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Ball analysis for an efficient sixth convergence order-scheme under weaker conditions

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Abstract

In this study we consider an efficient sixth order-scheme for solving Banach space valued equations. The convergence criteria in earlier studies involve higher order derivatives limiting applicability of these methods. In this study we use the first derivative only in our analysis to expand the usage of these schemes. The technique we use can be used on other schemes to obtain the same advantages. Numerical experiments compare favorably our results to earlier ones.

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1. Introduction

Let $F : D \subset B_1 \to B_2$ be a continuously differentiable nonlinear operator and D stand for an open non empty convex compact set of B_1. Here B_1 and B_2 stand for Banach spaces. Consider the problem of finding a solution x_* of the nonlinear equation

$$F(x) = 0. \tag{1}$$

It is desirable to obtain a unique solution x_* of (1). But this can rarely be achieved, so most researchers and practitioners develop iterative schemes which converge to x_*. In this paper we extend the convergence ball

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of a class of an efficient sixth order-scheme studied in [18]. Precisely, we consider the sixth order method defined in [18] for $n = 1, 2, \ldots,$ by

$$\begin{aligned} y_n &= x_n - \frac{2}{3}F'(x_n)^{-1}F(x_n) \\ z_n &= x_n - (9A_n^{-1}F'(x_n) + \frac{3}{2}(A_n^{-1}F'(x_n))^{-1} - \frac{13}{2}I)F'(x_n)^{-1}F(x_n) \\ x_{n+1} &= z_n - 2(3A_n^{-1} - F'(x_n)^{-1})F(z_n), \end{aligned} \tag{2}$$

where $A_n = F'(x_n) + F'(y_n)$.

The analysis in [18] uses assumptions on the sixth order derivatives of F and when $B_1 = B_2 = \mathbb{R}^m$. The assumptions on higher order derivatives reduce the applicability of method (2). For example: Let $B_1 = B_2 = \mathbb{R}$, $D = [-\frac{1}{2}, \frac{3}{2}]$. Define F on D by

$$F(x) = \begin{cases} x^3 \log x^2 + x^5 - x^4, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Then, we get

$$F'(x) = 3x^2 \log x^2 + 5x^4 - 4x^3 + 2x^2,$$
$$F''(x) = 6x \log x^2 + 20x^3 - 12x^2 + 10x,$$
$$F'''(x) = 6 \log x^2 + 60x^2 = 24x + 22,$$

and $x_* = 1$. Obviously $F'''(x)$ is not bounded on D. Hence, the convergence of scheme (2) is not guaranteed by the analysis in [18]. In this study we use only assumptions on the first derivative to prove our results. The advantages of our approach include: larger radius needed on method of convergence (i.e. more initial points), tighter upper bounds on $\|x_k - x_*\|$(i.e. fewer iterates to achieve a desired error tolerance). It is worth noting that these advantages are obtained without any additional conditions [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32].

Throughout this paper $U(x, r)$ stand for open ball with center at x and radius $r > 0$ and $\bar{U}(x, r)$ denote the closure of $U(x, r)$.

Rest of the paper is organized as follows. The convergence analysis of method (2) is given in Section 2 and examples are given in Section 3.

2. Ball analysis

We consider real functions and parameters to assist us in the convergence of method (2). Assume equation

$$w_0(t) - 1 = 0, \tag{1}$$

has a real positive root denoted as r_0, where for $I = [0, \infty)$, $\omega_0 : I \longrightarrow I$ is continuous and increasing with $\omega_0(0) = 0$. Moreover, consider for $I_0 = [0, r_0)$ functions $\omega : I_0 \longrightarrow I, \omega_1 : I_0 \longrightarrow I$ continuous and increasing with $\omega(0) = 0$. Define functions g_1, h_1 on I_0 as

$$g_1(s) = \frac{(\int_0^1 \omega((1-\tau)s)d\tau + \frac{1}{3}\int_0^1 \omega_1(\tau s)d\tau)}{1 - \omega_0(s)}$$

