 2.2 and Eqs. (2.1), (2.2), (2.3)
we obtain

$$x_{2k+5}=-x_{-(2k-3)}. \tag{2.4}$$

Similarly one can show from Lemma 2.2 and Lemma 2.3 that for $n=2$ and $m=1,2,3,4$

$$x_{4k+3}=\frac{-ax_{-2k}}{a+\prod_{i=0}^{2k} x_{-i}}, \tag{2.5}$$

$$x_{4k+4}=\frac{1}{a}\left[-x_{-(2k-1)}\left(-a+\prod_{i=0}^{2k} x_{-i}\right)\right], \tag{2.6}$$

$$x_{4k+5}=\frac{ax_{-(2k-2)}}{-a+\prod_{i=0}^{2k} x_{-i}}, \tag{2.7}$$

$$x_{4k+6}=\frac{1}{a}\left[x_{-(2k-3)}\left(a+\prod_{i=0}^{2k}x_{-i}\right)\right]. \tag{2.8}$$

Similarly one can show from Lemma 2.3 and Eqs. (2.5), (2.6), (2.7), (2.8) that for $n=3$ and $m=1,2,3,4$

$$x_{6k+4}=x_{-2k}, \tag{2.9}$$

$$x_{6k+5}=x_{-(2k-1)}, \tag{2.10}$$

$$x_{6k+6}=x_{-(2k-2)}, \tag{2.11}$$

$$x_{6k+7}=x_{-(2k-3)}. \tag{2.12}$$

Now suppose that our assumption holds for $n-1$.That is,
for $m\equiv 1(mod4)$,

$$x_{(2k+1)n-(2k+1-m)}=\begin{cases}\frac{ax_{-[2k-(m-1)]}}{a+\prod_{i=0}^{2k}x_{-i}}, n\equiv 1(mod4)\\ \frac{x_{-[2k-(m-1)]}\left(-a+\prod_{i=0}^{2k}x_{-i}\right)}{a+\prod_{i=0}^{2k}x_{-i}}, n\equiv 2(mod4)\\ -\frac{ax_{-[2k-(m-1)]}}{a+\prod_{i=0}^{2k}x_{-i}}, n\equiv 3(mod4)\\ x_{-[2k-(m-1)]}, n\equiv 0(mod4)\end{cases}, \tag{2.13}$$

for $m\equiv 2(mod4)$,

$$x_{(2k+1)n-(2k+1-m)}=\begin{cases}\frac{-x_{-[2k-(m-1)]}\left(a+\prod_{i=0}^{2k}x_{-i}\right)}{a}, n\equiv 1(mod4)\\ -x_{-[2k-(m-1)]}, n\equiv 2(mod4)\\ \frac{-x_{-[2k-(m-1)]}\left(-a+\prod_{i=0}^{2k}x_{-i}\right)}{a}, n\equiv 3(mod4)\\ x_{-[2k-(m-1)]}, n\equiv 0(mod4)\end{cases}, \tag{2.14}$$

for $m\equiv 3(mod4)$,

$$x_{(2k+1)n-(2k+1-m)}=\begin{cases}\frac{-ax_{-[2k-(m-1)]}}{-a+\prod_{i=0}^{2k}x_{-i}}, n\equiv 1(mod4)\\ \frac{x_{-[2k-(m-1)]}\left(a+\prod_{i=0}^{2k}x_{-i}\right)}{-a+\prod_{i=0}^{2k}x_{-i}}, n\equiv 2(mod4)\\ \frac{ax_{-[2k-(m-1)]}}{-a+\prod_{i=0}^{2k}x_{-i}}, n\equiv 3(mod4)\\ x_{-[2k-(m-1)]}, n\equiv 0(mod4)\end{cases}, \tag{2.15}$$

for $m\equiv 0(mod4)$,

$$x_{(2k+1)n-(2k+1-m)}=\begin{cases}\frac{x_{-[2k-(m-1)]}\left(-a+\prod_{i=0}^{2k}x_{-i}\right)}{a}, n\equiv 1(mod4)\\ -x_{-[2k-(m-1)]}, n\equiv 2(mod4)\\ \frac{x_{-[2k-(m-1)]}\left(a+\prod_{i=0}^{2k}x_{-i}\right)}{a}, n\equiv 3(mod4)\\ x_{-[2k-(m-1)]}, n\equiv 0(mod4)\end{cases}. \tag{2.16}$$