and

$$h_1(s) = g_1(s) - 1.$$

Assume

$$\frac{1}{3}\omega_1(s) - 1 < 0. \tag{2}$$

The existence of roots for real functions is based on the Intermediate Value Theorem (IVT). By these hypotheses and definition, we obtain $h_1(0) < 0$ and $h_1(s) \longrightarrow \infty$ with $s \longrightarrow r_0^-$. By IVT, there exists a least root for h_1 in $(0, r_0)$ denoted by R_1. Define function h, p on I_0 as

$$p(s) \;=\; \frac{1}{2}\left(\omega_0(s) + \omega_0(g_1(s)s)\right).$$

Assume equation

$$p(s) - 1 = 0, \tag{3}$$

has a least root in $(0, r_0)$ denoted by r_p. Define functions q, b, g_2 and h_2 on the interval $[0, r_p)$ as

$$q(s) = \frac{\omega_0(s) + \omega_0(g_1(s)s)}{4(1 - p(s))},$$

$$b(s) = \frac{3}{2}\frac{(\omega_1(s) + \omega_1(g_1(s)s))(6q(s)^2 + q(s) + 1)}{1 - \omega_0(s)},$$

$$g_2(s) = g_1(s) + \frac{b(s)\int_0^1 \omega_1(\tau s)d\tau}{1 - \omega_0(s)}$$

and

$$h_2(s) = g_2(s) - 1.$$

Assume

$$g_1(0) + b(0)\omega_1(0) - 1 < 0. \tag{4}$$

Then, we get again using (4) and the definitions: $h_2(0) < 0$ and $h_2(s) \longrightarrow \infty$ as $s \longrightarrow r_p^-$. By IVT, equation $h_2(s) = 0$ has a least root in $(0, r_p)$ denoted by R_2. Assume equation

$$\omega_0(g_2(s)s) - 1 = 0 \tag{5}$$

has a least root in $(0, r_p)$ denoted by r_1. Define functions c, g_3 and h_3 on $(0, r_1)$ as

$$c(s) = \frac{\omega_0(s) + \omega_0(g_2(s)s)}{(1 - \omega_0(s))(1 - \omega_0(g_2(s)s))} + \frac{3}{2}\frac{\omega_0(s) + \omega_0(g_2(s)s)}{(1 - \omega_0(s))(1 - p(s))},$$

$$g_3(s) = (g_1(g_2(s)s) + c(s)\int_0^1 \omega_1(\tau g_2(s)s)d\tau)g_1(s)$$

and

$$h_3(s) = g_3(s) - 1.$$

Assume

$$(g_1(0) + c(0)\omega_1(0))(g_1(0) + b(0)\omega_1(0)) - 1 < 0. \tag{6}$$

Then, we get $h_3(0) < 0$ and $h_3(s) \longrightarrow \infty$ as $s \longrightarrow r_1^-$. Denote by R_3 the least root of equation $h_3(s) = 0$ in $(0, r_1)$. Lastly, introduce a radius of convergence

$$R = \min\{R_i\},\ i = 1, 2, 3. \tag{7}$$

Notice that then, we have for $s \in [0, R)$

$$0 \le \omega_0(s) < 1,\ 0 \le \omega_0(g_2(s)s) < 1, \tag{8}$$

$$0 \le p(s) < 1 \tag{9}$$

and

$$0 \le g_i(s) < 1, i = 1, 2, 3. \tag{10}$$

Set $e_n = \|x_n - x_*\|$. The conditions (A) that follow shall be used in the ball convergence of method (2):

(A1) Operator $F : D \longrightarrow B_2$ is continuously differentiable and there exists a simple solution x_* of equation $F(x) = 0$.

(A2) There exists a continuous and increasing function ω_0 on I_0 with values on itself with $\omega_0(0) = 0$ such that for all $x \in D$

$$\|F'(x_*)^{-1}(F'(x) - F'(x_*))\| \leq \omega_0(\|x - x_*\|).$$

Let $D_0 = D \cap U(x_*, r_0)$, if r_0 exists and is given in (1).