We have from Eq. (1.1)

$$x_{(2k+1)n+m}=\frac{(-1)^{(2k+1)n+(1-m)}ax_{(2k+1)n-(2k+1-m)}}{a+(-1)^{(2k+1)n+(1-m)}\prod_{i=0}^{2k}x_{(2k+1)n+(i+1-m)}}. \tag{2.17}$$

Then for $n\equiv 1(mod4)$ and $m\equiv 1(mod4)$

$x_{(2k+1)n+m}=\frac{-ax_{(2k+1)n-(2k+1-m)}}{a-\prod_{i=0}^{2k}x_{(2k+1)n+(i+1-m)}}$

$=\frac{-ax_{(2k+1)n-(2k-4s)}}{a-x_{(2k+1)n-(-4s)}x_{(2k+1)n-(1-4s)}\ldots x_{(2k+1)n-(2k-1-4s)}x_{(2k+1)n-(2k-4s)}}$

where s is a positive integer and $m=4s+1$.Since $k\equiv 1(mod2)$ and $m\leq 2k+1$, we get from Eqs. (2.13), (2.14), (2.15), (2.16)

$x_{(2k+1)n+m}=\frac{-\frac{a^2x_{-[2k-(m-1)]}}{a+p}}{a-\frac{ap}{-a+p}}=\frac{-\frac{a^2x_{-[2k-(m-1)]}}{a+p}}{\frac{-a^2}{-a+p}}$

That is,

$$x_{(2k+1)n+m}=\frac{x_{-[2k-(m-1)]}\left(-a+\prod_{i=0}^{2k}x_{-i}\right)}{a+\prod_{i=0}^{2k}x_{-i}}. \tag{2.18}$$

Secondly it can be writen for $n\equiv 1(mod4)$ and $m\equiv 2(mod4)$

$x_{(2k+1)n+m}=\frac{-ax_{(2k+1)n-(2k-4s)}}{a+x_{(2k+1)n+1-(-4s)}x_{(2k+1)n-(-4s)}x_{(2k+1)n-(1-4s)}\ldots x_{(2k+1)n-(2k-2-4s)}x_{(2k+1)n-(2k-1-4s)}}.$

We get from Eqs. (2.13), (2.14), (2.15), (2.16),(2.18)

$x_{(2k+1)n+m}=\frac{-x_{-[2k-(m-1)]}(a+p)}{a+p}.$

That is,

$$x_{(2k+1)n+m}=-x_{-[2k-(m-1)]}. \tag{2.19}$$

We have for $n\equiv 1(mod4)$ and $m\equiv 3(mod4)$

$x_{(2k+1)n+m}=\frac{-ax_{(2k+1)n-(2k-4s)}}{a-x_{(2k+1)n+2-(4s)}x_{(2k+1)n+1-(-4s)}x_{(2k+1)n-(-4s)}\ldots x_{(2k+1)n-(2k-2-4s)}}.$

Then from Eqs. (2.13), (2.14), (2.15), (2.16),(2.18),(2.19)

$x_{(2k+1)n+m}=\frac{-\frac{a^2x_{-[2k-(m-1)]}}{a-p}}{a-\frac{ap}{a+p}}=\frac{x_{-[2k-(m-1)]}(a+p)}{-a+p}.$

That is,

$$x_{(2k+1)n+m}=\frac{x_{-[2k-(m-1)]}\left(a+\prod_{i=0}^{2k}x_{-i}\right)}{-a+\prod_{i=0}^{2k}x_{-i}}. \tag{2.20}$$

Lastly we take for $n\equiv 1(mod4)$ and $m\equiv 0(mod4)$

$x_{(2k+1)n+m}=\frac{ax_{(2k+1)n-(2k-4s)}}{a+x_{(2k+1)n+3-(4s)}x_{(2k+1)n+2-(-4s)}x_{(2k+1)n+1-(-4s)}\ldots x_{(2k+1)n-(2k-3-4s)}}.$

Then from Eqs. (2.13), (2.14), (2.15), (2.16),(2.18),(2.19),(2.20)

$x_{(2k+1)n+m}=\frac{-x_{-[2k-(m-1)]}(a-p)}{a-p}.$

That is,

$$x_{(2k+1)n+m}=-x_{-[2k-(m-1)]}. \tag{2.21}$$

Similarly one can obtained other situations for $n\equiv 2(mod4), n\equiv 3(mod4)$ and $n\equiv 0(mod4)$. Thus, the proof is complete. □

Theorem 2.6. *Let $\{x_n\}_{n=-2k}^{\infty}$ be a solution of $Eq.(1.1)$.Assume that $\prod_{i=0}^{2k}x_{-i}\neq\pm a$, k is a even positive integer and $1\leq m\leq 2k+1$. Then for $n=0,1,\ldots$all solutions of $Eq.(1.1)$ are of the form*

for $m\equiv 1(mod4)$,

$$x_{(2k+1)n+m}=\begin{cases}\frac{ax_{-[2k-(m-1)]}}{a+\prod_{i=0}^{2k}x_{-i}}, n\equiv 0(mod4)\\ -x_{-[2k-(m-1)]}, n\equiv 1(mod4)\\ \frac{-ax_{-[2k-(m-1)]}}{a-\prod_{i=0}^{2k}x_{-i}}, n\equiv 2(mod4)\\ x_{-[2k-(m-1)]}, n\equiv 3(mod4)\end{cases},$$

for $m\equiv 2(mod4)$,

$$x_{(2k+1)n+m}=\begin{cases}\frac{1}{a}\left[-x_{-[2k-(m-1)]}\left(a+\prod_{i=0}^{2k}x_{-i}\right)\right], n\equiv 0(mod4)\\ \frac{-x_{-[2k-(m-1)]}\left(a+\prod_{i=0}^{2k}x_{-i}\right)}{a-\prod_{i=0}^{2k}x_{-i}}, n\equiv 1(mod4)\\ \frac{1}{a}\left[x_{-[2k-(m-1)]}\left(a+\prod_{i=0}^{2k}x_{-i}\right)\right], n\equiv 2(mod4)\\ x_{-[2k-(m-1)]}, n\equiv 3(mod4)\end{cases},$$

for $m \equiv 3(mod4)$,

$$x_{(2k+1)n+m} = \begin{cases} \frac{ax_{-[2k-(m-1)]}}{a-\prod\limits_{i=0}^{2k} x_{-i}}, n \equiv 0(mod4) \\ -x_{-[2k-(m-1)]}, n \equiv 1(mod4) \\ \frac{-ax_{-[2k-(m-1)]}}{a+\prod\limits_{i=0}^{2k} x_{-i}}, n \equiv 2(mod4) \\ x_{-[2k-(m-1)]}, n \equiv 3(mod4) \end{cases},$$

for $m \equiv 0(mod4)$,

$$x_{(2k+1)n+m} = \begin{cases} \frac{1}{a}\left[-x_{-[2k-(m-1)]}\left(a-\prod\limits_{i=0}^{2k} x_{-i}\right)\right], n \equiv 0(mod4) \\ \frac{-x_{-[2k-(m-1)]}\left(a-\prod\limits_{i=0}^{2k} x_{-i}\right)}{a+\prod\limits_{i=0}^{2k} x_{-i}}, n \equiv 1(mod4) \\ \frac{1}{a}\left[x_{-[2k-(m-1)]}\left(a-\prod\limits_{i=0}^{2k} x_{-i}\right)\right], n \equiv 2(mod4) \\ x_{-[2k-(m-1)]}, n \equiv 3(mod4) \end{cases}.$$

Proof. It can be proved like proof of Theorem 2.5. □

Corollary 2.7. $Eq.(1.1)$ *has periodic solutions of period* $(8k+4)$.