(A3) There exist continuous and increasing functions ω and ω_1 on the interval I_0 with values on interval I_0 such that for each $x, y \in I_0$

$$\|F'(x_*)^{-1}(F'(y) - F'(x))\| \leq \omega(\|y - x\|)$$

and

$$\|F'(x_*)^{-1}F'(x)\| \leq \omega_1(\|x - x_*\|).$$

(A4) $\bar{U}(x_*, R) \subset D$, and items (1)-(7) are true, where R is defined in (7).

(A5) There exists $R_* \geq R$ such that

$$\int_0^1 \omega_0(\tau R_*)d\tau < 1.$$

Under these definitions and conditions we present the ball convergence of method (2).

Theorem 2.1. *Under the conditions (A) choose starting point $x_0 \in U(x_*, R)$. Then, the following items hold for all*

$$\{x_n\} \subset U(x_*, R), \tag{11}$$

$$\lim_{n \longrightarrow \infty} x_n = x_*, \tag{12}$$

$$\|y_n - x_*\| \leq g_1(e_n)e_n \leq e_n < R, \tag{13}$$

$$\|z_n - x_*\| \leq g_2(e_n)e_n \leq e_n, \tag{14}$$

$$\|x_{n+1} - x_*\| \leq g_3(e_n)e_n \leq e_n, \tag{15}$$

and x_* is the only solution of equation $F(x) = 0$ in the set D_1 given below condition (A5), and the functions g_i, h_i are defined previously.

Proof. Mathematical induction is used to show items (11)-(15). First we establish the existence of all iterates in $U(x_*, R)$. By (A1), (A2) and (1)-(3), we get for $x \in U(x_*, R)$

$$\|F'(x_*)^{-1}(F'(x) - F'(x_*))\| \leq \omega_0(\|x - x_*\|) < \omega_0(r) \leq \omega_0(R) < 1,$$

leading together with a lemma due to Banach on invertible operators [28] that $F'(x)^{-1} \in L(B_2, B_1)$ with

$$\|F'(x)^{-1}F'(x_*)\| \leq \frac{1}{1 - \omega_0(\|x - x_*\|)}. \tag{16}$$

Hence, if $x = x_0$ it follows by method (2) that iterate y_0 exists.

Next, we shall show that these iterates belong in the ball $U(x_*, R)$. We can write by the second condition in (A3) and (A1), since

$$F(x) = F(x) - F(x_*) = \int_0^1 F'(x_* + \tau(x - x_*))d\tau(x - x_*)$$

that

$$\|F'(x_*)^{-1}F'(x)\| \leq \int_0^1 \omega_1(\tau\|x-x_*\|)d\tau\|x-x_*\|. \tag{17}$$

By (2), (6) (for $i=0$), (16) (for $x=x_0$), (A1) the first condition in (A3) and the first substep of method (2) (for $n=0$), we get in turn that

$$\begin{aligned}
\|y_0-x_*\| &= \|x_0-x_*-F'(x_0)^{-1}F(x_0)+\frac{1}{3}F'(x_0)^{-1}F(x_0)\| \\
&= \|F'(x_0)^{-1}F'(x_*)(F'(x_*)^{-1}\int_0^1 (F'(x_*+\tau(x_0-x_*))-F'(x_0))(x_0-x_*)d\tau \\
&\quad +\frac{1}{3}F'(x_0)^{-1}F(x_0)\| \\
&\leq \frac{\int_0^1 \omega((1-\tau)e_0)d\tau e_0+\frac{1}{3}\int_0^1 \omega_1(\tau e_0)d\tau e_0}{1-\omega_0 e_0)} \\
&= g_1(e_0)e_0 \leq e_0 < R,
\end{aligned} \tag{18}$$

showing (11) for $n=0$ and $y_0 \in U(x_*,R)$.

Next, we show A_0^{-1} exists. Indeed by (7, (9), (A2), (18), and (18), we have

$$\begin{aligned}
\|(2F'(x_*))^{-1}(A_0-2F'(x_*))\| &\leq \frac{1}{2}\big(\|F'(x_*)^{-1}(F'(x_0)-F'(x_*))\| \\
&\quad +\|F'(x_*)^{-1}(F'(y_0)-F'(x_*))\|\big) \\
&\leq \frac{1}{2}(\omega_0(e_0)+\omega_0(\|y_0-x_*\|)) \\
&\leq \frac{1}{2}(\omega_0(e_0)+\omega_0(g_1(e_0)e_0)) = p(e_0) \leq p(R) < 1,
\end{aligned} \tag{19}$$

then,

$$\|A_0^{-1}F'(x_*)\| \leq \frac{1}{2(1-p(e_0))}, \tag{20}$$

so z_0, x_1 exist by method (2) for $n=0$. Then, we can write by method (2) (secod substep for $n=0$)

$$z_0-x_* = x_0-x_*-F'(x_0)^{-1}F(x_0)+B_0F'(x_0)^{-1}F(x_0), \tag{21}$$

where

$$\begin{aligned}
B_0 &= I-9A_0^{-1}F'(x_0)-\frac{3}{2}(A_0^{-1}F'(x_0))^{-1}+\frac{13}{2}I \\
&= -\frac{3}{2}F'(x_0)^{-1}A_0[6(A_0^{-1}F'(x_0)-\frac{1}{2}I)^2 \\
&\quad +(A_0^{-1}F'(x_0)-\frac{1}{2}I)+I].
\end{aligned} \tag{22}$$

We need the estimates

$$\begin{aligned}
\|A_0^{-1}F'(x_0)-\frac{1}{2}I\| &= \frac{1}{2}\|(A_0^{-1}F'(x_*)) \\
&\quad \times[F'(x_*)^{-1}(F'(x_0)-F'(x_*))+F'(x_*)^{-1}(F'(x_*)-F'(y_0))]\| \\
&\leq \frac{\|F'(x_*)^{-1}(F'(x_0)-F'(x_*))\|+\|F'(x_*)^{-1}(F'(y_0)-F'(x_*))\|}{4(1-p(e_0))} \\
&\leq \frac{\omega_0(e_0)+\omega_0(\|y_0-x_*\|)}{4(1-p(e_0))} \\
&\leq \frac{\omega_0(e_0)+\omega_0(g_1(e_0)e_0)}{4(1-p(e_0))} = q(e_0),
\end{aligned} \tag{23}$$

and

$$\begin{aligned}\|F'(x_*)^{-1}A_0\| &\leq \omega_1(e_0)+\omega_1(\|y_0-x_*\|)\\ &\leq \omega_1(e_0)+\omega_1(g_1(e_0)e_0).\end{aligned} \tag{24}$$

Then, by (16) and (22)-(24), we find

$$\begin{aligned}\|B_0\| &\leq \frac{3}{2}\|F'(x_0)^{-1}F'(x_*)\|\|F'(x_*)^{-1}A_0\|\\ &\quad\times[6\|A_0^{-1}F'(x_*)-\frac{1}{2}I\|^2+\|A_0^{-1}F'(x_0)-\frac{1}{2}I\|+\|I\|]\\ &\leq \frac{3}{2}\frac{(\omega_1(e_0)+\omega_1(g_1(e_0)e_0))}{1-\omega_0(e_0)}\\ &\quad\times(6q(e_0)^2+q(e_0)+1)=b(e_0).\end{aligned} \tag{25}$$