Proof. It is obvious from Theorem 2.4 and Theorem 2.5. □

3. Applications

In this section some applications given to verify theorem 2.4 and theorem 2.5. We choosed a, k and initial conditions arbitrarly. The figures in this section were drawn by Matlab packet program.

Example 2.8. Let x_n be a solution of the Eq.(1.1) with $a = 3, k = 5$. If the initial conditions $x_0 = 1$, $x_{-1} = 1,3$, $x_{-2} = 3$, $x_{-3} = 2,5$, $x_{-4} = 1,8$, $x_{-5} = 1,1$, $x_{-6} = 4$, $x_{-7} = 5,3$, $x_{-8} = 1,5$,$x_{-9} = 2,5$,, then the solution is given by Figure 2.9.

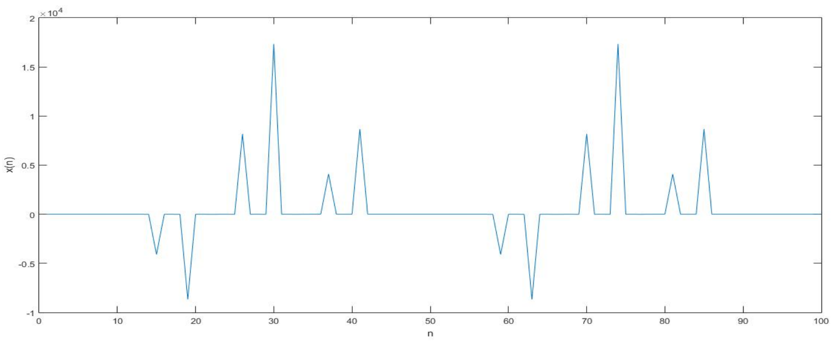

Figure 2.9 Note that the figure verifies the Theorem 2.4 and Corollary 2.

Example 2.10. Let x_n be a solution of the Eq.(1.1) with $a = -7, k = 4$. If the initial conditions $x_0 = 2$, $x_{-1} = 1,2$, $x_{-2} = 1$, $x_{-3} = 3,5$, $x_{-4} = 4,8$, $x_{-5} = 1,6$, $x_{-6} = 5$, $x_{-7} = 2,3$, $x_{-8} = 2,7$, then the solution is given by Figure 2.11.

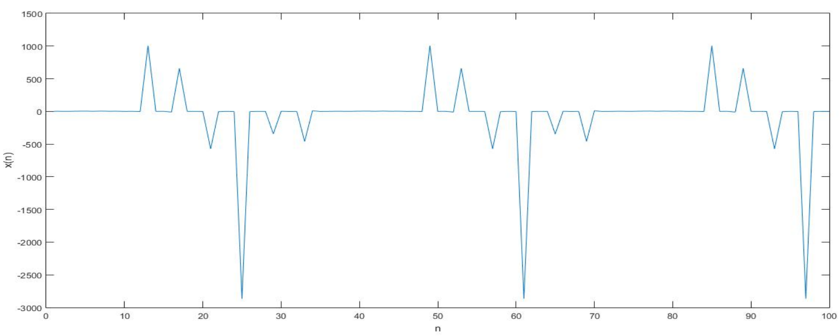

Figure 2.11 Note that the figure verifies the Theorem 2.5 and Corollary 2.

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