In view of (21) and (22)-(25), we can find

$$\begin{aligned}\|z_0-x_*\| &\leq \|y_0-x_*\|+\|B_0\|\|F'(x_0)^{-1}F'(x_*)\|\|F'(x_*)^{-1}F(x_0)\|\\ &\leq [g_1(e_0)+\frac{b(e_0)\int_0^1\omega_1(\tau e_0)d\tau}{1-\omega_0(e_0)}]e_0\\ &= g_2(e_0)e_0,\end{aligned} \tag{26}$$

so $z_0\in U(x_*,R)$ and (14) is true. Notice that by (7), (16 (for $x=z_0$), we get $F'(z_0)^{-1}$ exists and

$$\|F'(z_0)^{-1}F'(x_*)\|\leq\frac{1}{1-\omega_0(\|z_0-x_*\|)}. \tag{27}$$

Next, it follows by method (2) (third step for $n=0$) that we can write

$$x_1-x_*=z_0-x_*-F'(z_0)^{-1}F(z_0)+C_0F'(x_*)F'(x_*)^{-1}F(z_0), \tag{28}$$

where

$$\begin{aligned}C_0 &= F'(z_0)^{-1}-F'(x_0)^{-1}-6[A_0^{-1}-\frac{1}{2}F'(x_0)^{-1}]\\ &= F'(z_0)^{-1}(F'(x_0)-F'(z_0))F'(x_0)^{-1}\\ &\quad -3A_0^{-1}[F'(x_0)-F'(x_*)+F``(x_*)-F'(y_0)]F'(x_0)^{-1},\end{aligned} \tag{29}$$

leading using the triangle inequality to

$$\begin{aligned}\|C_0F'(x)\| &\leq \frac{\omega_0(e_0)+\omega_0(\|z_0-x_*\|)}{(1-\omega_0(e_0))(1-\omega_0(\|z_0-x_*\|))}\\ &\quad+\frac{3}{2}\frac{(\omega_0(e_0)+\omega_0(\|z_0-x_*\|))}{(1-\omega_0(e_0))(1-\omega_0(\|z_0-x_*\|))}\\ &\leq c(e_0)e_0,\end{aligned} \tag{30}$$

where we also use the identities

$$\begin{aligned}F'(z_0)^{-1}-F'(x_0)^{-1} &= F'(z_0)^{-1}[(F'(x_0)-F'(x_*))\\ &\quad+(F'(x_*)-F'(z_0))]F'(x_0)^{-1}\end{aligned} \tag{31}$$

and

$$\begin{aligned}A_0^{-1}-\frac{1}{2}F'(x_0)^{-1} &= (F'(x_0)+F'(y_0))^{-1}-\frac{1}{2}F'(x_0)^{-1}\\ &= A_0^{-1}(F'(x_0)-\frac{1}{2}(F'(x_0)+F'(y_0)))F'(x_0)^{-1}\\ &= \frac{1}{2}A_0^{-1}[(F'(x_0)-F'(x_*))\\ &\quad+(F'(x_*)-F'(y_0))]F'(x_0)^{-1}\end{aligned} \tag{32}$$

By using (28)-(31), (15) (for $m = 3$) and the triangle inequality we find

$$\begin{aligned}\|x_1 - x_*\| &\leq [g_1(\|z_0 - x_*\|) + c(e_0)\int_0^1 \omega_1(\tau\|z_0 - x_*\|)d\tau]\|z_0 - x_*\| \\ &\leq g_2(e_0)e_0, \end{aligned} \tag{33}$$

showing $x_1 \in U(x_*, R)$ as well as (15) to be true. Hence, the verification of estimates (11), (12)-(15) for $n = 0$ is finished. Assuming (11)-(15) are true for $j = 0, 1, 2, \ldots, n-1$, and simply switching x_0, y_0, z_0, x_1 by x_j, y_j, z_j, x_{j+1} in the previous estimates, we immediately obtain that these estimates hold for $j = n$. Then, the induction for these estimates is terminated. We also have in particular

$$\|x_{n+1} - x_*\| \leq \mu e_0 < R, \tag{34}$$

with $\mu = g_3(e_0) \in [0, 1)$, so $\lim_{n\longrightarrow\infty} x_n = x_*$ and $x_{n+1} \in U(x_*, R)$. It is left to show the uniqueness of the solution x_* in the set D_1. Consider $v \in D_1$ with $F(v) = 0$ and let $M = \int_0^1 F'(v + \tau(x_* - v))d\tau$. Then, by (A1) and (A5) we obtain

$$\|F'(x_*)^{-1}(M - F'(x_*))\| \leq \int_0^1 \omega_0((1-\tau)\|x_* - v\|)d\tau \leq \int_0^1 \omega_0(\tau R_*)d\tau < 1,$$

so the invertability is implied leading together with the estimate $0 = F(x_*) - F(v) = M(x_* - v)$ to the conclusion that $x_* = v$.

□

Remark 2.2. *We can compute [24] the computational order of convergence (COC) defined by*

$$\xi = \ln\left(\frac{\|x_{n+1} - x^*\|}{\|x_n - x^*\|}\right) / \ln\left(\frac{\|x_n - x^*\|}{\|x_{n-1} - x^*\|}\right)$$

or the approximate computational order of convergence

$$\xi_1 = \ln\left(\frac{\|x_{n+1} - x_n\|}{\|x_n - x_{n-1}\|}\right) / \ln\left(\frac{\|x_n - x_{n-1}\|}{\|x_{n-1} - x_{n-2}\|}\right).$$

This way we obtain in practice the order of convergence without resorting to the computation of higher order derivatives appearing in the method or in the sufficient convergence criteria usually appearing in the Taylor expansions for the proofs of those results.

3. Numerical Examples

Example 3.1. *Let us consider a system of differential equations governing the motion of an object and given by*

$$F_1(x) = e^x,\ F_2(y) = (e-1)y + 1,\ F_3(z) = 1$$

with initial conditions $F_1(0) = F_2(0) = F_3(0) = 0$. *Let* $F = (F_1, F_2, F_3)$. *Let* $B_1 = B_2 = \mathbb{R}^3, D = \bar{U}(0, 1), p = (0, 0, 0)^T$. *Define function* F *on* D *for* $w = (x, y, z)^T$ *by*

$$F(w) = (e^x - 1, \frac{e-1}{2}y^2 + y, z)^T.$$

The Fréchet-derivative is defined by

$$F'(v) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Notice that using the $(\mathcal{A})$ *conditions, we get for* $\alpha = 1$, $\omega_0(t) = (e-1)t, \omega(t) = e^{\frac{1}{e-1}}t, \omega_1(t) = e^{\frac{1}{e-1}}$. *The radii are*

$$R_1 = 0.15440695, R_2 = 3.13632884,$$

$$R_3 = 0.00895286 \text{ and } R = R_3.$$

Example 3.2. *Let $B_1 = B_2 = C[0,1]$, the space of continuous functions defined on $[0,1]$ be equipped with the max norm. Let $D = \overline{U}(0,1)$. Define function F on D by*

$$F(\varphi)(x) = \varphi(x) - 5\int_0^1 x\theta\varphi(\theta)^3 d\theta. \tag{1}$$

We have that

$$F'(\varphi(\xi))(x) = \xi(x) - 15\int_0^1 x\theta\varphi(\theta)^2\xi(\theta)d\theta, \text{ for each } \xi \in D.$$

Then, we get that $x^* = 0$, so $\omega_0(t) = 7.5t, \omega(t) = 15t$ and $\omega_1(t) = 2$. Then the radii are

$$R_1 = 0.02222, R_2 = 0.435938,$$

$$R_3 = 0.0473229 \text{ and } R = R_1.$$

Example 3.3. *Returning back to the motivational example at the introduction of this study, we have $\omega_0(t) = \omega(t) = 96.6629073t$ and $\omega_1(t) = 2$. The parameters for method (2) are*

$$R_1 = 0.00229894, R_2 = 0.0364927,$$

$$R_3 = 0.000091765 \text{ and } R = R_3.$$

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